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Topological Functors via Closure Operators

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Özet. Bu makalede, verilen bir $\mathcal{X}$ kategorisi için, verilen bir $\mathcal{M} \subseteq \mathcal{X}_1$ derlemesi üzerindeki kapanış operatörlerinin belli kategorilerini, $\mathcal{X}$ üzerindeki önsınıf-değerli esnek öndemetlerin belli kategorilerinin içine tam gömüyoruz. Daha sonra, $\mathcal{X}$ üzerindeki önsınıf-değerli esnek öndemetlerin biraz önce bahsi geçen kategorilerini, $\mathcal{X}$ üzerindeki topolojik izleçlerin belli kategorilerinin içine tam gömüyoruz. Elde edilen dolu gömmeleri birleştirerek, verilen bir kapanış operatöründen bir topolojik izleç inşa ediyoruz.[†]

Anahtar Kelimeler. Kapanış operatörü, esnek öndemet, esnek doğal dönüşüm, (tam) önsıralı ya da kısmi sıralı sınıf, (zayıf) topolojik izleç.

Abstract. In this article for a given category $\mathcal{X}$, we fully embed certain categories of closure operators on a given collection $\mathcal{M} \subseteq \mathcal{X}_1$, in certain categories of preclass-valued lax presheaves on $\mathcal{X}$. We then fully embed the just mentioned categories of preclass-valued lax presheaves on $\mathcal{X}$, in certain categories of topological functors on $\mathcal{X}$. Combining the full embeddings obtained, we construct a topological functor from a given closure operator.

Keywords. Closure operator, lax presheaf, lax natural transformation, (complete) preordered or partially ordered class, (weak) topological functor.

1. Introduction

The categorical notion of closure operators has unified several notions in different areas of mathematics, [12]. It is studied in connection with many other notions as well as the notion of topological functors. Closure operators and/or topological functors have been investigated in [1] to show full functors and topological functors form a weak factorization system in the category of small categories, in [3], to characterize the notions of compactness, perfectness, separation, minimality and absolute closedness with respect to certain closure operators in certain topological categories, in [4] to show that the category of MerTop is topological over Top and to

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study certain related closure operators, in [5] to verify that there is a bicoreflective general process available for carrying out certain constructions and that the bicoreflector can be adapted to respect a closure operator when the topological construct is endowed with such, in [6] to prove certain categories are topological, in [8] to define connectedness with respect to a closure operator in a category and to show that under appropriate hypotheses, most classical results about topological connectedness can be generalized to this setting, in [9] to define and compare an internal notion of compact objects relative to a closure operator and relative to a class of morphisms, in [10] to show that $\mathrm{Alg}(T)$ as well as some other categories are topological, in [11] to provide a product theorem for c-compact objects which gives the known Tychonoff's Theorem, in [13] to investigate epireflective subcategories of topological categories by means of closure operators, in [14] to study initial closure operators which include both regular and normal closure operators, in [15] to study the concepts of isolated submodule, honest submodule, and relatively divisible submodule, in [16] in connection with semitopologies, in [17] to show certain fuzzy categories are topological and extended fuzzy topologies are given dually as a certain fuzzy closure operators, in [18] to study the notions of closed, open, initial and final morphism with respect to a closure operator, in [19] to give a connection between closure operators, weak Lawvere-Tierney topologies and weak Grothendieck topologies and in [21] to prove for a topological functor over B, every cocontinuous left action of $B(b,b)$ on any cocomplete poset can be realized as the final lift action associated to a canonically defined topological functor over B; to mention a few.

The categories we consider in this paper are generally quasicategories in the sense of [2], however we refer to them as categories.

For a given category $\mathcal{X}$, in Section 2 of the paper, we introduce the categories, $\mathrm{Cl}(\mathcal{X})$ ($\mathrm{Cl}_s(\mathcal{X})$), of closure operators (respectively, semi-idempotent closure operators) and we show they can be fully embedded in the categories, $\mathrm{Prcls}_{\mathrm{LL}}^{\mathcal{X}^{\mathrm{op}}}$ (respectively, $\mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$), of preclass-valued lax presheaves (respectively, preclass-valued semi presheaves). We also consider the cases where the domain of the closure operator is a complete preordered class, or a complete partially ordered class and fully embed the corresponding categories in complete preclass-valued lax presheaves, etc. In Section 3, we show the category $\mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$ can be fully embedded in the category $\mathrm{CAT}(\mathcal{X})$ of concrete categories over $\mathcal{X}$. In Section 4, we fully embed the category $\mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$ in the category, $\mathrm{WTop}_1(\mathcal{X})$, of weak 1-topological categories over $\mathcal{X}$. We also prove if the semi presheaves are complete preclass valued, then the embedding factors through the category, WTop($\mathcal{X}$), of weak topological categories over $\mathcal{X}$; and that if they are poclass valued, then the embedding factors through the category, Top($\mathcal{X}$), of topological categories over $\mathcal{X}$. We conclude this section by combining the previously obtained full embeddings to get (weak) topological categories from given closure operators. Finally, in Section 5, we give several examples.

2. Lax Presheaves via Closure Operators

For a category $\mathcal{X}$, we denote the collection of objects by $\mathcal{X}_0$ and the collection of morphisms by $\mathcal{X}_1$.

Definition 2.1. Let $\mathcal{X}$ be a category and for $x \in \mathcal{X}_0$, $\mathcal{X}_1/x$ be the class of all morphisms to x. Define a preorder on $\mathcal{X}_1/x$, by $f \leq g$ if there is a morphism α such that $f = g \circ \alpha$ and let "$\sim$" be the equivalence relation generated by "$\leq$", so that $f \sim g$ if and only if $f \leq g$ and $g \leq f$. For $\mathcal{M} \subseteq \mathcal{X}_1$, the above preorder and equivalence relation on $\mathcal{X}_1/x$ can be passed over to $\mathcal{M}/x$. Also we write $m \sim \mathcal{M}/x$ ($m \sim \mathcal{M}$) if there is $n \in \mathcal{M}/x$ ($n \in \mathcal{M}$) such that $m \sim n$.

Denoting a pullback of g along f by $f^{-1}(g)$, one can easily verify:

Lemma 2.2. *Let $f : x \longrightarrow y$ be a morphism and $g, h \in \mathcal{X}_1/y$ such that $f^{-1}(g)$ and $f^{-1}(h)$ exist.*

(i) *If $g \leq h$, then $f^{-1}(g) \leq f^{-1}(h)$.*
(ii) *If $g \sim h$, then $f^{-1}(g) \sim f^{-1}(h)$.*

Definition 2.3. $\mathcal{M}$ has $\mathcal{X}$-pullbacks if for all $f : x \longrightarrow y$ in $\mathcal{X}_1$, whenever $m \in \mathcal{M}/y$, then a pullback, $f^{-1}(m)$, of m along f exists and $f^{-1}(m) \in \mathcal{M}/x$.

Definition 2.4. Let $\mathcal{M} \subseteq \mathcal{X}_1$ have $\mathcal{X}$-pullbacks. A closure operator $c_{\mathcal{M}}$ on $\mathcal{M}$ is a family of $\{c^x_{\mathcal{M}} : \mathcal{M}/x \longrightarrow \mathcal{M}/x\}_{x \in \mathcal{X}_0}$ of functions with the following properties:

(i) For every $m \in \mathcal{M}/x$, $m \leq c^x_{\mathcal{M}}(m)$ (expansiveness),
(ii) For $m, n \in \mathcal{M}/x$ with $m \leq n$, $c^x_{\mathcal{M}}(m) \leq c^x_{\mathcal{M}}(n)$ (order preservation),
(iii) For every $f : x \longrightarrow y \in \mathcal{X}_1$ and $m \in \mathcal{M}/y$, $c^x_{\mathcal{M}}(f^{-1}(m)) \leq f^{-1}(c^y_{\mathcal{M}}(m))$ (continuity).

Sometimes we use the notations $\bar{f}$ or $c_{\mathcal{M}}(f)$ instead of $c^x_{\mathcal{M}}(f)$.

Definition 2.5. Let $\mathcal{X}$ be a category with a closure operator $c_{\mathcal{M}}$ on it.

(i) An object $m \in \mathcal{M}$ is called semi-closed if $\overline{m} \sim m$. A closure operator $c_{\mathcal{M}}$ is called semi-identity if all the members of $\mathcal{M}$ are semi-closed.
(ii) An object $m \in \mathcal{M}$ is called semi-idempotent if $\overline{m}$ is semi-closed. A closure operator $c_{\mathcal{M}}$ is called semi-idempotent if all the members of $\mathcal{M}$ are semi-idempotent.

Lemma 2.6. *Let $c_{\mathcal{M}}$ be a closure operator.*

(i) *If $m \in \mathcal{M}$ is semi-closed, then so is $f^{-1}(m)$.*
(ii) *If $m \in \mathcal{M}$ is semi-idempotent, then $f^{-1}(\overline{m})$ is semi-closed.*

Proof. (i) By Lemma 2.2 (ii), $f^{-1}(m) \leq \overline{f^{-1}(m)} \leq f^{-1}(\overline{m}) \sim f^{-1}(m)$. The result follows.

(ii) Follows from part (i) and the fact that $\overline{m}$ is semi-closed. □

Definition 2.7. A closure morphism, $c : c_{\mathcal{M}} \longrightarrow c_{\mathcal{N}}$, from a closure operator $c_{\mathcal{M}}$ to a closure operator $c_{\mathcal{N}}$ is a family of order preserving maps $\{c^x : \mathcal{M}/x \longrightarrow \mathcal{N}/x\}_{x\in\mathcal{X}_0}$ such that for each $f : x \longrightarrow y$ in $\mathcal{X}_1$ and each m in $\mathcal{M}/y$, $c^x(f^{-1}(\overline{m})) \leq f^{-1}(\overline{c^y(m)})$.

The collection of the identities form a closure morphism $1_{c_{\mathcal{M}}} : c_{\mathcal{M}} \longrightarrow c_{\mathcal{M}}$ and for morphisms $c : c_{\mathcal{M}} \longrightarrow c_{\mathcal{N}}$ and $c' : c_{\mathcal{N}} \longrightarrow c_{\mathcal{K}}$, $c' \circ c(f^{-1}(\overline{m})) \leq c'(f^{-1}(\overline{c(m)})) \leq (f^{-1}(\overline{c'(c(m))}))$. Hence $c' \circ c$ is a closure morphism. So we have:

Lemma 2.8. *The closure operators in a category $\mathcal{X}$ whose domain has $\mathcal{X}$-pullbacks, together with the closure morphisms form a category.*

We denote the category of Lemma 2.8, whose objects are the closure operators in a category $\mathcal{X}$ for which the domain has $\mathcal{X}$-pullbacks, and whose morphisms are the closure morphisms, by $\mathrm{Cl}(\mathcal{X})$. The full subcategory of $\mathrm{Cl}(\mathcal{X})$ whose objects are semi-idempotent is denoted by $\mathrm{Cl}_s(\mathcal{X})$.

With Prcls the category of preclasses with order preserving maps, we have:

Definition 2.9. (a) A preclass valued lax presheaf $M : \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Prcls}$ is a map that satisfies the following two conditions:
(i) For each $x \in \mathcal{X}$, $1_{M(x)} \leq M(1_x)$.
(ii) For each $f, g \in \mathcal{X}_1$, $M(f \circ g) \leq M(g) \circ M(f)$.
A preclass valued semi presheaf is a preclass valued lax presheaf satisfying

(ii)′ For each $f, g \in \mathcal{X}_1$, $M(f \circ g) \sim M(g) \circ M(f)$.

(b) A lax natural transformation $\varphi : M \longrightarrow M'$ is a transformation such that for each morphism $f : x \longrightarrow y$, one has $\varphi_x \circ M(f) \leq M'(f) \circ \varphi_y$.

If $\varphi : M \longrightarrow M'$ and $\psi : M' \longrightarrow M''$ are lax natural transformations, then for each morphism $f : x \longrightarrow y$ we have $(\psi \circ \varphi)_x \circ M(f) \leq \psi_x \circ M'(f) \circ \varphi_y \leq M''(f) \circ \psi_y \circ \varphi_y = M''(f) \circ (\psi \circ \varphi)_y$. So $\psi \circ \varphi$ is a lax natural transformation. It follows that:

Lemma 2.10. *Lax presheaves and lax natural transformations on $\mathcal{X}$ form a category.*

We denote the category of Lemma 2.10 by $\mathrm{Prcls}_{\mathrm{LL}}^{\mathcal{X}^{\mathrm{op}}}$ and its full subcategory whose objects are semi presheaves by $\mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$.

Definition 2.11. For $c_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ in $\mathrm{Cl}(\mathcal{X})$, let $M_c : \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Prcls}$ be the mapping that takes $f : x \longrightarrow y$ to $M_c(f) : \mathcal{M}/y \longrightarrow \mathcal{M}/x$, where $M_c(f)(m) = f^{-1}(\overline{m})$ for f the identity morphism, we pick f^{-1} to act like identity.

Proposition 2.12. *M_c is a lax presheaf.*

Proof. Since $\mathcal{M}$ has $\mathcal{X}$-pullbacks, $M_c(f)$ is well-defined. For $m, n \in \mathcal{M}/y$ with $m \leq n$, $\overline{m} \leq \overline{n}$ and consequently for each $f : x \longrightarrow y$, $f^{-1}(\overline{m}) \leq f^{-1}(\overline{n})$. So $M_c(f)$ is a morphism in Prcls.

For $m \in M_c(x)$ and morphisms $f : x \longrightarrow y$ and $g : y \longrightarrow z$, we have $m \leq \overline{m} = M_c(1)(m)$ and $M_c(g \circ f)(m) = (g \circ f)^{-1}(\overline{m}) \sim f^{-1} \circ g^{-1}(\overline{m}) \leq f^{-1}(\overline{g^{-1}(\overline{m})}) = M_c(f) \circ M_c(g)(m)$. So $M_c : \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Prcls}$ is a lax presheaf. □

Definition 2.13. For $c : c_{\mathcal{M}} \longrightarrow c_{\mathcal{N}}$ in $\mathrm{Cl}(\mathcal{X})$, let $\theta_c : M_c \longrightarrow N_c$ be the transformation defined by the collection $\{c^x : \mathcal{M}/x \longrightarrow \mathcal{N}/x\}_{x \in \mathcal{X}_0}$, so that $(\theta_c)_x = c^x$.

Proposition 2.14. *θ_c is a lax natural transformation.*

Proof. For each m, we have $(\theta_c)_x \circ M_c(f)(m) = (\theta_c)_x(f^{-1}(\overline{m})) = c^x(f^{-1}(\overline{m})) \leq f^{-1}(\overline{c^y(m)}) = N_c(f)(c^y(m)) = N_c(f) \circ (\theta_c)_y(m)$. Hence θ_c is a lax natural transformation. □

Theorem 2.15. (i) *The mapping $\mathbb{L} : \mathrm{Cl}(\mathcal{X}) \longrightarrow \mathrm{Prcls}_{\mathrm{LL}}^{\mathcal{X}^{\mathrm{op}}}$, that takes the object $c_{\mathcal{M}}$ to M_c and the morphism $c : c_{\mathcal{M}} \longrightarrow c_{\mathcal{N}}$ to θ_c, is a full embedding.*

(ii) *The full embedding $\mathbb{L}$ restricted to $\mathrm{Cl}_s(\mathcal{X})$ factors through $\mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$, yielding a full embedding $\mathbb{L}_s : \mathrm{Cl}_s(\mathcal{X}) \longrightarrow \mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$.*

Proof. (i) One can easily verify that $\mathbb{L}$ is a faithful functor.

Now we show $\mathbb{L}$ is one to one on objects. For this aim let $\mathbb{L}(c_{\mathcal{M}}) = \mathbb{L}(c_{\mathcal{N}})$. So for each $x \in \mathcal{X}_0$ we have $\mathcal{M}/x = \mathcal{N}/x$, and therefore $\mathcal{M} = \mathcal{N}$. Also for $1_x : x \longrightarrow x$ and each $m \in \mathcal{M}$ we have $M_c(1_x)(m) = N_c(1_x)(m)$, i.e $c_{\mathcal{M}}(m) = c_{\mathcal{N}}(m)$, consequently $c_{\mathcal{M}} = c_{\mathcal{N}}$.

Faithfulness and the fact that $\mathbb{L}$ is one to one on objects renders $\mathbb{L}$ an embedding.

Finally to show $\mathbb{L}$ is full, let $\theta : M_c \longrightarrow N_c$ be in $\hom(\mathbb{L}(c_{\mathcal{M}}), \mathbb{L}(c_{\mathcal{N}}))$. Define $c : c_{\mathcal{M}} \longrightarrow c_{\mathcal{N}}$ by $c(f) = \theta(f)$. Since $c(f^{-1}(\overline{m})) = \theta(f^{-1}(\overline{m})) = \theta(M(f)(m)) \leq N(f)(\theta(m)) = f^{-1}(\overline{c(m)})$, c is in $\hom(c_{\mathcal{M}}, c_{\mathcal{N}})$ and it easily follows that $\mathbb{L}(c) = \theta$.

(ii) We first need to show that for each object $c_{\mathcal{M}}$ in $\mathrm{Cl}_s(\mathcal{X})$, $\mathbb{L}(c_{\mathcal{M}})$ is a semi presheaf. Let $c_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ be in $\mathrm{Cl}(\mathcal{X})$. For $m \in M_c(x)$, we have $m \leq \overline{m} \sim M_c(1)(m)$; and for morphisms $f : x \longrightarrow y$ and $g : y \longrightarrow z$, since $c_{\mathcal{M}}$ is a semi-idempotent closure operator, Lemma 2.6 implies, $M_c(g \circ f)(m) = (g \circ f)^{-1}(\overline{m}) \sim f^{-1} \circ g^{-1}(\overline{m}) \sim f^{-1}(\overline{g^{-1}(\overline{m})}) = M_c(f) \circ M_c(g)(m)$. Hence M_c is a semi presheaf.

The fact that $\mathbb{L}$ is an embedding will easily imply that so is $\mathbb{L}_s$. $\square$

Definition 2.16. Let $\mathcal{M}$ be a collection of morphisms in $\mathcal{X}$ and $c_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ be a closure operator.

(i) $\mathcal{M}$ is locally complete if for all $x \in \mathcal{X}$, $\mathcal{M}/x$ is complete, i.e. it has meets.
(ii) $\mathcal{M}$ is stably locally complete if it is complete, it has $\mathcal{X}$-pullbacks, and for all morphisms $f : x \longrightarrow y$, $f^{-1} : \mathcal{M}/y \longrightarrow \mathcal{M}/x$ preserves meets.
(iii) $c_{\mathcal{M}}$ is meet preserving if $\mathcal{M}$ is stably locally complete and for all x, the mapping $c^x_{\mathcal{M}} : \mathcal{M}/x \longrightarrow \mathcal{M}/x$ preserves meets.

We denote by $\mathrm{CmCl}_s(\mathcal{X})$ (respectively $\mathrm{CmPoCl}_s(\mathcal{X})$, the full subcategory of $Cl_s(\mathcal{X})$ whose objects are meet preserving (respectively meet preserving with domain a poset). Also let 'Cmprcls' (respectively 'Cmpocls') be the subcategory of 'Prcls' whose objects are complete (respectively complete and partially ordered) and whose morphisms are meet preserving and denote by $\mathrm{Cmprcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$ (respectively $\mathrm{Cmpocls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$) the category whose objects are semi presheaves $M : \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Cmprcls}$ (respectively $M : \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Cmpocls}$). We have:

Corollary 2.17. *The full embedding* $\mathbb{L}_s : Cl_s(\mathcal{X}) \longrightarrow \mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$ *restricts to give:*

(i) *the full embedding* $\mathbb{L}_s : \mathrm{CmCl}_s(\mathcal{X}) \longrightarrow \mathrm{Cmprcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$.
(ii) *the full embedding* $\mathbb{L}_s : \mathrm{CmPoCl}_s(\mathcal{X}) \longrightarrow \mathrm{Cmpocls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$.

Proof. Follows easily. $\square$

3. Concrete Functors via Lax Presheaves

Definition 3.1. For $M : \mathcal{X}^{\text{op}} \longrightarrow \text{Prcls}$ in $\text{Prcls}_{\text{SL}}^{\mathcal{X}^{\text{op}}}$, let $\int_{\mathcal{X}} M$ have objects (x, a) with $a \in M(x)$ and morphisms $\tilde{f} : (x, a) \longrightarrow (y, b)$ corresponding to morphisms $f : x \longrightarrow y$ in $\mathcal{X}$ for which $a \leq M(f)(b)$. Also define $\dot{M} : \int_{\mathcal{X}} M \longrightarrow \mathcal{X}$ to take $\tilde{f} : (x, a) \longrightarrow (y, b)$ to $f : x \longrightarrow y$.

Proposition 3.2. *$(\int_{\mathcal{X}} M, \dot{M})$ is a concrete category.*

Proof. For each $a \in M(x)$ we have $a \leq M(1)(a)$, so $\tilde{1}_x : (x, a) \longrightarrow (x, a)$ is a morphism. Also if $\tilde{f} : (x, a) \longrightarrow (y, b)$ and $\tilde{g} : (y, b) \longrightarrow (z, c)$ are morphisms, then $a \leq M(f)(b) \leq M(f) \circ M(g)(c) \sim M(g \circ f)(c)$ meaning $\tilde{g} \circ \tilde{f}$ is a morphism. Hence $\int_{\mathcal{X}} M$ is a category. It follows easily that $\dot{M}$ is a faithful functor. □

The category $\int_{\mathcal{X}} M$ is a generalization of the category of elements as defined in [20].

Definition 3.3. For $\theta : M \longrightarrow N$ in $\text{Prcls}_{\text{SL}}^{\mathcal{X}^{\text{op}}}$, let $\dot{\theta} : \int_{\mathcal{X}} M \longrightarrow \int_{\mathcal{X}} N$ be defined by taking $\tilde{f} : (x, a) \longrightarrow (y, b)$ in $\int_{\mathcal{X}} M$ to $\tilde{f} : (x, \theta_x(a)) \longrightarrow (y, \theta_y(b))$ in $\int_{\mathcal{X}} N$.

Proposition 3.4. *$\dot{\theta} : \dot{M} \longrightarrow \dot{N}$ is a concrete functor.*

Proof. Obviously $\dot{\theta}$ is well-defined on objects. To show it is well-defined on morphisms, let $\tilde{f} : (x, a) \longrightarrow (y, b)$ be given in $\int_{\mathcal{X}} M$. So $a \leq M(f)(b)$. Since θ_x preserves order, $\theta_x(a) \leq \theta_x(M(f)(b))$. Since θ is lax, $\theta_x(M(f)(b)) \leq N(f)(\theta_y(b))$. Therefore $\theta_x(a) \leq N(f)(\theta_y(b))$, implying the morphism $f : x \longrightarrow y$ lifts uniquely to $\tilde{f} : (x, \theta_x(a)) \longrightarrow (y, \theta_y(b))$ in $\int_{\mathcal{X}} N$. It then follows easily that $\dot{\theta}$ is a concrete functor. □

With $\text{CAT}(\mathcal{X})$ denoting the category whose objects are the concrete categories over $\mathcal{X}$ and whose morphisms are the concrete functors between them, we have:

Theorem 3.5. *The mapping $\mathbb{C} : \text{Prcls}_{\text{SL}}^{\mathcal{X}^{\text{op}}} \longrightarrow \text{CAT}(\mathcal{X})$ that takes the morphism $\theta : M \longrightarrow N$ to $\dot{\theta} : \dot{M} \longrightarrow \dot{N}$ is a full embedding.*

Proof. It follows easily that $\mathbb{C}$ is a functor. To show it is faithful, let $M \underset{\theta'}{\overset{\theta}{\rightrightarrows}} N$ be morphisms in $\text{Prcls}_{\text{SL}}^{\mathcal{X}^{\text{op}}}$ such that $\dot{\theta} = \dot{\theta}'$. Then $\dot{\theta}(x, a) = \dot{\theta}'(x, a)$, and so $(x, \theta_x(a)) = (x, \theta'_x(a))$. Therefore $\theta_x(a) = \theta'_x(a)$, implying $\theta = \theta'$.

Next we show $\mathbb{C}$ is one to one on objects. So suppose $\dot{M} = \dot{N}$. It follows that $\int_{\mathcal{X}} M = \int_{\mathcal{X}} N$. Now if $a \in M(x)$, then $(x, a) \in \int_{\mathcal{X}} M$ and so $(x, a) \in \int_{\mathcal{X}} N$, which implies $a \in N(x)$. Therefore $M(x) \subseteq N(x)$. Similarly $N(x) \subseteq M(x)$. Hence $M = N$. It now follows that $\mathbb{C}$ is an embedding.

Finally to show fullness, let $F : \dot{M} \longrightarrow \dot{N}$ be a morphism in $\mathrm{CAT}(\mathcal{X})$. Since $\dot{N} \circ F = \dot{M}$, if $F(x,a) = (y,b)$, then $y = x$. We define $\theta : M \longrightarrow N$ so that $\theta_x(a)$ is the second component of $F(x,a)$. Therefore we have $F(x,a) = (x, \theta_x(a))$. To show θ is lax, let $f : x \longrightarrow y$ be a morphism in $\mathcal{X}$ and $b \in M(y)$. Then f lifts to $\tilde{f} : (x, M(f)(b)) \longrightarrow (y,b)$ in $\int_{\mathcal{X}} M$ and so $F(\tilde{f}) : (x, \theta_x(M(f)(b))) \longrightarrow (y, \theta_y(b))$ is in $\int_{\mathcal{X}} N$. Therefore, with $\tilde{g} = F(\tilde{f})$, $\theta_x(M(f)(b)) \leq N(g)(\theta_y(b))$. But $\dot{N} \circ F(\tilde{f}) = \dot{M}(\tilde{f})$ implies $g = f$ and so $\theta_x(M(f)(b)) \leq N(f)(\theta_y(b))$. Hence θ is lax.

It is obvious that $\dot{\theta} = F$. $\square$

4. Topological Functors via Closure Operators

Definition 4.1. A functor $G : \mathcal{C} \longrightarrow \mathcal{X}$ is said to be weak (1-)topological if every structured (1-)source $(f_i : x \longrightarrow y_i = G(b_i))_I$ has an initial lift $(\widetilde{f_i} : a \longrightarrow b_i)_I$.

Proposition 4.2. (i) *For* $M \in \mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$, $\dot{M} : \int_{\mathcal{X}} M \longrightarrow \mathcal{X}$ *is weak 1-topological.*
(ii) *For* $M \in \mathrm{Cmprcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$, $\dot{M} : \int_{\mathcal{X}} M \longrightarrow \mathcal{X}$ *is weak topological.*
(iii) *For* $M \in \mathrm{Cmpocls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}}$, $\dot{M} : \int_{\mathcal{X}} M \longrightarrow \mathcal{X}$ *is topological.*

Proof. (i) If $f : x \longrightarrow y = \dot{M}(y,a)$ is an $\dot{M}$-structured morphism, then obviously $\widetilde{f} : (x, M(f)(a)) \longrightarrow (y,a)$ is a lift of f. To show $\widetilde{f} : (x, M(f)(a)) \longrightarrow (y,a)$ is initial, suppose $g : z \longrightarrow x$ is such that $f \circ g$ has a lift $\widetilde{f \circ g} : (z,c) \longrightarrow (y,a)$, then $c \leq M(f \circ g)(a) \sim M(g)(M(f)(a))$. Hence there is a lift $\widetilde{g} : (z,c) \longrightarrow (x, M(f)(a))$ of g.

(ii) Consider an $\dot{M}$-structured source $S = (f_i : x \longrightarrow y_i = \dot{M}(y_i, a_i))_I$ over I. For each $i \in I$, $M(f_i)(a_i) \in M(x)$ which is a complete preclass. Let a be a meet of $M(f_i)(a_i)$. We show that $\widetilde{S} = (\widetilde{f_i} : (x,a) \longrightarrow (y_i, a_i))_I$ is an initial lift of the source S. If $g : z \longrightarrow x$ is such that $S \circ g$ has a lift $P = (\widetilde{f_i \circ g} : (z,c) \longrightarrow (y_i, a_i))_I$, then for each i we have $c \leq M(f_i \circ g)(a_i) \sim M(g)(M(f_i)(a_i))$. Since $M(g)$ is a morphism in Cmprcls, it preserves meets. Hence we have $c \leq M(g)(a)$, i.e. there is a lift $\widetilde{g} : (z,c) \longrightarrow (x,a)$ of g.

(iii) If $(x,a) \sim (x,b)$ in $\dot{M}^{-1}(x)$, then $a \sim b$ in $M(x)$ and so $a = b$. Therefore $\dot{M}$ is amnestic. By part (ii) $\dot{M}$ is weak topological, hence it is topological. $\square$

Denoting by $\mathrm{WTop}_1(\mathcal{X})$ (respectively $\mathrm{WTop}(\mathcal{X})$, $\mathrm{Top}(\mathcal{X})$) the full subcategory of $\mathrm{CAT}(\mathcal{X})$ whose objects are weak 1-topological (respectively weak topological, topological), we have:

Theorem 4.3. *We have:*

(i) *The full embedding* $\mathbb{C} : \mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}} \longrightarrow \mathrm{CAT}(\mathcal{X})$ *factors through* $\mathrm{WTop}_1(\mathcal{X})$, *yielding a full embedding* $\mathbb{C} : \mathrm{Prcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}} \longrightarrow \mathrm{WTop}_1(\mathcal{X})$.

(ii) *The full embedding* $\mathbb{C} : \mathrm{Cmprcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}} \longrightarrow \mathrm{CAT}(\mathcal{X})$ *factors through* $\mathrm{WTop}(\mathcal{X})$, *yielding a full embedding* $\mathbb{C} : \mathrm{Cmprcls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}} \longrightarrow \mathrm{WTop}(\mathcal{X})$.

(iii) *The full embedding* $\mathbb{C} : \mathrm{Cmpocls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}} \longrightarrow \mathrm{CAT}(\mathcal{X})$ *factors through* $Top(\mathcal{X})$, *yielding a full embedding* $\mathbb{C} : \mathrm{Cmpocls}_{\mathrm{SL}}^{\mathcal{X}^{\mathrm{op}}} \longrightarrow \mathrm{Top}(\mathcal{X})$.

Proof. Follows from Theorem 3.5 and Proposition 4.2. □

Corollary 4.4. *We have the following full embeddings.*

(i) $W_1 : \mathrm{Cl}_s(\mathcal{X}) \longrightarrow \mathrm{WTop}_1(\mathcal{X})$.

(ii) $W : \mathrm{CmCl}_s(\mathcal{X}) \longrightarrow \mathrm{WTop}(\mathcal{X})$.

(iii) $T : \mathrm{CmPoCl}_s(\mathcal{X}) \longrightarrow \mathrm{Top}(\mathcal{X})$.

Proof. Composing the full embeddings given in Theorem 2.15, Corollary 2.17 and Theorem 4.3 yields the given full embeddings. □

5. Examples

Lemma 5.1. *Let* $U : \mathcal{X} \longrightarrow Set$ *be a construct, Epi be the collection of all the epis in* $\mathcal{X}$ *and* $Iinc = \{i : a \longrightarrow x : i$ *is initial and* $U(i)$ *is the inclusion*$\}$. *Suppose* $\mathcal{X}$ *has pullbacks and unique* $(Epi, Iinc)$*-factorization that is pullback stable. If the collection* $\mathcal{M} \supseteq Iinc$ *has* $\mathcal{X}$*-pullbacks and satisfies:* $m = i \circ e$ *with* $m \in \mathcal{M}$, $e \in Epi$ *and* $i \in Iinc$, *implies* e *is a retraction, then:*

(i) $\mathcal{M}$ *is (stably) locally complete if Iinc is.*

(ii) *any closure operator* $\overline{(\ )} : Iinc \longrightarrow Iinc$ *extends to a closure operator on* $\mathcal{M}$ *such as* $c : \mathcal{M} \longrightarrow \mathcal{M}$. *Furthermore* c *is idempotent if* $\overline{(\ )}$ *is.*

Proof. (i) Suppose Iinc is locally complete. Given any collection $m_\alpha \in \mathcal{M}/x$ for some x, let $m_\alpha = i_\alpha \circ e_\alpha$ be the factorization of m_α. Using the fact that e_α is a retraction, one can easily verify that any meet of the collection i_α is a meet of the collection m_α.

Now suppose Iinc is stably locally complete. Given a morphism $f : x \longrightarrow y$ and a collection $m_\alpha : b_\alpha \longrightarrow y$ in $\mathcal{M}/y$, let $m_\alpha = i_{m_\alpha} \circ e_{m_\alpha}$ be the factorization of m_α, and n_α be the pullback of m_α along f. Since factorizations are pullback stable,

$i_{n_\alpha} = f^{-1}(i_{m_\alpha})$. So $\wedge n_\alpha = \wedge i_{n_\alpha} = \wedge f^{-1}(i_{m_\alpha}) = f^{-1}(\wedge i_{m_\alpha}) = f^{-1}(\wedge m_\alpha)$, as required.

(ii) Given $m : a \longrightarrow x$ in $\mathcal{M}/x$, let $m = i_m \circ e_m$ with $e_m \in$ Epi and $i_m \in$ Iinc. Define $c(m) = \overline{i_m}$. Since $m \leq i_m$ and $i_m \leq \overline{i_m}$, $m \leq c(m)$. If $m \leq n$ via α (i.e. $m = n \circ \alpha$), then $i_m \leq i_n$ via $e_n \circ \alpha \circ s_m$, where s_m is the right inverse of e_m which exists since $m \in \mathcal{M}$. So $(m) = \overline{i_m} \leq \overline{i_n} = c(n)$. Finally suppose $f : x \longrightarrow y$ is a morphism in $\mathcal{X}$ and $m \in \mathcal{M}/y$. Let n be the pullback of m along f. Since factorizations are pullback stable, $i_n = f^{-1}(i_m)$. So $c(n) = \overline{i_n} = \overline{f^{-1}(i_m)} \leq f^{-1}(\overline{i_m}) = f^{-1}(c(m))$, as desired. Hence c is a closure operator on $\mathcal{M}$. If $m \in$ Iinc, then $i_m = m$ and so $c(m) = \overline{i_m} = \overline{m}$. Hence c is an extension of the given closure operator.

Also with $m \in \mathcal{M}$, we have $c(m) = \overline{i_m} \in$ Iinc. So $c(c(m)) = c(\overline{i_m}) = \overline{\overline{i_m}}$, rendering c idempotent if $\overline{(\ )}$ is. $\square$

Example 5.2. Consider the category Set as a construct over Set via the identity functor. The collection Iinc of Lemma 5.1 is the collection Inc of all the inclusions which is stably locally complete. So if $\mathcal{M}$ is a class of morphisms that has $\mathcal{X}$-pullbacks and contains all the inclusions ($\mathcal{M}$ can be the collection of inclusions, the collection of monos, or the collection of all the morphisms, among others), then all the conditions of Lemma 5.1 are met, and so $\mathcal{M}$ is stably locally complete.

Next consider the identity closure operator on Inc. By Lemma 5.1, we get an idempotent closure operator c on M. $c(m)$ is just the image of m. Note that each inclusion is closed and every morphism $m \in \mathcal{M}$ is semi-closed (because $m = i_m \circ e_m$ and e_m is a retraction). Hence c is a semi-identity closure operator.

The associated category $\int M$, related to this closure operator, has objects (X, m), where X is a set and $m : A \longrightarrow X$ is in $\mathcal{M}$ for some set A; and has morphisms $f : (X, m) \longrightarrow (Y, n)$, where $f : X \longrightarrow Y$ is a function such that $m \leq f^{-1}(c(n))$ or equivalently $Im_{f \circ m} \subset Im_n$ or equivalently $f \circ m \leq n$. This category over Set is, by Corollary 4.4 (ii), a weak topological construct.

Example 5.3. Consider the category Top of topological spaces and continuous functions as a construct over Set via the forgetful functor. The collection Iinc of Lemma 5.1 is the collection Inc of all the inclusions (with the subspace topology) which is stably locally complete. So if $\mathcal{M}$ is a class of morphisms that has $\mathcal{X}$-pullbacks and contains all the inclusions such that in the (Epi, Inc)-factorization of each m in $\mathcal{M}$, the epi factor is a retraction ($\mathcal{M}$ can be the collection of inclusions, the collection of embeddings (i.e., initial monos), among others), then all the conditions of Lemma 5.1 are met, and so $\mathcal{M}$ is stably locally complete.

Consider the following closure operators on Inc, that take the inclusion map $i : A \longrightarrow X$ to the inclusion map $\bar{i} : \overline{A} \longrightarrow X$, [7], where $\overline{A}$ is:

(i) the intersection of all closed subsets of X containing A.
(ii) the intersection of all clopen subsets of X containing A.
(iii) the union of A with all connected subsets of X that intersect A.
(iv) the set of all $x \in X$ such that for every neighborhood U of x, $A \cap \bar{\{x\}} \cap U \neq \emptyset$, that $\bar{\{x\}}$ is the topological closure of the subset $\{x\}$.
(v) the set of all $x \in X$ such that for every neighborhood U of x, $A \cap \bar{U} \neq \emptyset$, that $\bar{U}$ is the topological closure of the subset U.

By Lemma 5.1, each of the above closure operators yield a closure operator c on $\mathcal{M}$, where $c(m) = \overline{i_m}$, with i_m the image of m. All the above closure operators are idempotent except the one in part (v). So in cases (i) to (iv), we may consider the categories $\int M$ related to these closure operators. Objects of these categories are (X, m), where $m : A \longrightarrow X$ is in $\mathcal{M}$ and morphisms are $f : (X, m) \longrightarrow (Y, n)$, where $f : X \longrightarrow Y$ is a continuous function such that $m \leq f^{-1}(c(n))$ or equivalently $f \circ m \leq \overline{i_n}$. These categories over Top are, by Corollary 4.4 (ii), weak topological.

Example 5.4. Consider the category Grp of groups and group homomorphisms as a construct over Set via the forgetful functor. The collection Iinc of Lemma 5.1 is the collection Inc of all the inclusions (with the subgroup structure) which is stably locally complete. So if $\mathcal{M}$ is a class of morphisms that has $\mathcal{X}$-pullbacks and contains all the inclusions such that in the (Epi, Inc)-factorization of each m in $\mathcal{M}$, the epi factor is a retraction ($\mathcal{M}$ can be the collection of inclusions, the collection of initial monos, among others), then all the conditions of Lemma 5.1 are met, and so $\mathcal{M}$ is stably locally complete.

Consider the following closure operators on Inc, that take the inclusion map $i : A \longrightarrow X$ to the inclusion map $\bar{i} : \overline{A} \longrightarrow X$, [7], where $\overline{A}$ is:

(i) the intersection of all normal subgroups of X containing A.
(ii) the intersection of all normal subgroups K of X containing A such that X/K is Abelian.
(iii) the intersection of all normal subgroups K of X containing A such that X/K is torsion-free.
(iv) the subgroup generated by A and by all perfect subgroups of X.

By Lemma 5.1, each of the above closure operators yield a closure operator c on $\mathcal{M}$, where $c(m) = \overline{i_m}$, with i_m the image of m. All the above closure operators are idempotent except the one in part (iv). So in cases (i) to (iii), we may consider the categories $\int M$ related to these closure operators. Objects of these categories are (X, m), where $m : A \longrightarrow X$ is in $\mathcal{M}$ and morphisms are $f : (X, m) \longrightarrow (Y, n)$, where $f : X \longrightarrow Y$ is a group homomorphism such that $m \leq f^{-1}(c(n))$ or equivalently $f \circ m \leq \overline{i_n}$. These categories over Grp are, by Corollary 4.4 (ii), weak topological.

Example 5.5. Consider the category Set_* of pointed sets and point preserving functions. Let $\mathcal{M}$ be any collection of morphisms that has $\mathcal{X}$-pullbacks and is stably locally complete. Define $c : \mathcal{M} \longrightarrow \mathcal{M}$ to take the morphism $m : (A, a_0) \longrightarrow (X, x_0)$ to $m \oplus \hat{x}_0 : (A \coprod 1, a_0) \longrightarrow (X, x_0)$, where 1 is the terminal and $\hat{x}_0 : 1 \longrightarrow X$ is the map taking the point to x_0. Now $m \leq m \oplus \hat{x}_0$ via $\nu_1 : A \longrightarrow A \coprod 1$, the first injection to the coproduct. If $m \leq n$ via ϕ, then $m \oplus \hat{x}_0 \leq n \oplus \hat{x}_0$ via $\phi \coprod 1$. Finally, given $f : (X, x_0) \longrightarrow (Y, y_0)$ and $m : (B, b_0) \longrightarrow (Y, y_0)$, let $n : (A, a_0) \longrightarrow (X, x_0)$ be the pullback of m along f. Then $c(f^{-1}(m)) = c(n) = n \oplus \hat{x}_0$ and $f^{-1}(c(m)) = f^{-1}(m \oplus \hat{y}_0) = n \oplus i$, where $i : (f^{-1}(y_0), x_0) \longrightarrow (X, x_0)$ is the inclusion. But $n \oplus \hat{x}_0 \leq n \oplus i$ via $1 \coprod \hat{x}_0$. Hence c is a closure operator.

Now for $m : (A, a_0) \longrightarrow (X, x_0)$ in $\mathcal{M}$, $c(m) = m \oplus \hat{x}_0$ and $c(c(m)) = m \oplus \hat{x}_0 \oplus \hat{x}_0$. Since $m \oplus \hat{x}_0 \oplus \hat{x}_0 \leq m \oplus \hat{x}_0$ via $1 \coprod (1 \oplus 1) : (A \coprod 1 \coprod 1, a_0) \longrightarrow (A \coprod 1, a_0)$, $m \oplus \hat{x}_0 \sim m \oplus \hat{x}_0 \oplus \hat{x}_0$. Hence c is semi-idempotent but obviously not idempotent.

The corresponding weak topological category can be constructed as in the previous examples.

Example 5.6. Let $(X, \leq)$ be a complete partially ordered set and $\mathcal{X} = C(X, \leq)$ be the associated category. With $\mathcal{M} = \mathcal{X}_1$ and c the identity closure operator on $\mathcal{M}$, the corresponding category $\int M$ has objects (x, x') with $x' \leq x$ and there is a unique morphism $f : (x, x') \longrightarrow (y, y')$ if and only if $x \leq y$ and $y' \wedge x \leq x'$. By Corollary 4.4 (iii), this category is topological over $\mathcal{X}$.

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