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New Representation of Quaternions Lie Group and SU(2)

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Özet. Bu makalede $\mathbb{R}^4$ için dış çarpım kavramı incelenmektedir. Bu dış çarpım kullanılarak $\mathbb{R}^5$'te yeni bir çarpım ortaya atılmaktadır. $\mathbb{R}^5$, standard toplama, skalerle çarpma ve bu çarpımla birlikte, birleşmeli bir cebir oluşturur. Bu cebir yoluyla kuaterniyonlar için bir Lie grubu olarak yeni bir temsil sunulur. Ayrıca SU(2) için bir temsil çıkarılır.[†]

Anahtar Kelimeler. Dikgen Lie grubu, dış çarpım, temsil.

Abstract. In this paper the concept of outer product for $\mathbb{R}^4$ is considered. By using this outer product a new product on $\mathbb{R}^5$ is introduced. $\mathbb{R}^5$ with this product and usual addition and scalar multiplication is an associative algebra. Via this algebra a new representation for quaternions as a Lie group is presented. Moreover a representation for SU(2) is deduced.

Keywords. Orthogonal Lie group, outer product, representation.

1. Introduction

In this paper we introduce an outer product on $\mathbb{R}^4$ which is a generalization of the outer product of $\mathbb{R}^3$. In fact we prove that $\mathbb{R}^4$ with this outer product is a noncommutative Lie algebra. A Lie algebra homomorphism via this outer product is introduced. This homomorphism determines a representation of three dimensional Lie subalgebras of SO(4). Moreover its exponential is similar to Rodrigues formula. We also introduce a concept of curl for the vector fields of $\mathbb{R}^4$, and we will show that it can deduce by differential forms via Hodge star operator. An associative algebra in $\mathbb{R}^5$ by using of the outer product deduced. The representation of this algebra under a special conditions is isomorphic to quaternions Lie group. This determines a class of representations for quaternions Lie group and the Lie group SU(2).

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2. Outer Product on $\mathbb{R}^4$

Let us begin this section by a new definition of outer product on $\mathbb{R}^4$.

Definition 2.1. Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be three vectors in $\mathbb{R}^4$ and let λ be a fixed nonzero vector. Then we define the outer product of a and b by

$$\begin{aligned} a \times b &= \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \qquad (1) \\ &= [\lambda_2(a_3b_4 - a_4b_3) - \lambda_3(a_2b_4 - a_4b_2) + \lambda_4(a_2b_3 - a_3b_2)]\, e_1 \\ &- [\lambda_1(a_3b_4 - a_4b_3) - \lambda_3(a_1b_4 - a_4b_1) + \lambda_4(a_1b_3 - a_3b_1)]\, e_2 \\ &+ [\lambda_1(a_2b_4 - a_4b_2) - \lambda_2(a_1b_4 - a_4b_1) + \lambda_4(a_1b_2 - a_2b_1)]\, e_3 \\ &- [\lambda_1(a_2b_3 - a_3b_2) - \lambda_2(a_1b_3 - a_3b_1) + \lambda_3(a_1b_2 - a_2b_1)]\, e_4, \end{aligned}$$

where e_1, e_2, e_3 and e_4 are standard members of basis of $\mathbb{R}^4$. We denote the usual inner product of a and b by $a \cdot b$ and we define $a \circ b$ by $a \circ b = (\lambda \cdot \lambda)(a \cdot b) - (\lambda \cdot a)(\lambda \cdot b)$. Obviously $a \circ \lambda = \lambda \circ a = b \circ \lambda = \lambda \circ b = 0$.

Theorem 2.1. *If $a, b, c \in \mathbb{R}^4$ and $t \in \mathbb{R}$, then*

(i) $a \times a = 0$ *and* $a \times b = -b \times a$*;*
(ii) $ta \times b = t(a \times b)$*;*
(iii) $a \times (b + c) = a \times b + a \times c$*;*
(iv) $(a \times b) \cdot c = a \cdot (b \times c)$ *and* $(a \times b) \circ c = a \circ (b \times c)$*;*
(v) $(a \times b) \cdot a = b \cdot (a \times b) = 0$ *and* $(a \times b) \circ a = b \circ (a \times b) = 0$*;*
(vi) $(a \times b) \times c = (c \circ a)b - (c \circ b)a + ((\lambda \cdot a)(c \cdot b) - (\lambda \cdot b)(c \cdot a))\lambda$*;*
(vii) $(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0$*;*
(viii) $(a \times b) \circ (a \times b) = (a \circ a)(b \circ b) - (a \circ b)^2$*;*
(ix)

$$(a \times b) \cdot (a \times b) = \begin{vmatrix} a \cdot a & a \cdot \lambda & a \cdot b \\ \lambda \cdot a & \lambda \cdot \lambda & \lambda \cdot b \\ b \cdot a & b \cdot \lambda & b \cdot b \end{vmatrix}; \qquad (2)$$

(x) *If we define on $\mathbb{R}^4$, $[a, b] = a \times b$ then $\mathbb{R}^4$ with this bracket is a Lie algebra.*

Proof. One can prove (i), (ii), (iii), (iv) and (v) by direct calculation. We prove other parts.

To prove (vi) we know that if $a \times b = (x_1, x_2, x_3, x_4)$ then we only restrict ourself to x_3 which is $x_3 = \lambda_1(a_2b_4 - a_4b_2) - \lambda_2(a_1b_4 - a_4b_1) + \lambda_4(a_1b_2 - a_2b_1)$. Let $(a \times b) \times c = (y_1, y_2, y_3, y_4)$. Then

$$\begin{aligned}
y_3 =& - [\lambda_1(a_3b_4 - a_4b_3) - \lambda_3(a_1b_4 - a_4b_1) + \lambda_4(a_1b_3 - a_3b_1)](\lambda_1c_4 - \lambda_4c_1) \\
&- [\lambda_2(a_3b_4 - a_4b_3) - \lambda_3(a_2b_4 - a_4b_2) + \lambda_4(a_2b_4 - a_4b_2)](\lambda_2c_4 - \lambda_4c_2) \\
&- [\lambda_1(a_2b_3 - a_3b_2) - \lambda_2(a_1b_3 - a_3b_1) + \lambda_3(a_1b_2 - a_2b_1)](\lambda_2c_1 - \lambda_1c_2) \\
=& - a_3(-\lambda_1\lambda_4b_1c_4 + \lambda_4^2b_1c_1 + \lambda_2^2b_4c_4 - \lambda_2\lambda_4b_4c_2 - \lambda_1\lambda_2b_2c_1 + \lambda_1^2b_2c_2) \\
&+ b_3(-\lambda_1\lambda_4a_1c_4 + \lambda_4^2a_1c_1 + \lambda_2^2a_4c_4 - \lambda_2\lambda_4a_4c_2 - \lambda_1\lambda_2a_2c_1 + \lambda_1^2a_2c_2) \\
&+ \lambda_3(\lambda_1a_1b_4c_4 - \lambda_4b_4a_1c_1 - \lambda_1b_1a_4c_4 + \lambda_4a_4b_1c_1 + \lambda_2a_2b_4c_4 - \lambda_4b_4a_2c_2 \\
&- \lambda_2b_2a_4c_4 + \lambda_4a_4b_2c_2) \\
=& - a_3((\lambda \cdot \lambda)(b \cdot c) - (\lambda \cdot b)(\lambda \cdot c)) + b_3((\lambda \cdot \lambda)(a \cdot c) - (\lambda \cdot a)(\lambda \cdot c)) \\
&+ \lambda_3((\lambda \cdot a)(c \cdot b) - (\lambda \cdot b)(c \cdot a)) \\
=&(c \circ a)b_3 - (c \circ b)a_3 + ((\lambda \cdot a)(c \cdot b) - (\lambda \cdot b)(c \cdot a))\lambda_3.
\end{aligned}$$

For the cases x_1, x_2 and x_4 one can present the similar proofs for y_1, y_2 and y_4 respectively.

Proof of (vii).

$$\begin{aligned}
(a \times b) \times c + (b \times c) \times a + (c \times a) \times b& \\
= (c \circ a)b - (c \circ b)a + ((\lambda \cdot a)(c \cdot b) - (\lambda \cdot b)(c \cdot a))\lambda& \\
+ (a \circ b)c - (a \circ c)b + ((\lambda \cdot b)(a \cdot c) - (\lambda \cdot c)(a \cdot b))\lambda& \\
+ (b \circ c)a - (b \circ a)c + ((\lambda \cdot c)(a \cdot b) - (\lambda \cdot a)(b \cdot c))\lambda = 0.&
\end{aligned}$$

Proof of (viii).

$$\begin{aligned}
(a \times b) \circ (a \times b) &= ((a \times b) \times a) \circ b \\
&= ((a \circ a)b - (a \circ b)a + ((\lambda \cdot a)(a \cdot b) - (\lambda \cdot b)(a \cdot a))\lambda) \circ b \\
&= (a \circ a)(b \circ b) - (a \circ b)(a \circ b) + ((\lambda \cdot a)(a \cdot b) - (\lambda \cdot b)(a \cdot a))(\lambda \circ b) \\
&= (a \circ a)(b \circ b) - (a \circ b)(a \circ b) \\
&= (a \circ a)(b \circ b) - (a \circ b)^2.
\end{aligned}$$

Proof of (ix).

$$
\begin{aligned}
(a \times b) \cdot (a \times b) &= ((a \times b) \times a)) \cdot b \\
&= ((a \circ a)b - (a \circ b)a + ((\lambda \cdot a)(a \cdot b) - (\lambda \cdot b)(a \cdot a))\lambda) \cdot b \\
&= (a \circ a)(b \cdot b) - (a \circ b)(a \cdot b) + ((\lambda \cdot a)(a \cdot b) - (\lambda \cdot b)(a \cdot a))(\lambda \cdot b) \\
&= ((\lambda \cdot \lambda)(a \cdot a) - (\lambda \cdot a)^2)(b \cdot b) - ((\lambda \cdot \lambda)(a \cdot b) - (\lambda \cdot a)(\lambda \cdot b))(a \cdot b) \\
&\quad + ((\lambda \cdot a)(a \cdot b) - (\lambda \cdot b)(a \cdot a))(\lambda \cdot b).
\end{aligned}
$$

Thus we have

$$
(a \times b) \cdot (a \times b) = \begin{pmatrix} a \cdot a & a \cdot \lambda & a \cdot b \\ \lambda \cdot a & \lambda \cdot \lambda & \lambda \cdot b \\ b \cdot a & b \cdot \lambda & b \cdot b \end{pmatrix}. \tag{3}
$$

The proof of (x) follows from (ii), (iv), (v) and (vii). □

We recall that a Lie algebra $L = M + K$ is the direct sum of two subalgebras M and K if it is the vector sum of them and each elements of M commutes with all elements of K, i.e. if $X \in M$ and $Y \in K$ then $[X, Y] = 0$ (see [8, 9]).

Theorem 2.2. *Let* λ *be a fixed member of* $\mathbb{R}^4$, $M = \{t\lambda : \ t \in \mathbb{R}, \ \lambda \in \mathbb{R}^4\}$, $K = \{a \in \mathbb{R}^4 : \lambda \cdot a = 0\}$. *Then* $\mathbb{R}^4$ *is the direct sum of* M *and* K.

Proof. The orthogonality condition implies each element of $\mathbb{R}^4$ is the direct sum of two elements of M and K respectively. Let $X \in M$ and $Y \in K$. Then $X = t\lambda$ for some $t \in \mathbb{R}$ and $Y = a, \ a \in \mathbb{R}^4$ and $\lambda \cdot a = 0$. Hence

$$
[X, Y] = [t\lambda, a] = t(\lambda \times a) = 0.
$$

□

Let V be a real or complex vector space and $\mathrm{GL}(V)$ be the group of all nonsingular linear transformations of V onto itself. A representation of a group G with the representation space V is a homomorphism $T : \ g \mapsto T(g)$ of G into $\mathrm{GL}(V)$. The dimension of the representation is dimension of V. As a consequence of definition we have: $T(g_1 g_2) = T(g_1)T(g_2), \ T(g^{-1}) = T(g)^{-1}, \ T(e) = E$, where $g_1, \ g_2, \ g \in G$, e is the identity of G and E is the identity operator on V (see [2, 3]).

Now we define a representation via a vector space homomorphism $T : \mathbb{R}^4 \to \mathrm{SO}(4)$, by:

$$T(a) = A = \begin{pmatrix} 0 & \lambda_3 a_4 - \lambda_4 a_3 & -\lambda_2 a_4 + \lambda_4 a_2 & \lambda_2 a_3 - \lambda_3 a_2 \\ -\lambda_3 a_4 + \lambda_4 a_3 & 0 & \lambda_1 a_4 - \lambda_4 a_1 & -\lambda_1 a_3 + \lambda_3 a_1 \\ \lambda_2 a_4 - \lambda_4 a_2 & -\lambda_1 a_4 + \lambda_4 a_1 & 0 & \lambda_1 a_2 - \lambda_2 a_1 \\ -\lambda_2 a_3 + \lambda_3 a_2 & \lambda_1 a_3 - \lambda_3 a_1 & -\lambda_1 a_2 + \lambda_2 a_1 & 0 \end{pmatrix}. \quad (4)$$

As a consequence $T(a)(b) = a \times b$, where $T(a)(b)$ is product of skew-symmetric matrix $T(a)$ with the column vector b. Moreover this liner transformation is a Lie algebra homomorphism because

$$\begin{aligned}[T(a), T(b)](c) &= (T(a)T(b) - T(b)T(a))(c) = T(a)(b \times c) - T(b)(a \times c) \\ &= a \times (b \times c) - b \times (a \times c) = (a \times b) \times c = T(a \times b)(c) = T[a, b](c).\end{aligned}$$

So T is a Lie algebra homomorphism. In fact the image $T(\mathbb{R}^4)$ of T is a 3 dimensional Lie subalgebra of SO(4).

The characteristic equation of $T(a)$ is

$$\det(T(a) - rE) = (r^2(r^2 + (\lambda_2 a_1 - \lambda_1 a_2)^2 + (\lambda_3 a_1 - \lambda_1 a_3)^2 + (\lambda_4 a_1 - \lambda_1 a_4)^2 + (\lambda_2 a_3 - \lambda_3 a_2)^2 + (\lambda_2 a_4 - \lambda_4 a_2)^2 + (\lambda_3 a_4 - \lambda_4 a_3)^2)) = 0.$$

If we put

$$t^2 = ((\lambda_2 a_1 - \lambda_1 a_2)^2 + (\lambda_3 a_1 - \lambda_1 a_3)^2 + (\lambda_4 a_1 - \lambda_1 a_4)^2 + (\lambda_2 a_3 - \lambda_3 a_2)^2 + (\lambda_2 a_4 - \lambda_4 a_2)^2 + (\lambda_3 a_4 - \lambda_4 a_3)^2),$$

then by Cayley Hamilton theorem (see [6]) we have $A^2(A^2 + t^2 I) = 0$ or $A^4 = -t^2 A^2$. Thus $A^5 = -t^2 A^3,\ A^6 = t^4 A^2,\ A^7 = t^4 A^3,\ A^8 = -t^6 A^2,\ A^9 = -t^6 A^3, \ldots$.

We give the following explicit formula for $\exp(A)$ which is similar to Rodrigues formula (see [1, 6, 7]).

$$\exp(A) = I + A + \frac{A^2}{t^2}(1 - \cos t) + \frac{A^3}{t^3}(1 - \sin t)$$

Because

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \frac{A^5}{5!} + \frac{A^6}{6!} + \frac{A^7}{7!} + \frac{A^8}{8!} + \frac{A^9}{9!} + \ldots$$

$$\begin{aligned}
&= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{-t^2A^2}{4!} + \frac{-t^2A^3}{5!} \\
&\qquad\qquad + \frac{t^4A^2}{6!} + \frac{t^4A^3}{7!} + \frac{-t^6A^2}{8!} + \frac{-t^6A^3}{9!} + ... \\
&= I + A + \frac{A^2}{t^2}\left(\frac{t^2}{2!} - \frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^8}{8!} + ...\right) + \frac{A^3}{t^3}\left(\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} - \frac{t^9}{9!} + ...\right) \\
&= I + A + \frac{A^2}{t^2}(1 - \cos t) + \frac{A^3}{t^3}(1 - \sin t).
\end{aligned}$$

Now we would like to define the concept of curl in $\mathbb{R}^4$.

Definition 2.2. Let $F : \mathbb{R}^4 \to \mathbb{R}^4$ be a vector field on $\mathbb{R}^4$, we define the curl of F by

$$\begin{aligned}
\operatorname{curl} F = &[\lambda_2(\frac{\partial F_4}{\partial x_3} - \frac{\partial F_3}{\partial x_4}) - \lambda_3(\frac{\partial F_4}{\partial x_2} - \frac{\partial F_2}{\partial x_4}) + \lambda_4(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3})]e_1 \\
&- [\lambda_1(\frac{\partial F_4}{\partial x_3} - \frac{\partial F_3}{\partial x_4}) - \lambda_3(\frac{\partial F_4}{\partial x_1} - \frac{\partial F_1}{\partial x_4}) + \lambda_4(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3})]e_2 \\
&+ [\lambda_1(\frac{\partial F_4}{\partial x_2} - \frac{\partial F_2}{\partial x_4}) - \lambda_2(\frac{\partial F_4}{\partial x_1} - \frac{\partial F_1}{\partial x_4}) + \lambda_4(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2})]e_3 \\
&- [\lambda_1(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}) - \lambda_2(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3}) + \lambda_3(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2})]e_4,
\end{aligned}$$

where F_1, F_2, F_3 and F_4 are the components of F.

This concept can be reformulated in the words of differential forms. In fact if

$$\begin{aligned}
\omega = &(\lambda_2F_1 - \lambda_1F_2)dx_1dx_2 + (\lambda_3F_1 - \lambda_1F_3)dx_1dx_3 + (\lambda_4F_1 - \lambda_1F_4)dx_1dx_4 \\
&+ (\lambda_3F_2 - \lambda_2F_3)dx_2dx_3 + (\lambda_4F_2 - \lambda_2F_4)dx_2dx_4 + (\lambda_4F_3 - \lambda_3F_4)dx_3dx_4 \qquad (5)
\end{aligned}$$

then

$$\begin{aligned}
d\omega = &[\lambda_1(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}) - \lambda_2(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3}) + \lambda_3(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2})]dx_1dx_2dx_3 \\
&+ [\lambda_1(\frac{\partial F_4}{\partial x_2} - \frac{\partial F_2}{\partial x_4}) - \lambda_2(\frac{\partial F_4}{\partial x_1} - \frac{\partial F_1}{\partial x_4}) + \lambda_4(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2})]dx_1dx_2dx_4 \\
&+ [\lambda_1(\frac{\partial F_4}{\partial x_3} - \frac{\partial F_3}{\partial x_4}) - \lambda_3(\frac{\partial F_4}{\partial x_1} - \frac{\partial F_1}{\partial x_4}) + \lambda_4(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3})]dx_1dx_3dx_4 \\
&+ [\lambda_2(\frac{\partial F_4}{\partial x_3} - \frac{\partial F_3}{\partial x_4}) - \lambda_3(\frac{\partial F_4}{\partial x_2} - \frac{\partial F_2}{\partial x_4}) + \lambda_4(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3})]dx_2dx_3dx_4.
\end{aligned}$$

If we set $d\omega = G_4dx_1dx_2dx_3 + G_3dx_1dx_2dx_4 + G_2dx_1dx_3dx_4 + G_1dx_2dx_3dx_4$ then $*d\omega = -G_4dx_4 + G_3dx_3 - G_2dx_2 + G_1dx_1$ where G_1, G_2, G_3, G_4 are components of

$d\omega$ as mentioned. Let $\varphi : T_x\mathbb{R}^4 \to T_x^*\mathbb{R}^4$ be an isomorphism between the vector space $T_x\mathbb{R}^4$ and its dual space which is defined by

$$\varphi(v_x)(w_x) = v_x.w_x$$

where dot is usual inner product in $\mathbb{R}^4$. For a given $x \in \mathbb{R}^4$ we deduce:

$$\varphi^{-1}(*d\omega) = (G_1, -G_2, G_3, -G_4) = \operatorname{curl} F.$$

Theorem 2.3. *For each vector field $F : \mathbb{R}^4 \to \mathbb{R}^4$ and each scalar filed $f : \mathbb{R}^4 \to \mathbb{R}$ we have* $\operatorname{div}(\operatorname{curl} F) = 0$ *and* $\operatorname{curl}(\operatorname{grad} f) = 0$ *where*

$$\operatorname{div}(F) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} + \frac{\partial F_4}{\partial x_4} \text{ and } \operatorname{grad}(f) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}).$$

Proof.

$$\begin{aligned}
&\operatorname{div}(\operatorname{curl} F)\\
&= [\lambda_2(\frac{\partial^2 F_4}{\partial x_1 \partial x_3} - \frac{\partial^2 F_4}{\partial x_1 \partial x_3}) - \lambda_3(\frac{\partial^2 F_4}{\partial x_1 \partial x_2} - \frac{\partial^2 F_2}{\partial x_1 \partial x_4}) + \lambda_4(\frac{\partial^2 F_3}{\partial x_1 \partial x_2} - \frac{\partial^2 F_2}{\partial x_1 \partial x_3})]\\
&- [\lambda_1(\frac{\partial^2 F_4}{\partial x_2 \partial x_3} - \frac{\partial^2 F_3}{\partial x_2 \partial x_4}) - \lambda_3(\frac{\partial^2 F_4}{\partial x_2 \partial x_1} - \frac{\partial^2 F_1}{\partial x_2 \partial x_4}) + \lambda_4(\frac{\partial^2 F_3}{\partial x_2 \partial x_1} - \frac{\partial F_1}{\partial x_2 \partial x_3})]\\
&+ [\lambda_1(\frac{\partial^2 F_4}{\partial x_3 \partial x_2} - \frac{\partial^2 F_2}{\partial x_3 \partial x_4}) - \lambda_2(\frac{\partial^2 F_4}{\partial x_3 \partial x_1} - \frac{\partial^2 F_1}{\partial x_3 \partial x_4}) + \lambda_4(\frac{\partial^2 F_2}{\partial x_3 \partial x_1} - \frac{\partial^2 F_1}{\partial x_3 \partial x_2})]\\
&- [\lambda_1(\frac{\partial^2 F_3}{\partial x_4 \partial x_2} - \frac{\partial^2 F_2}{\partial x_4 \partial x_3}) - \lambda_2(\frac{\partial^2 F_3}{\partial x_4 \partial x_1} - \frac{\partial^2 F_1}{\partial x_4 \partial x_3}) + \lambda_3(\frac{\partial^2 F_2}{\partial x_4 \partial x_1} - \frac{\partial^2 F_1}{\partial x_4 \partial x_2})]\\
&= 0.
\end{aligned}$$

Direct calculation implies $\operatorname{curl}(\operatorname{grad} f) = 0$ and we left its proof. □

3. An Associative Algebra on $\mathbb{R}^5$

In this section we introduce an associative algebra on $\mathbb{R}^5$ which is look alike the Clifford algebra (see [4, 7, 10]). If $a_0, b_0 \in \mathbb{R}$ and $p = (a_1, a_2, a_3, a_4)$, $q = (b_1, b_2, b_3, b_4)$ $\in \mathbb{R}^4$ then we denote $x = (a_0, a_1, a_2, a_3, a_4)$ and $y = (b_0, b_1, b_2, b_3, b_4)$ by $a_0 + p$ and $b_0 + q$ respectively. With these notations we have the next definition.

Definition 3.1. We define the product of x and y by:

$$x \star y = (a_0 + p) \star (b_0 + q) = a_0 b_0 - p \circ q + a_0 q + b_0 p + p \times q + (p \cdot q)\lambda.$$

Since $x = (a_0, a_1, a_2, a_3, a_4) = (a_0, 0, 0, 0, 0) + (0, a_1, a_2, a_3, a_4) = a_0 + p$, then the above definition of $x * y$ is well-defined.

Theorem 3.1. $(\mathbb{R}^5, +, \star)$ *is an associative algebra with an identity element, where its scalar product is the usual scalar product of* $\mathbb{R}^5$.

Proof. We only prove the associativity. The proof of the other properties are simple.

$$
\begin{aligned}
&((x \star y) \star z) - x \star (y \star z) \\
&\quad = (a_0b_0 - p \circ q + a_0q + b_0p + p \times q + (p \cdot q)\lambda)(c_0 + r) \\
&\quad\quad - (a_0 + p)(b_0c_0 - q \circ r + b_0r + c_0q + q \times r + (q \cdot r)\lambda) \\
&\quad = [(a_0b_0 - p \circ q)c_0 - (a_0q \circ r + b_0p \circ r + +(p \cdot q)(\lambda \circ r)) \\
&\quad\quad + (a_0b_0 - p \circ q)r + c_0(a_0q + b_0p + p \times q + (p \cdot q)\lambda) + a_0(q \times r) \\
&\quad\quad + b_0(p \times r) + (p \times q) \times r) + (p.q)(\lambda \times r) + (a_0q + b_0p + p \times q + (p \cdot q) \cdot r)\lambda \\
&\quad\quad - [a_0(b_0c_0 - q \circ r) - b_0(p \circ r) - q \circ r) - c_0(p \circ q) - p \circ (\times r) - (q \cdot r)(p \circ \lambda) \\
&\quad\quad + a_0b_0r + a_0c_0q + a_0(q \times r) + a_0(q \cdot r)\lambda + b_0(p \times r) + c_0(p \times q) \\
&\quad\quad + p \times (q \times r) + (q \cdot r)(p \times \lambda) + (b_0c_0 - q \circ r)p + (b_0(p \cdot r) + +c_0(p \cdot q) \\
&\quad\quad + p \circ (q \times r) + (q \cdot r)(p \cdot \lambda))\lambda].
\end{aligned}
$$

By using of the equalities

$$p \circ (q \times r) = (p \times q) \circ r,\ p \cdot (q \times r) = (p \times q) \cdot r,\ p \times \lambda = q \times \lambda = 0, p \circ \lambda = q \circ \lambda = 0$$

and

$$(p \times q) \times r - p \times (q \times r) = (p \times q) \times r + (q \times r) \times p = (p \times r) \times q$$

we deduce

$$
\begin{aligned}
&((x \star y) \star z) - x \star (y \star z) \\
&\quad = -(p \circ q)r + (p \times q) \times r + (p \circ q)(\lambda \cdot r)\lambda \\
&\quad\quad - p \times (q \times r) + (q \circ r)p + (q \circ r)(\lambda \cdot p)\lambda \\
&\quad = (p \times r) \times q - (q \circ p)r + (q \circ r)p + [(p \circ q)(\lambda \cdot r) - (q \circ r)(\lambda \cdot p)]\lambda = 0.
\end{aligned}
$$

□

Now by an example we show that $(\mathbb{R}^5, +, \star)$ is not a commutative algebra.

Let $e_0 = (1, 0, 0, 0, 0)$, $e_1 = (0, 1, 0, 0, 0)$, $e_2 = (0, 0, 1, 0, 0)$, $e_3 = (0, 0, 0, 1, 0)$, $e_4 = (0, 0, 0, 0, 1)$. Then

$x \star y = [a_0b_0 - (\sum \lambda_i^2)(\sum a_ib_i) + (\sum \lambda_ia_i)(\sum \lambda_ib_i)]e_0 + [a_0b_1 + a_1b_0 + \lambda_2(a_3b_4 - a_4b_3) - \lambda_3(a_2b_4 - a_4b_2) + \lambda_4(a_2b_3 - a_3b_2) + \lambda_1(\sum a_ib_i)]e_1 + [a_0b_2 + a_2b_0 - \lambda_1(a_3b_4 - a_4b_3) + \lambda_3(a_1b_4 - a_4b_1) - \lambda_4(a_1b_3 - a_3b_1) + \lambda_2(\sum a_ib_i)]e_2 + [a_0b_3 + a_3b_0 + \lambda_1(a_2b_4 - a_4b_2) -$

$\lambda_2(a_1b_4 - a_4b_1) + \lambda_4(a_1b_2 - a_2b_1) + \lambda_3(\sum a_ib_i]e_3 + [a_0b_4 + a_4b_0 - \lambda_1(a_2b_3 - a_3b_2) + \lambda_2(a_1b_3 - a_3b_1) - \lambda_3(a_1b_2 - a_2b_1 + \lambda_4(\sum a_ib_i]e_4$.

Hence

$$e_1^2 = e_2^2 = e_3^2 = e_4^2 = \lambda_1e_1 + \lambda_2e_2 + \lambda_3e_3 + \lambda_4e_4$$

and

$$e_i \star e_j = \lambda_i\lambda_je_0 + (-1)^{i+j}(\lambda_ke_l - \lambda_le_k), e_j \star e_i = \lambda_i\lambda_je_0 + (-1)^{i+j+1}(\lambda_ke_l - \lambda_le_k)$$

where $i < j$ and $k,\, l \in \{1, 2, 3, 4\} - \{i, j\}$ and $k < l$. So $e_i \star e_j \neq e_j \star e_i$. In the above algebra e_0 is an identity element.

4. Main Results

Let A be an associative algebra with multiplicative identity e and let W be a complex vector space. A representation ρ of A on W is determined by a set of linear operators $\rho(a)$ on W such that

1. $\rho(\gamma a + \mu b) = \gamma\rho(a) + \mu\rho(b), \quad a,\, b \in A, \quad \gamma,\ \mu \in \mathbb{C}$,
2. $\rho(ab) = \rho(a)\rho(b)$,
3. $\rho(e) = E$, where E is the identity operator of W (see [5]).

If $x = (a_0, a_1, a_2, a_3, a_4) \in \mathbb{R}^5$ then we define $\rho : (\mathbb{R}^5, +, \star) \to M(5, \mathbb{R})$ by: $\rho(x) = A$, where

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ ta_1 + \lambda_1 s & a_0 + \lambda_1 a_1 & w_3^4 + \lambda_2 a_1 & -w_2^4 + \lambda_3 a_1 & w_2^3 + \lambda_4 a_1 \\ ta_2 + \lambda_2 s & -w_3^4 + \lambda_1 a_2 & a_0 + \lambda_2 a_2 & w_1^4 + \lambda_3 a_2 & -w_1^3 + \lambda_4 a_2 \\ ta_3 + \lambda_3 s & w_2^4 + \lambda_1 a_3 & -w_1^4 + \lambda_2 a_3 & a_0 + \lambda_3 a_3 & w_1^2 + \lambda_4 a_3 \\ ta_4 + \lambda_4 s & -w_2^3 + \lambda_1 a_4 & w_1^3 + \lambda_2 a_4 & -w_1^2 + \lambda_3 a_4 & a_0 + \lambda_4 a_4 \end{pmatrix} \qquad (6)$$

and

$$t = -(\sum_{i=1}^{4} \lambda_i^2), \quad s = \sum_{i=1}^{4} \lambda_i a_i \quad \text{and} \quad w_i^j = \lambda_i a_j - \lambda_j a_i, \quad i, j = 1, 2, 3, 4. \qquad (7)$$

Theorem 4.1. *With the above assumptions ρ is a representation of the algebra $(\mathbb{R}^5, +, \star)$.*

For the proof see appendix.

Theorem 4.2. *If $G = \{\rho(x) = A \in GL(5, \mathbb{R}) :\ x \in (\mathbb{R}^5, +, \star),\ \det(A) = 1\}$ then G is a group which is isomorphic to quaternions Lie group Q.*

Proof. Obviously G is a group (see [6]). We define $\psi : G \longrightarrow Q$ by:

$$\begin{aligned}\psi(\rho(x)) &= \psi(A) = \psi(\rho(a_0, a_1, a_2, a_3, a_4)) = a_0 + (\lambda_1 a_2 - \lambda_2 a_1 + \lambda_3 a_4 - \lambda_4 a_3)i \\ &+ (\lambda_1 a_4 - \lambda_4 a_1 + \lambda_2 a_3 - \lambda_3 a_2)j + (\lambda_1 a_3 - \lambda_3 a_1 - \lambda_2 a_4 + \lambda_4 a_2)k.\end{aligned}$$

Since ρ is an isomorphism, G isomorphic to $\overline{G} = \{x \in (\mathbb{R}^5, +, \star) : \det(\rho(x)) = 1\}$. If we put $\varphi(x) = \psi(\rho(x))$, then we prove that φ is an isomorphism. We have $\varphi(x + y) = \varphi(x) + \varphi(y)$. If $x = a_0 + p$ and $y = b_0 + q$ then

$$\begin{aligned}\varphi(x)\varphi(y) &= (\varphi(a_0) + \varphi(p))(\varphi(b_0) + \varphi(q)) \\ &= \varphi(a_0)\varphi(b_0) + \varphi(a_0)\varphi(q) + \varphi(p)\varphi(b_0) + \varphi(p)\varphi(q).\end{aligned}$$

Moreover

$$\begin{aligned}\varphi(xy) &= \varphi((a_0 + p) \star (b_0 + q)) = \varphi(a_0 b_0 - p \circ q + a_0 q + b_0 p + p \times q + (p \cdot q)\lambda) \\ &= \varphi(a_0 b_0) - \varphi(p \circ q) + \varphi(a_0 q) + \varphi(b_0 p) + \varphi(p \times q) + \varphi((p \cdot q)\lambda).\end{aligned}$$

We also have

$$\varphi(a_0 q) = a_0 \varphi(q) \text{ and } \varphi(\lambda) = 0,\ \varphi(p \times q) = \varphi(p)\varphi((q) + p \circ q.$$

So φ is a homomorphism. We see that

$$\begin{aligned}\det(\rho(x)) &= [a_0 + a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4][a_0^2 + (\lambda_1 a_2 - \lambda_2 a_1 + \lambda_3 a_4 - \lambda_4 a_3)^2 \\ &+ (\lambda_1 a_4 - \lambda_4 a_1 + \lambda_2 a_3 - \lambda_3 a_2)^2 + (\lambda_1 a_3 - \lambda_3 a_1 - \lambda_2 a_4 + \lambda_4 a_2)^2].\end{aligned}$$

Then $\det A = 1$ and $\varphi(x) = 1$ imply that $a_0 = 1$ and

$$\begin{aligned}a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 &= 0 \\ a_2\lambda_1 - a_1\lambda_2 - a_3\lambda_4 + a_4\lambda_3 &= 0 \\ a_4\lambda_1 - a_1\lambda_4 + a_3\lambda_2 - a_2\lambda_3 &= 0 \\ a_3\lambda_1 - a_1\lambda_3 - a_4\lambda_2 + a_2\lambda_4 &= 0.\end{aligned}$$

Since determinant of coefficient matrix of a_1, a_2, a_3, a_4 is non zero (It is equal to $(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)^2$) then $a_1 = a_2 = a_3 = a_4 = 0$. So $\ker(\varphi) = \{1\}$. Thus φ is an isomorphism. □

Theorem 4.3. *Let* $H = \{A = \rho(x) \in GL(5, \mathbb{R}) : \ x \in (\mathbb{R}^5, +, \star) : \ \det(A) = 1,$ $a_0 + a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 = 1\}$. *Then* H *is isomorphic to* SU(2).

Proof. We know that SU(2) is isomorphic to three dimensional unit sphere S^3 (see [6]). Since ψ in the previous theorem is an isomorphism then $\psi^{-1}(S^3)$ is a subgroup of G. If $\psi(\rho(x)) \in S^3$ then

$$\begin{aligned}\| \psi(\rho(x) \|^2 = 1 &= a_0^2 + (\lambda_2 a_1 - \lambda_1 a_2)^2 + (\lambda_3 a_1 - \lambda_1 a_3)^2 + (\lambda_4 a_1 - \lambda_1 a_4)^2 \\ &+ (\lambda_2 a_3 - \lambda_3 a_2)^2 + (\lambda_2 a_4 - \lambda_4 a_2)^2 + (\lambda_3 a_4 - \lambda_4 a_3)^2,\end{aligned}$$

where $\| \, . \, \|$ is the standard norm of $\mathbb{R}^4$. We also have

$$\begin{aligned}\det(\rho(x)) = {} & [a_0 + a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4] \\ & [a_0^2 + (\lambda_2^2 + \lambda_3^2 + \lambda_4^2)a_1^2 + (\lambda_1^2 + \lambda_3^2 + \lambda_4^2)a_2^2 \\ & + (\lambda_1^2 + \lambda_2^2 + \lambda_4^2)a_3^2 + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)a_4^2 - 2\lambda_1\lambda_2 a_1 a_2 - 2\lambda_1\lambda_3 a_1 a_3 \\ & - 2\lambda_1\lambda_4 a_1 a_4 - 2\lambda_2\lambda_3 a_2 a_3 - 2\lambda_2\lambda_4 a_2 a_4 - 2\lambda_3\lambda_4 a_3 a_4)]^2 \\ = {} & [a_0 + a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4][a_0^2 + (\lambda_2 a_1 - \lambda_1 a_2)^2 + (\lambda_3 a_1 - \lambda_1 a_3)^2 \\ & + (\lambda_4 a_1 - \lambda_1 a_4)^2 + (\lambda_2 a_3 - \lambda_3 a_2)^2 + (\lambda_2 a_4 - \lambda_4 a_2)^2 + (\lambda_3 a_4 - \lambda_4 a_3)^2]^2.\end{aligned}$$

The facts $\det(\rho(x)) = \det(A) = 1$ and $\psi(\rho(x)) \in S^3$ imply that

$$a_0 + a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 = 1.$$

Thus $H = \psi^{-1}(S^3)$. □

Remark 4.4. If $x = (a_0, a_1, a_2, a_3, a_4)$, $y = (b_0, b_1, b_2, b_3, b_4) \in \mathbb{R}^5$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4$, then we can define the inner product of x, y by

$$g(x, y) = a_0^2 + (\sum_{i=1}^{4} \lambda_i^2)(\sum_{i=0}^{4} a_i b_i) - (\sum_{i=1}^{4} a_i \lambda_i)(\sum_{i=1}^{4} b_i \lambda_i) \tag{8}$$

g is degenerate and its associated matrix with respect to the usual basis of $\mathbb{R}^5$ is

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2^2 + \lambda_3^2 + \lambda_4^2 & -\lambda_1\lambda_2 & -\lambda_1\lambda_3 & -\lambda_1\lambda_4 \\ 0 & -\lambda_2\lambda_1 & \lambda_1^2 + \lambda_3^2 + \lambda_4^2 & -\lambda_2\lambda_3 & -\lambda_2\lambda_4 \\ 0 & -\lambda_3\lambda_1 & -\lambda_3\lambda_2 & \lambda_1^2 + \lambda_2^2 + \lambda_4^2 & -\lambda_3\lambda_4 \\ 0 & -\lambda_4\lambda_1 & -\lambda_4\lambda_2 & -\lambda_4\lambda_3 & \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \end{pmatrix}. \tag{9}$$

We see that for each $A \in (\mathbb{R}^5, +, \star)$ presented in Equation (6) we have $AJA^t = g(x, x)J$ for all vectors $x \in \mathbb{R}^5$ where $A = \rho(x)$ and so for $A \in H$ presented in Theorem 4.3 we have $AJA^t = J$. Also simple calculations show that for each V in the Lie algebra of G we have $VJ + JV^t = 0$.

5. Conclusion

Theorems 4.2 and 4.3 present new representations of quaternions Lie group and SU(2). This representations are based on the new concept of outer product in $\mathbb{R}^4$, and special attention to $\mathbb{R}^5$ as an associative algebra.

Appendix

Proof of Theorem 4.1. Let $V = \{\rho(x) = A : x \in (\mathbb{R}^5, +, \star)\}$. Then V is a vector space. We have $\rho(x+y) = \rho(x) + \rho(y)$, $\rho(tx) = t\rho(x)$ and $\rho(e) = \rho(1,0,0,0,0) = I$, $\rho(x)\rho(y) = [a_0b_0 + a_1(-(\sum \lambda_i^2)b_1 + \lambda_1(\sum \lambda_i b_i)) + a_2(-(\sum \lambda_i^2)b_2 + \lambda_2(\sum \lambda_i b_i)) +$ $a_3(-(\sum \lambda_i^2)b_3+\lambda_3(\sum \lambda_i b_i))+a_4(-(\sum \lambda_i^2)b_4+\lambda_4(\sum \lambda_i b_i))]E_1^1+[a_0b_1+a_1(b_0+\lambda_1 b_1)+$ $a_2(-\lambda_3b_4+\lambda_4b_3+\lambda_1b_2)+a_3(\lambda_2b_4-\lambda_4b_2+\lambda_1b_3)+a_4(\lambda_2b_3-\lambda_3b_2+\lambda_1b_4)]E_1^2+[a_0b_2+$ $a_1(\lambda_3b_4-\lambda_4b_3+\lambda_2b_1)+a_2(b_0+\lambda_2b_2)+a_3(-\lambda_1b_4+\lambda_4b_1+\lambda_2b_3)+a_4(\lambda_1b_3-\lambda_3b_1+$ $\lambda_2b_4)]E_1^3+[a_0b_3+a_1(-\lambda_2b_4+\lambda_4b_2+\lambda_3b_1)+a_2(\lambda_1b_4-\lambda_4b_1+\lambda_3b_2)+a_3(b_0+\lambda_3b_3)+$ $a_4(-\lambda_1b_2+\lambda_2b_1+\lambda_3b_4)]E_1^4+[a_0b_4+a_1(\lambda_2b_3-\lambda_3b_2+\lambda_4b_1)+a_2(-\lambda_1b_3-\lambda_3b_1+$ $\lambda_4b_2)+a_3(\lambda_1b_2-\lambda_2b_1+\lambda_4b_3)+a_4(b_0+\lambda_4b_4)]E_1^5+[(-(\sum\lambda_i^2)a_1+\lambda_1(\sum\lambda_ia_i))b_0+$ $(a_0+\lambda_1a_1)(-(\sum\lambda_i^2)b_1+\lambda_1(\sum\lambda_ib_i))+(\lambda_3a_4-\lambda_4a_3+\lambda_2a_1)(-(\sum\lambda_i^2)b_2+\lambda_2(\sum\lambda_ib_i))+$ $(-\lambda_2a_4+\lambda_4a_2+\lambda_3a_1)(-(\sum\lambda_i^2)b_3+\lambda_3(\sum\lambda_ib_i))+(\lambda_2a_3-\lambda_3a_2+\lambda_4a_1)(-(\sum\lambda_i^2)b_4+$ $\lambda_4(\sum\lambda_ib_i))]E_2^1+[(-(\sum\lambda_i^2)a_1+\lambda_1(\sum\lambda_ia_i)b_1+(a_0+\lambda_1a_1)(b_0+\lambda_1b_1)+(\lambda_3a_4-\lambda_4a_3+$ $\lambda_2a_1)(-\lambda_3b_4+\lambda_4b_3+\lambda_1b_2)+(-\lambda_2a_4+\lambda_4a_2+\lambda_3a_1)(\lambda_2b_4-\lambda_4b_2+\lambda_1b_3)+(\lambda_2a_3-$ $\lambda_3a_2+\lambda_4a_1)(\lambda_2b_3-\lambda_3b_2+\lambda_1b_4)]E_2^2+[(-(\sum\lambda_i^2)a_1+\lambda_1(\sum\lambda_ia_i)b_2+(a_0+\lambda_1a_1)(\lambda_3b_4-$ $\lambda_4b_3+\lambda_2b_1)+(-\lambda_2a_4+\lambda_4a_2+\lambda_3a_1)(b_0+\lambda_2b_2)+(\lambda_2a_3-\lambda_3a_2+\lambda_4a_1)(-\lambda_1b_4+\lambda_4b_1+$ $\lambda_2b_3)+(\lambda_2a_3-\lambda_3a_2+\lambda_4a_1)(\lambda_1b_3-\lambda_3b_1+\lambda_2b_4)]E_2^3+[(-(\sum\lambda_i^2)a_1+\lambda_1(\sum\lambda_ia_i))b_3+$ $(a_0+\lambda_1a_1)(-\lambda_2b_4+\lambda_4b_2+\lambda_3b_1)+(\lambda_3a_4-\lambda_4a_3+\lambda_2a_1)(\lambda_1b_4-\lambda_4b_1+\lambda_3b_2)+(-\lambda_2a_4+$ $\lambda_4a_2+\lambda_3a_1)(b_0+\lambda_3b_3)+(\lambda_2a_3-\lambda_3a_2+\lambda_4a_1)(-\lambda_1b_2+\lambda_2b_1+\lambda_3b_4)]E_2^4+[(-(\sum\lambda_i^2)a_1+$ $\lambda_1(\sum\lambda_ia_i))b_4+(a_0+\lambda_1a_1)(\lambda_2b_3-\lambda_3b_2+\lambda_4b_1)+(\lambda_3a_4-\lambda_4a_3+\lambda_2a_1)(-\lambda_1b_3+\lambda_3b_1+$ $\lambda_4b_2)+(-\lambda_2a_4+\lambda_4a_2+\lambda_3a_1)(\lambda_1b_2-\lambda_2b_1+\lambda_4b_3)+(\lambda_2a_3-\lambda_3a_2+\lambda_4a_1)(b_0+\lambda_4b_4)]E_2^5+$ $[(-(\sum\lambda_i^2)a_2+\lambda_2(\sum\lambda_ia_i))b_0+(-\lambda_3a_4+\lambda_4a_3+\lambda_1a_2)(-(\sum\lambda_i^2)b_1+\lambda_1(\sum\lambda_ib_i))+(a_0+$ $\lambda_2a_2)(-(\sum\lambda_i^2)b_2+\lambda_2(\sum\lambda_ib_i))+(\lambda_1a_4-\lambda_4a_1+\lambda_3a_2)(-(\sum\lambda_i^2)b_3+\lambda_3(\sum\lambda_ib_i))+$ $(-\lambda_1a_3+\lambda_3a_1+\lambda_4a_2)(-(\sum\lambda_i^2)b_4+\lambda_4(\sum\lambda_ib_i))]E_3^1+[(-(\sum\lambda_i^2)a_2+\lambda_2(\sum\lambda_ia_i))b_1+$ $(-\lambda_3a_4+\lambda_4a_3+\lambda_1a_2)(b_0+\lambda_1b_1)+(a_0+\lambda_2a_2)(-\lambda_3b_4+\lambda_4b_3+\lambda_1b_2)+(\lambda_1a_4-\lambda_4a_1+$ $\lambda_3a_2)(\lambda_2b_4-\lambda_4b_2+\lambda_1b_3)+(-\lambda_1a_3+\lambda_3a_1+\lambda_4a_2)(\lambda_2b_3-\lambda_3b_2+\lambda_1b_4)]E_3^2+[(-(\sum\lambda_i^2)a_2+$ $\lambda_2(\sum\lambda_ia_i))b_2+(\lambda_3a_4+\lambda_4a_3+\lambda_1a_2)(\lambda_3b_4-\lambda_4b_3+\lambda_2b_1)+(a_0+\lambda_2a_2)(b_0+\lambda_2b_2)+$ $(\lambda_1a_4-\lambda_4a_1+\lambda_3a_2)(-\lambda_1b_4+\lambda_4b_1+\lambda_2b_3)+(-\lambda_1a_3+\lambda_3a_1+\lambda_4a_2)(\lambda_1b_3-\lambda_3b_1+$ $\lambda_2b_4)]E_3^3+[(-(\sum\lambda_i^2)a_2+\lambda_2(\sum\lambda_ia_i))b_3+(-\lambda_3a_4+\lambda_4a_3+\lambda_1a_2)(-\lambda_2b_4+\lambda_4b_2+$

$\lambda_3b_1)+(a_0+\lambda_2a_2)(\lambda_1b_4-\lambda_4b_1+\lambda_3b_2)+(\lambda_1a_4-\lambda_4a_1+\lambda_3a_2)(b_0+\lambda_3b_3)+(-\lambda_1a_3+\lambda_3a_1+\lambda_4a_2)(-\lambda_1b_2+\lambda_2b_1+\lambda_3b_4)]E_3^4+[(-(\sum\lambda_i^2)a_2+\lambda_2(\sum\lambda_ia_i))b_4+(-\lambda_3a_4+\lambda_4a_3+\lambda_1a_2)(\lambda_2b_3-\lambda_3b_2+\lambda_4b_1)+(a_0+\lambda_2a_2)(-\lambda_1b_3-\lambda_3b_1+\lambda_4b_2)+(\lambda_1a_4-\lambda_4a_1+\lambda_3a_2)(\lambda_1b_2-\lambda_2b_1+\lambda_4b_3)+(-\lambda_1a_3+\lambda_3a_1+\lambda_4a_2)(b_0+\lambda_4b_4)]E_3^5+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i))b_0+(\lambda_2a_4-\lambda_4a_2+\lambda_1a_3)(-(\sum\lambda_i^2)b_1+\lambda_1(\sum\lambda_ib_i))+(-\lambda_1a_4+\lambda_4a_1+\lambda_2a_3)(-(\sum\lambda_i^2)b_2+\lambda_2(\sum\lambda_ib_i))+(a_0+\lambda_3a_3)(-(\sum\lambda_i^2)b_3+\lambda_3(\sum\lambda_ib_i))+(\lambda_1a_2-\lambda_2a_1+\lambda_4a_3)(-(\sum\lambda_i^2)b_4+\lambda_4(\sum\lambda_ib_i))]E_4^1+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i))b_1+(\lambda_2a_4-\lambda_4a_2+\lambda_1a_3)(b_0+\lambda_1b_1)+(-\lambda_1a_4+\lambda_4a_1+\lambda_2a_3)(-\lambda_3b_4+\lambda_4b_3+\lambda_1b_2)+(a_0+\lambda_3a_3)(\lambda_2b_4-\lambda_4b_2+\lambda_1b_3)+(\lambda_1a_2-\lambda_2a_1+\lambda_4a_3)(\lambda_2b_3-\lambda_3b_2+\lambda_1b_4)]E_4^2+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i))b_2+(\lambda_3(\sum\lambda_ia_i))(\lambda_3b_4-\lambda_4b_3+\lambda_2b_1)+(-\lambda_1a_4+\lambda_4a_1+\lambda_2a_3)(b_0+\lambda_2b_2)+(a_0+\lambda_3a_3)(-\lambda_1b_4+\lambda_4b_1+\lambda_2b_3)+(\lambda_1a_2-\lambda_2a_1+\lambda_4a_3)(\lambda_1b_3-\lambda_3b_1+\lambda_2b_4)]E_4^3+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i))b_3+(\lambda_2a_4-\lambda_4a_2+\lambda_1a_3)(-\lambda_2b_4+\lambda_4b_2+\lambda_3b_1)+(\lambda_1a_4+\lambda_4a_1+\lambda_2a_3)(\lambda_1b_4-\lambda_4b_1+\lambda_3b_2)+(a_0+\lambda_2a_2)(b_0+\lambda_3b_3)+(\lambda_1a_2-\lambda_2a_1+\lambda_4a_3)(-\lambda_1b_2+\lambda_2b_1+\lambda_3b_4)]E_4^4+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i))b_4+(\lambda_2a_4-\lambda_4a_2+\lambda_1a_3)(\lambda_2b_3-\lambda_3b_2+\lambda_4b_1)+(-\lambda_1a_4+\lambda_4a_1+\lambda_2a_3)(-\lambda_1b_3-\lambda_3b_1+\lambda_4b_2)+(a_0+\lambda_3a_3)(\lambda_1b_2-\lambda_2b_1+\lambda_4b_3)+(\lambda_1a_2-\lambda_2a_1+\lambda_4a_3)(b_0+\lambda_4b_4)]E_4^5+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i))b_0+(\lambda_2a_3-\lambda_3a_2+\lambda_1a_4)(-(\sum\lambda_i^2)b_1+\lambda_1(\sum\lambda_ib_i))+(\lambda_1a_3-\lambda_3a_1+\lambda_2a_4)(-(\sum\lambda_i^2)b_2+\lambda_2(\sum\lambda_ib_i))+(-\lambda_1a_2+\lambda_2a_1+\lambda_3a_4)(-(\sum\lambda_i^2)b_3+\lambda_3(\sum\lambda_ib_i))+(a_0+\lambda_4a_4)(-(\sum\lambda_i^2)b_4+\lambda_4(\sum\lambda_ib_i))]E_5^1+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i)b_1+(\lambda_2a_3-\lambda_3a_2+\lambda_1a_4)(b_0+\lambda_1b_1)+(\lambda_1a_3-\lambda_3a_1+\lambda_2a_4)(-\lambda_3b_4+\lambda_4b_3+\lambda_1b_2)+(-\lambda_1a_2+\lambda_2a_1+\lambda_3a_4)(\lambda_2b_4-\lambda_4b_2+\lambda_1b_3)+(a_0+\lambda_4a_4)(\lambda_2b_3-\lambda_3b_2+\lambda_1b_4)]E_5^2+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i))b_2+(\lambda_2a_4-\lambda_4a_2+\lambda_1a_3)(\lambda_3b_4-\lambda_4b_3+\lambda_2b_1)+(\lambda_1a_3-\lambda_3a_1+\lambda_2a_4)(b_0+\lambda_2b_2)+(-\lambda_1a_2+\lambda_2a_1+\lambda_3a_4)(-\lambda_1b_4+\lambda_4b_1+\lambda_2b_3)+(a_0+\lambda_4a_4)(\lambda_1b_3-\lambda_3b_1+\lambda_2b_4)]E_5^3+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i))b_3+(\lambda_2a_4-\lambda_4a_2+\lambda_1a_3)(-\lambda_2b_4+\lambda_4b_2+\lambda_3b_1)+(\lambda_1a_3-\lambda_3a_1+\lambda_2a_4)(\lambda_1b_4-\lambda_4b_1+\lambda_3b_2)+(-\lambda_1a_2+\lambda_2a_1+\lambda_3a_4)(b_0+\lambda_3b_3)+(a_0+\lambda_4a_4)(-\lambda_1b_2+\lambda_2b_1+\lambda_3b_4)]E_5^4+[(-(\sum\lambda_i^2)a_3+\lambda_3(\sum\lambda_ia_i))b_4+(\lambda_2a_4-\lambda_4a_2+\lambda_1a_3)(\lambda_2b_3-\lambda_3b_2+\lambda_4b_1)+(\lambda_1a_3-\lambda_3a_1+\lambda_2a_4)(-\lambda_1b_3-\lambda_3b_1+\lambda_4b_2)+(-\lambda_1a_2+\lambda_2a_1+\lambda_3a_4)(\lambda_1b_2-\lambda_2b_1+\lambda_4b_3)+(a_0+\lambda_4a_4)(b_0+\lambda_4b_4)]E_5^5.$

If we put

$z_0=a_0b_0-(\sum\lambda_i^2)(\sum a_ib_i)+(\sum\lambda_ia_i)(\sum\lambda_ib_i)$

$z_1=a_0b_1+a_1b_0+\lambda_2(a_3b_4-a_4b_3)-\lambda_3(a_2b_4-a_4b_2)+\lambda_4(a_2b_3-a_3b_2)+\lambda_1(\sum a_ib_i)$

$z_2=a_0b_2+a_2b_0-\lambda_1(a_3b_4-a_4b_3)+\lambda_3(a_1b_4-a_4b_1)-\lambda_4(a_1b_3-a_3b_1)+\lambda_2(\sum a_ib_i)$

$z_3=a_0b_3+a_3b_0+\lambda_1(a_2b_4-a_4b_2)-\lambda_2(a_1b_4-a_4b_1)+\lambda_4(a_1b_2-a_2b_1)+\lambda_3(\sum a_ib_i)$

$z_4=a_0b_4+a_4b_0-\lambda_1(a_2b_3-a_3b_2)+\lambda_2(a_1b_3-a_3b_1)-\lambda_3(a_1b_2-a_2b_1+\lambda_4(\sum a_ib_i)$

then by suitable factorization in the above formula we have:

$\rho(x)\rho(y) = [z_0]E_1^1 + [z_1]E_1^2 + [z_2]E_1^3 + [z_3]E_1^4 + [z_4]E_1^5 + [-(\sum \lambda_i^2)z_1 + \lambda_1(\sum \lambda_i z_i)]E_2^1 + [[z_0 + \lambda_1 z_1]E_2^2 + [\lambda_3 z_4 - \lambda_4 z_3 + \lambda_2 z_1]E_2^3 + +[-\lambda_2 z_4 + \lambda_4 z_2 + \lambda_3 z_1]E_2^4 + [\lambda_2 z_3 - \lambda_3 z_2 + \lambda_4 z_1]E_2^5 + [-(\sum \lambda_i^2 z_2 + \lambda_2(\sum \lambda_i z_i)]E_3^1 + [-\lambda_3 z_4 + \lambda_4 z_3 + \lambda_1 z_2]E_3^2 + [z_0 + \lambda_2 z_2]E_3^3 + [\lambda_1 z_4 - \lambda_4 z_1 + \lambda_3 z_2]E_3^4 + [-\lambda_1 z_3 + \lambda_3 z_1 + \lambda_4 z_2]E_3^5 + [-(\sum \lambda_i^2)z_3 + \lambda_3(\sum \lambda_i z_i)]E_4^1 + [\lambda_2 z_4 - \lambda_4 z_2 + \lambda_1 z_3]E_4^2 + [-\lambda_1 z_4 + \lambda_4 z_1 + \lambda_2 z_3]E_4^3 + [z_0 + \lambda_3 z_3]E_4^4 + [\lambda_1 z_2 - \lambda_2 z_1 + \lambda_4 z_3]E_4^5 + [-(\sum \lambda_i^2)z_3 + \lambda_3(\sum \lambda_i z_i)]E_5^1 + [\lambda_2 z_3 - \lambda_3 z_2 + \lambda_1 z_4]E_5^2 + [\lambda_1 z_3 - \lambda_3 z_1 + \lambda_2 z_4]E_5^3 + [-\lambda_1 z_2 + \lambda_2 z_1 + \lambda_3 z_4]E_5^4 + [z_0 + \lambda_4 z_4]E_5^5 = \rho(x \star y).$ □

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