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# On the Injectivity and Essential Extension of Dynamical Systems

Mahboobeh Mohamadhasani

*Department of Mathematics, Hormozgan University, P.O. Box 3995, Bandarabbas, Iran*
*ma.mohamadhasani@gmail.com, m.mohamadhasani@hormozgan.ac.ir*

**Özet.** Dinamik sistemlerin birebirliği ve herhangi bir dinamik sistemin esas genişlemesi bu makaledeki en önemli kavramlardır. Herhangi bir dinamik sistem için birebir genişlemenin varlığını kanıtlıyoruz. Ayrıca, ast esas genişlemesi olmayan her dinamik sistemin birebir olduğunu anlıyoruz.[1]

**Anahtar Kelimeler.** Birebir dinamik sistem, geri çekme, denklik bağıntısı, esas genişleme.

**Abstract.** Injectivity of dynamical systems and essential extension of any dynamical system are the most important notions in this paper. We prove the existence of injective extension for any dynamical system. Also we see that every dynamical system without any proper essential extension is injective.

**Keywords.** Injective dynamical system, retract, congruence relation, essential extension.

## 1. Introduction

Motivated by some algebraic notions like injectivity in Module theory [2] we thought about injectivity in dynamical systems. In view of this goal, by using the definition of extended morphism, the notion of injectivity in dynamical systems is introduced. It is definitely very important to show the existence of an injective extension for any dynamical system, which is a useful tool for proving some theorems in the paper. Also this note proves the existence of an injective dynamical system.

Retraction and the relation between it and injectivity is the next notion.

In Section 3, essential extension of dynamical systems is a new notion to introduce. For continuation, in the same section we illustrated the use of congruence relation

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[1] Türkçe özet ve anahtar kelimeler, orijinal İngilizce metindeki ilgili kısmın doğrudan tercümesi olup *Çankaya University Journal of Science and Engineering* editörlüğü tarafından yazılmıştır. | Turkish abstract and the keywords are written by the editorial staff of *Çankaya University Journal of Science and Engineering* which are the direct translations of the related original English text.

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and generated congruence relation by a subset of cartesian product of state space in dynamical system. In a theorem, we see every dynamical system which has no proper essential extension, is injective.

## 2. Injectivity in Dynamical Systems

We see a dynamical system as this view [1]: $\{A, \{\varphi^i\}_{i\in Z}\}$ where $\varphi^i : A \to A$ is a mapping such that $\varphi^i \circ \varphi^j = \varphi^{i+j}$ and $\varphi^0 = \mathrm{id}_A$.

In [3] we stated the definition of extended morphism between two dynamical systems as the following:

**Definition 2.1.** [3] Let $\{A, \{\varphi^i\}_{i\in Z}\}$ and $\{B, \{\psi^i\}_{i\in Z}\}$ be dynamical systems. The mapping $f : A \to B$ is called an extended morphism between two dynamical systems $\{A, \{\varphi^i\}_{i\in Z}\}$ and $\{B, \{\psi^i\}_{i\in Z}\}$ if $f(\varphi^i(a)) = \psi^i(f(a))$, for all $i \in Z$ and $a \in A$.

If $f$ is one to one or onto then it is called an extended monomorphism or an extended epimorphism, respectively. If $f$ is one to one and onto then it is called an extended isomorphism

Obviously the composition of two extended morphism is an extended morphism as well.

**Definition 2.2.** [3] Let $\{A, \{\varphi^i\}_{i\in Z}\}$ be a dynamical system and $A' \subseteq A$, then $\{A', \{\varphi^i \mid_{A'}\}_{i\in Z}\}$ is called a subdynamical system of the dynamical system $\{A, \{\varphi^i\}_{i\in Z}\}$ if $\varphi^i(a) \in A'$ for all $i$ and for all $a \in A'$.

**Example 2.3.** (a) $\{R^+, \{\varphi^i \mid_{R^+}\}_{i\in Z}\}$ is a subdynamical system of $\{R, \{\varphi^i\}_{i\in Z}\}$ where

$$\begin{aligned} \varphi^i : R &\longrightarrow R \\ x &\longrightarrow xe^{i\theta} \end{aligned}$$

and $\theta \in R$ is a fixed element.

(b) $f : R \longrightarrow R$, where $f(x) = 2x$ is an extended morphism on the dynamical system $\{R^+, \{\varphi^i \mid_{R^+}\}_{i\in Z}\}$.

We define injective dynamical system as follows:

**Definition 2.4.** Let $\{A, \{\varphi^i\}_{i\in Z}\}$, $\{B, \{\psi^i\}_{i\in Z}\}$ and $\{C, \{\Phi^i\}_{i\in Z}\}$ be dynamical systems. The dynamical system $\{A, \{\varphi^i\}_{i\in Z}\}$ is said to be injective if for every extended monomorphism $h : B \to C$ and each extended morphism $f : B \to A$ there exists a mapping $g : C \to A$ such that $gh = f$.

The following proposition is an important key for the other theorems and propositions. In this proposition, we see any dynamical system has an injective extension. Also it shows the existence of injective dynamical system.

**Proposition 2.5.** *Any dynamical system has an injective extension.*

*Proof.* Let $\{A, \{\varphi^i\}_{i\in Z}\}$ be a dynamical system. $A$ can be embedded in $A^Z$ by $\rho : A \longrightarrow A^Z$ where $\rho(a) = f_a$ and $f_a(n) = a$, for all $n$ ($\rho$ is an extended monomorphism). We note that $\{A^Z, \{\psi^i\}_{i\in Z}\}$ can be considered as an dynamical system where

$$\psi^i(f) = \begin{cases} f_{\varphi^i(a)} & f = f_a \\ f & f \neq f_a \end{cases}.$$

If $\{B, \{\omega^i\}_{i\in Z}\}$ and $\{C, \{\tau^i\}_{i\in Z}\}$ be dynamical systems and $k : B \longrightarrow C$ and $g : B \longrightarrow A^Z$ be extended monomorphism and extended morphism, respectively, between dynamical systems. We consider $h : C \longrightarrow A^Z$ where

$$h(c)(n) = \begin{cases} g(k^{-1}(c))(n) & c \in k(B) \\ f_0(n) & c \notin k(B) \end{cases}$$

and $f_0 : Z \longrightarrow A$ is a fixed point in $A^Z$ . Clearly $h \circ k = g$. $\square$

**Corollary 2.6.** *Any dynamical system can be embedded in an injective dynamical system.*

We define the product of dynamical systems as the following:

Let $\{A_j, \{\varphi_j^i\}_{i\in Z}\}_{j\in J}$ be a family of dynamical systems then

$$\left\{\prod_{j\in J} A_j, \left\{(\prod_{j\in J}\varphi_j)^i\right\}_{i\in Z}\right\}$$

is a dynamical system where

$$\begin{aligned} (\textstyle\prod_{j\in J}\varphi_j)^i : \textstyle\prod_{j\in J} A_j &\to \textstyle\prod_{j\in J} A_j \\ \{a_j\}_{j\in J} &\mapsto \{\varphi_j^i(a_j)\}_{j\in J} \end{aligned}$$

It is easy to see that:

**Proposition 2.7.** *Let $\{\{Q_j, \{\varphi_j^i\}_{i\in Z}\}, j \in J\}$ be a family of injective dynamical systems then $\{\prod Q_j, \{(\prod \varphi_j)^i\}_{i\in Z}\}$ is an injective dynamical system.*

**Definition 2.8.** The subdynamical system $\{A, \{\varphi^i\mid_A\}_{i\in Z}\}$ of the dynamical system $\{B, \{\varphi^i\}_{i\in Z}\}$ is said to be retract of dynamical system $\{B, \{\varphi^i\}_{i\in Z}\}$ if there exists an extended morphism $f : B \to A$ such that $f\mid_A = \text{id}$.

**Example 2.9.** In Example 2.3, the subdynamical system $\{R^+, \{\varphi^i\mid_{R^+}\}_{i\in Z}\}$ is a retract of dynamical system of $\{R, \{\varphi^i\}_{i\in Z}\}$. Because there exists extended morphism $f : R \longrightarrow R^+$ where $f(x) = |x|$ and $f\mid_{R^+} = \text{id}$.

A dynamical system is said to be absolutely retract if it is retract of any its extension.

**Lemma 2.10.** *Any retract of an injective dynamical system is injective.*

*Proof.* Let $\{A, \{\varphi^i\mid_A\}_{i\in I}\}$ be retract of injective dynamical system $\{B, \{\varphi^i\}_{i\in I}\}$. Then there exists extended morphism $g : B \longrightarrow A$ such that $g\mid_A = \text{id}$. Also let $\{C, \{\psi^i\}_{i\in Z}\}$ and $\{D, \{\rho^i\}_{i\in Z}\}$ be dynamical systems and $f' : C \longrightarrow D$, $f'' : C \longrightarrow A$ be extended monomorphism and extended morphism, respectively. Now there exists a mapping $f : D \longrightarrow B$ such that $ff' = if''$ where $i : A \longrightarrow B$ is inclusion. Hence $gff' = f''$. □

**Corollary 2.11.** *Any absolutely retract dynamical system is injective.*

*Proof.* By Proposition 2.5, $\{A, \{\varphi^i\mid_A\}_{i\in Z}\}$ has an injective extension like $\{D, \{\varphi^i\}_{i\in Z}\}$. Then there exists extended morphism $h : D \longrightarrow A$ such that $h\mid_A = \text{id}$. Let $f : B \to C$ and $g : B \to A$ be extended monomorphism and extended morphism, respectively. Hence there exists $\varphi : C \longrightarrow D$ such that $\varphi f = \subseteq g$, where $\subseteq : A \hookrightarrow D$ is inclusion. Therefore $h\varphi f = g$. □

## 3. Essential Dynamic System and Injectivity

In this section we begin with the definition of congruence relation and an important theorem. Then we define essential dynamical system and get some interesting notes about essentiality and injectivity.

**Definition 3.1.** Let $\{A, \{\varphi^i\}_{i\in Z}\}$ be a dynamical system and $\rho$ be an equivalence relation on $A$. We say $\rho$ is a congruence relation on the above dynamical system if for all $a, a' \in A$ where $a\ \rho\ a'$, we have $\varphi^i(a)\ \rho\ \varphi^i(a')$ for every $i \in Z$.

**Example 3.2.** Let

$$\begin{aligned} \varphi^i : R &\to R \\ x &\mapsto xe^{i\theta} \end{aligned}$$

where $\theta$ is a constant. $\{R, \varphi^i\}_{i\in Z}$ is a dynamical system. We define an equivalence relation $\equiv$ on $R$ as the following:

$x \equiv y$ if and only if $x, y \geq 0$ or $x, y < 0$.

It is obvious that $x \equiv y \Leftrightarrow \varphi^i(x) \equiv \varphi^i(y)$, for all $i$. Then $\equiv$ is a congruence relation.

If $\theta$ is a congruence relation on dynamical system $\{B, \{\varphi^i\}_{i\in Z}\}$ then by considering $B/\theta = \{[b]_\theta \mid b \in B\}$ and

$$\begin{aligned} \psi^i : B/\theta &\to B/\theta \\ [b]_\theta &\to [\varphi^i(b)]_\theta \end{aligned}$$

$\{B/\theta, \{\psi^i\}_{i\in Z}\}$ is a dynamical system.

**Proposition 3.3.** *Let $\{A, \{\varphi^i\}_{i\in Z}\}$ be a dynamical system, $X \subseteq A\times A$ and $\rho = \rho(X)$ be the generated congruence relation by $X$. Then for every $a, b \in A$, $a\,\rho\, b$ if and only if $a = b$ or there exist a natural number $n$ and a sequence $a = \varphi^{t_1}(c_1)$,*

$$\varphi^{t_1}(d_1) = \varphi^{t_2}(c_2),\ \varphi^{t_2}(d_2) = \varphi^{t_3}(c_3),\ ...,\ \varphi^{t_{n-1}}(d_{n-1}) = \varphi^{t_n}(c_n),\ \varphi^{t_n}(d_n) = b$$

*such that $t_1, ..., t_n \in Z$ and $((c_i, d_i) \in X$ or $(d_i, c_i) \in X)$, for all $i \in \{1, 2, ..., n\}$.*

*Proof.* We show that the above defined congruence relation is the smallest congruence relation which is included in $X$. It is clear to see $\rho$ is reflexive and symmetric. Also if $a\,\rho\, b$ and $b\,\rho\, c$ then there exist natural numbers $n$ and $k$ and two sequences

$$a = \varphi^{t_1}(c_1),$$

$$\varphi^{t_1}(d_1) = \varphi^{t_2}(c_2),\ \varphi^{t_2}(d_2) = \varphi^{t_3}(c_3),\ ...,\ \varphi^{t_{n-1}}(d_{n-1}) = \varphi^{t_n}(c_n),$$

$$\varphi^{t_n}(d_n) = b$$

and

$$b = \varphi^{t'_1}(c'_1),$$

$$\varphi^{t'_1}(d'_1) = \varphi^{t'_2}(c'_2),\ \varphi^{t'_2}(d'_2) = \varphi^{t'_3}(c'_3),\ ...,\ \varphi^{t'_{k-1}}(d'_{k-1}) = \varphi^{t'_k}(c'_k),$$

$$\varphi^{t'_k}(d'_k) = c$$

such that for all $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, $t_i, t'_j \in Z$ and $((c_i, d_i) \in X$ or $(d_i, c_i) \in X)$. Also $(c'_j, d'_j) \in X$ or $(d'_j, c'_j) \in X$.

Now by choosing $d_{n+j} := d'_j, c_{n+j} := c'_j$ and $t_{n+j} := t'_j$ we have the following sequence

$$a = \varphi^{t_1}(c_1),$$

$$\varphi^{t_1}(d_1) = \varphi^{t_2}(c_2),\ ...,\ \varphi^{t_n}(d_n) = \varphi^{t_{n+1}}(c_{n+1}),\ ...,\ \varphi^{t_{n+k-1}}(d_{n+k-1}) = \varphi^{t_{n+k}}(c_{n+k}),$$

$$\varphi^{t_{n+k}}(d_{n+k}) = c$$

such that for $i \in \{1, 2, ..., n+k\}$, $t_i \in Z$ and $((c_i, d_i) \in X$ or $(d_i, c_i) \in X)$. It shows $a\,\rho\,c$. Also if $a = b$ or $b = c$ then $a\,\rho\,c$. Hence $\rho$ is an equivalence relation. It is clear that if $a\,\rho\,b$ then $\varphi^i(a)\,\rho\,\varphi^i(b)$, for all $i \in Z$.

Let $\theta$ be a congruence relation on $A$ which is included in $X$ and $(a, b) \in \rho(X)$.

If $a = b$ then by the reflexivity property $(a, b) \in \theta$.

Now let $a = \varphi^{t_1}(c_1)$, $\varphi^{t_1}(d_1) = \varphi^{t_2}(c_2)$, ..., $\varphi^{t_{n-1}}(d_{n-1}) = \varphi^{t_n}(c_n)$, $\varphi^{t_n}(d_n) = b$ where for all $i \in \{1, 2, ..., n\}$, $((c_i, d_i) \in X$ or $(d_i, c_i) \in X)$ and $t_i \in Z$. Hence by respect to $X \subseteq \theta$, where $(\varphi^{t_1}(c_1), \varphi^{t_2}(d_2)) \in \theta$.

Now let $(\varphi^{t_1}(c_1), \varphi^{t_{i-1}}(d_{i-1})) \in \theta$ for all $i \leq n$. Because of $\varphi^{t_n}(c_n)\theta\varphi^{t_n}(d_n)$ and $\varphi^{t_n}(c_n) = \varphi^{t_{n-1}}(d_{n-1})$ and transitivity's property of $\theta$, $(a, b) \in \theta$. □

**Corollary 3.4.** *Let $\{A, \{\varphi^i\}_{i\in Z}\}$ be a dynamical system and $a_1, a_2 \in A$. Then the smallest relation on $A$ which is included $\{(a_1, a_2)\}$ is defined as the following and is denoted by $[a_1, a_2]$:*

$$x_1[a_1, a_2]x_2$$

*if and only if*

$$(x_1 = a_i, x_2 = a_j, i, j \in \{1, 2\})$$

*or*

$$x_1 = \varphi^{s_1}(a_{j_1}),\ \varphi^{s_1}(a_{j'_1}) = \varphi^{s_2}(a_{j_2}),\ ...,\ \varphi^{s_n}(a_{j'_n}) = x_2,$$

*where $(a_{j_k}, a_{j'_k}) = (a_1, a_2)$ or $(a_{j_k}, a_{j'_k}) = (a_2, a_1)$, for all $k = 1, 2, ..., n$, $s_1, ..., s_n \in Z$.*

Now let $f : A \longrightarrow B$ be a extended morphism between dynamical systems. We can define the kernel as the following:

$$a\ \ker f\ a' \Leftrightarrow f(a) = f(a').$$

**Definition 3.5.** Let $\{A, \{\varphi^i\mid_A\}_{i\in Z}\}$ is a subdynamical system of dynamical system $\{B, \{\varphi^i\}_{i\in Z}\}$. $B$ is an essential extension of $A$ if every extended morphism $h : B \longrightarrow C$ is monomorphism if $h\mid_A$ is monomorphism, for every dynamical system $\{C, \{\psi^i\}_{i\in Z}\}$.

**Example 3.6.** Let $\{A, \{\varphi^i\mid_A\}_{i\in Z}\}$ be a subdynamical system of $\{B, \{\varphi^i\}_{i\in Z}\}$. Then by Zorn's lemma the set $L = \{\theta \in \text{con}(B) \mid \theta \cap A^2 = \Delta_A\}$ has a maximal element $\theta$. Now $B/\theta$ is essential extension of $A/\theta$. (see Theorem 3.9)

**Lemma 3.7.** *Let $\{A, \{\varphi^i\mid_A\}_{i\in Z}\}$ be a subdynamical system of dynamical system $\{B, \{\varphi^i\}_{i\in Z}\}$. Then the following conditions are equivalent:*

(1) *$B$ is essential extension of $A$,*

(2) *for every $\theta \in \mathrm{con}(B)$ such that $A \hookrightarrow B \longrightarrow B/\theta$ is a monomorphism then $\theta = \Delta_B$, where $\Delta_B = \{(a,a) \mid a \in B\}$,*

(3) *for every $\theta \in con(B)$ if $\theta \cap A^2 = \Delta_A$ then $\theta = \Delta_B$.*

*Proof.* $(1 \Rightarrow 2 \Rightarrow 3)$: is clear.

$(3 \Rightarrow 1)$: Let $h : B \longrightarrow C$ be an extended morphism of dynamical systems such that $h\mid_A$ is monomorphism. Now we consider $\theta := \ker h$. Since $\Delta_A = \ker h\mid_A = \theta \cap A^2$, we have $\theta = \Delta_B$. □

**Theorem 3.8.** *If $\{A, \{\varphi^i \mid_A\}_{i\in Z}\}$ and $\{C, \{\varphi^i \mid_C\}_{i\in Z}\}$ are subdynamical systems of $\{B, \{\varphi^i\}_{i\in Z}\}$. Also $A \subseteq C$ and $B$ is essential extension of $C$. Then the following conditions are equivalent:*

(1) *$B$ is essential extension of $A$,*

(2) *If $\theta$ is congruence relation on $B$ such that $\theta \neq \Delta_B$ then $\theta \cap A^2 \neq \Delta_A$,*

(3) *For all $b_1, b_2 \in B$ such that $b_1 \neq b_2$ there exist $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $a_1[b_1, b_2]a_2$,*

(4) *Condition (2) is correct for every congruence relation on $C$,*

(5) *$C$ is essential extension of $A$.*

*Proof.* $(1\Rightarrow 2)$: is the same as $(1\Rightarrow 3)$ of the previous Lemma.

$(2\Rightarrow 3)$: Since $b_1 \neq b_2$ we have $[b_1, b_2] \neq \Delta_B$. This completes proof.

$(3\Rightarrow 4)$: Let $\theta$ is a congruence relation on $C$, where $\theta \neq \Delta_C$. Since there exists $b_1, b_2 \in C$ where $b_1 \neq b_2$ and $b_1\,\theta\,b_2$, there exists $a_1, a_2 \in A$ where $a_1[b_1, b_2]a_2$ and $a_1 \neq a_2$. Then by Corollary 3.4, $a_1\,\theta\,a_2$.

$(4\Rightarrow 5)$: Let $\varphi : C \longrightarrow D$ is an extended morphism where is not one to one. Now by consideration $[\varphi] = \ker \varphi$ we have $[\varphi] \neq \Delta_C$, by hypothesis $[\varphi] \cap A^2 \neq \Delta_A$.

$(5\Rightarrow 1)$: It is clear. □

Then essentiality has transitive property, meaning if $B$ is essential extension of $A$ and $C$ is essential extension of $B$ then $C$ is essential extension of $A$. It is clear that essentiality is an equivalence relation.

**Theorem 3.9.** *Let $\{A, \{\varphi^i \mid_A\}_{i\in Z}\}$ be a subdynamical system of dynamical system $\{B, \{\varphi^i\}_{i\in Z}\}$ and $\Theta$ be the maximal relation on $B$ in the following set $L = \{\theta \in \mathrm{con}(B) \mid \theta \cap A^2 = \Delta_A\}$. Then $B/\Theta$ is essential extension of $A/\Theta$. Also $A/\Theta$ is isomorphic to $A$.*

*Proof.* The natural mapping $A \longrightarrow A/\Theta$ is one to one, because

$$[a]_\Theta = [b]_\Theta \Rightarrow (a,b) \in \Theta \cap A^2 = \Delta_A \Rightarrow a = b.$$

Let $\eta/\Theta$ is congruence relation on $B/\Theta$ such that $\eta/\Theta \cap (A/\Theta)^2 = \Delta_{A/\theta}$ then $\eta$ is the congruence relation on $B$ such that $\Theta \subseteq \eta,\ \eta \cap A^2 = \Delta_A$. Because for all $a_1, a_2 \in A$ where $a_1\,\eta\,a_2$ we have $[a_1]_\Theta(\eta/\Theta)[a_2]_\Theta$. Now with respect to $(\eta/\Theta) \cap (A/\Theta)^2 = \Delta_{A/\theta}$ we have $[a_1]_\Theta = [a_2]_\Theta$. Hence $a_1 = a_2$. Since $\Theta$ is maximal in $L$, we have $\eta = \Theta$ and $\eta/\Theta = \Delta_{B/\theta}$. Then by Lemma 3.7 ($3 \Rightarrow 1$), $B/\Theta$ is essential extension of $A/\Theta$. □

**Theorem 3.10.** *Every dynamical system $\{A, \{\varphi^i\}_{i\in Z}\}$ without any proper essential extension is injective.*

*Proof.* Let $B$ be proper extension of $A$. By Corollary 2.11, it is enough to prove $A$ is retract of $B$. Using Lemma 3.7, there exists $\theta' \in \mathrm{con}(B)$ such that $\theta' \cap A^2 = \Delta_A$ and $\theta' \neq \Delta_B$. By Zorn's lemma, $L = \{\theta \in \mathrm{con}(B) \mid \theta \cap A^2 = \Delta_A\}$ has a maximal element $\theta_0$. By Theorem 3.9, $B/\theta_0$ is essential extension of $A/\theta_0$ and $A/\theta_0$ is isomorphic to $A$. Hence $A/\theta_0 = B/\theta_0$ and for all $b \in B$ there exists unique $a \in A$ such that $[a]_{\theta_0} = [b]_{\theta_0}$. Now there exists an extended morphism $\psi : B \longrightarrow A$ such that $\psi\mid_A = \mathrm{id}$. □

## 4. Conclusion

In this paper we introduced the new notions of dynamical systems. Injectivity, retraction and essentiality were introduced and in some theorems we considered their interesting relations. In one of the most important theorem we saw that every dynamical system without any proper essential extension is injective. In the future we want to work on the essentiality and its relation with the other types of injectivity.

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