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Kress Smoothing Transformation for Weakly Singular Fredholm Integral Equations of Second Kind

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Özet. Makale, ikinci tip, $k(x,y)$ zayıf tekil çekirdekli, özel olarak $k(x,y)=\ln|x-y|$ ve $k(x,y)=|x-y|^{-\alpha}$, $-1\le x,\ y\le 1$, $0<\alpha<1$, durumlu, Fredholm integral denklemi için nümerik bir metodu incelemektedir. Bu tür denklemlerin çözümleri uç noktaların, $x=\pm 1$, komşuluklarında tekil davranışlar gösterebilir. İkinci tip zayıf tekillikte Fredholm integral denklemlerini çözmek için Kress dönüşümünü temel alan ve daha sonra standart olarak Hermite düzleştirme dönüşümünü kullanan yeni bir düzleştirme dönüşümü sunuyoruz. Bu dönüşümle hala zayıf tekillikte fakat çözümü daha düzgün bir denklem elde edilir. Dönüştürülmüş denklem daha sonra interpole eden polinomlarla çarpım integralleme metodu kullanılarak nümerik olarak çözülür. İki tip interpole eden polinom, yani Gauss-Legendre ve Chebyshev polinomları, kullanılır. İlkinin performansını incelemek için nümerik örnekler sunulmuştur.[†]

Anahtar Kelimeler. İntegral denklemi, zayıf tekillikte integral denklemi, düzgünleştirme dönüşümü.

Abstract. The paper investigates a numerical method for the second kind Fredholm integral equation with weakly singular kernel $k(x,y)$, in particular, when $k(x,y)=\ln|x-y|$, and $k(x,y)=|x-y|^{-\alpha}$, $-1\le x,\ y\le 1$, $0<\alpha<1$. The solutions of such equations may exhibit a singular behaviour in the neighbourhood of the endpoints $x=\pm 1$. We introduce a new smoothing transformation based on the Kress transformation for solving weakly singular Fredholm integral equations of the second kind, and then using the Hermite smoothing transformation as a standard. With the transformation an equation which is still weakly singular is obtained, but whose solution is smoother. The transformed equation is then solved numerically by product integration methods with interpolating polynomials. Two types of interpolating polynomials, namely the Gauss-Legendre and Chebyshev polynomials, have been used. Numerical examples are presented to investigate the performance of the former.

Keywords. Integral equation, weakly singular integral equation, smoothing transformation.

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1. Introduction

Consider weakly singular Fredholm integral equation of second kind of the form

$$f(x) - \lambda \int_{-1}^{1} k(x,y) f(y) dy = g(x), \quad -1 \leq x \leq 1, \tag{1.1}$$

with weakly singular kernels of one of the following forms:
Abel kernel

$$k(x,y) = |x-y|^{-\alpha},\ 0 < \alpha < 1, \tag{1.2}$$

logarithmic kernel

$$k(x,y) = \ln|x-y|, \tag{1.3}$$

where $-1 \leq x \leq 1$.

The numerical solution of (1.1) is closely related to the solution of a linear algebraic system. Indeed, the main goal of the numerical methods to solve (1.1) is to reduce it approximately to a linear algebraic system, then the linear algebraic system is solved to obtain an approximate solution of (1.1). The numerical treatment of weakly singular integral equations should take into account the nature of the singularities at the endpoints $x = \pm 1$, and the singularities of the input function. Some of the methods that can be used to solve these integral equations are as follows.

1. Canceling the singularity (of the kernel).
2. Modified quadrature method.
3. Smoothing the kernel.
4. Approximating the kernel by a degenerate kernel.
5. Expansion methods (Galerkin and collocation methods).
6. Product integration.

We will use product integration method for solving (1.1).

2. The Smoothing Transformations

Since the rate of convergence of a numerical method depends on the regularity of the solution of equation (1.1), the knowledge of the behaviour of the solution is very important in the choice of the method; for this reason we shall discuss the analysis of the properties of the solution of (1.1).

When $g(x)$ is sufficiently smooth, the solutions of (1.1) with kernels (1.2), or (1.3) have first derivatives which behave, respectively, like $(x+1)^{-\alpha}$ and $\ln|x+1|$ near

$x = -1$, and have equivalent singularties near $x = 1$. When $0 < \alpha < 1$, the function $(x+1)^{-\alpha}$ certainly belongs to the space $L_p[-1, 1]$ for any p in the range $1 \le p < 1/\alpha$, and $\ln|x+1|$ is in the space $L_p[-1, 1]$ for any p in the range $1 \le p < \infty$; see Graham [16].

When the input function $g(x)$ is smooth, say $g \in C^{p+1}[-1, 1]$, the solution $f(x)$ has only endpoints mild singularities, that is, $f \in C^p(-1, 1)$, in the case of the equation (1.1) with the kernel (1.3), the solution $f(x)$ admits an expansion containing a finite number of terms of the form

$$(1 \pm x)^i \big[\ln|1 \pm x|\big]^j, \quad i, j = 1, 2, \ldots, p, \quad i \ge j, \tag{2.1}$$

plus a function of class $C^p(-1, 1)$. If the input function or one of its first derivatives has, for example, simple jumps at a finite number of points in $(-1, 1)$ and smooth elsewhere, the solution $f(x)$ may be expressed as a linear combination of $g(x)$ and a finite number of terms which are mildly singular, as those in (2.1), either at ± 1 or at the jump points of $g(x)$, plus an unknown smooth function; see Monegato and Scuderi [11].

We need to use smoothing transformation to reduce (1.1) to an equivalent equation which has smoother solution.

2.1. Hermite smoothing transformation. Consider equation (1.1) with its kernels (1.2), and (1.3); furthermore consider that the input function $g(x)$ has finite singularities $-1 < x_1 < x_2 < \ldots < x_M < 1$. Then the Hermite transformation $w(t) = H_M(t)$ associated with the partition $-1 = x_0 < x_1 < x_2 < \ldots < x_M < x_{M+1} = 1$ of $[-1, 1]$, and defined in each subinterval $[x_k, x_{k+1}]$, $k = 0, \ldots, M$ by the conditions

$$\begin{cases} H_M(x_j) = x_j,\ j = k, k+1, \\ H^{(i)}{}_M(x_j) = 0,\ j = k, k+1,\ i = 1, \ldots, \alpha_j - 1, \alpha_j \ge 2, \end{cases} \tag{2.2}$$

and the integers α_k, $k = 0, \ldots, M+1$ are chosen accordingly to the smoothing effect that $w(t)$ ought to produce at the points x_k, $k = 0, \ldots, M+1$.

The construction and evaluation of $H_M(t)$ and $H'_M(t)$ is not as trivial as it might appear at first, particularly if we want to have an automatic program where the α_k may be arbitrarily chosen. A numerically stable and efficient procedure is the following one.

Since we know a priori that in $[x_k, x_{k+1}]$

$$H'_M(t) = c_k(t - x_k)^{\alpha_k - 1}(x_{k+1} - t)^{\alpha_{k+1} - 1} \tag{2.3}$$

where c_k is a suitable constant, we can use this expression to derive the following representation for $H_M(t)$:

$$H_M(t) = c_k \int_{x_k}^{t} (y - x_k)^{\alpha_k - 1}(x_{k+1} - y)^{\alpha_{k+1} - 1} dy + x_k,\ t \in [x_k, x_{k+1}]. \tag{2.4}$$

By imposing the conditions $H_M(x_{k+1}) = x_{k+1}$, $k = 0, \dots, M$ we determine the coefficients c_k as

$$c_k = (x_{k+1} - x_k)^{2 - \alpha_k - \alpha_{k+1}} \frac{(\alpha_k + \alpha_{k+1} - 1)!}{(\alpha_k - 1)!(\alpha_{k+1} - 1)!},\ k = 0, \dots, M. \tag{2.5}$$

Hence, the Hermite smoothing transformation over $[-1, 1]$ is

$$w(t) = H_M(t),\ t \in [x_k, x_{k+1}],\ k = 0, 1, \dots, M, \tag{2.6}$$

where $H_M(t)$ is as in (2.4). For more details see Monegato and Scuderi [11].

2.2. Kress smoothing transformation. Consider equation (1.1) with its kernels (1.2), and (1.3). Then Kress transformation is

$$w(t) = \frac{[v(t)]^p - [v(-t)]^p}{[v(t)]^p + [v(-t)]^p}, \quad -1 \le t \le 1, \tag{2.7}$$

where

$$v(t) = \left(\frac{1}{2} - \frac{1}{p}\right)t^3 + \frac{t}{p} + \frac{1}{2}.$$

This transformation is bijective, strictly monotonically increasing and infinitely differentiable, in addition to that the derivatives of w vanish up to a certain order at the endpoints of integration. It can be shown that

$$w'(t) = 2p\frac{[v(t)]^p[v(-t)]^{p-1}v'(-t) + [v(-t)]^p[v(t)]^{p-1}v'(t))}{\big[[v(t)]^p + [v(-t)]^p\big]^2},\ -1 \le t \le 1. \tag{2.8}$$

We note that

$$\begin{aligned} w(-1) &= -1, \quad w(1) = 1, \\ w'(-1) &= w'(1) = 0. \end{aligned} \tag{2.9}$$

For more details see Kress [24].

2.3. Modified Kress smoothing transformation. Consider equation (1.1) with its kernels (1.2), and (1.3); furthermore consider that the input function $g(x)$ has finite singularities $-1 < x_1 < x_2 < \ldots < x_M < 1$; let $x_0 = -1$, $x_{M+1} = 1$. We need to define a new 1-1 transformation $w_k = w_k(t)$ on the interval $[x_k, x_{k+1}]$ for $k = 0, 1, \ldots, M$ such that the following conditions are satisfied:

1. $w_k(x_k) = x_k$, $w_k(x_{k+1}) = x_{k+1}$.
2. $w_k'(x_k) = w_k'(x_{k+1}) = 0$, $w_k'\left(\frac{x_{k+1}+x_k}{2}\right) = 2$.

We will use the Kress transformation (2.7) to define a new transformation. Let

$$w_k(t) = a + bw(s(t)), \ t \in [w_k, x_{k+1}], \tag{2.10}$$

with

$$s(t) = c + dt,$$

such that: $w = -1$ implies $w_k = x_k$; $w = 1$ implies $w_k = x_{k+1}$; $t = x_k$ implies $s = -1$; and $t = x_{k+1}$ implies $s = 1$.

These give:

$$a = \frac{x_{k+1} + x_k}{2}, \qquad b = \frac{x_{k+1} - x_k}{2},$$
$$c = \frac{x_{k+1} + x_k}{x_{k+1} - x_k}, \qquad d = \frac{2}{x_{k+1} - x_k}.$$

Then the transformation (2.10) becomes

$$\begin{aligned} w_k(t) =& \frac{1}{2}\left[(x_{k+1} + x_k) + (x_{k+1} - x_k)w\left(\frac{2t - (x_{k+1} + x_k)}{x_{k+1} - x_k}\right)\right], \\ & t \in [x_k, x_{k+1}], \text{ for } k = 0, 1, \ldots, M. \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} w_k'(t) =& w'\left(\frac{2t - (x_{k+1} + x_k)}{x_{k+1} - x_k}\right), \\ & t \in [x_k, x_{k+1}], \text{ for } k = 0, 1, \ldots, M. \end{aligned} \tag{2.12}$$

Using (2.9), (2.11), and (2.12) we obtain the following:

1. $w_k(x_k) = x_k$, $w_k(x_{k+1}) = x_{k+1}$.
2. $w_k'(x_k) = w'(-1) = 0$, $w_k'(x_{k+1}) = w'(1) = 0$.

The new transformation can be defined on $[-1, 1]$ as the following

$$w(t) = w_k(t), \ t \in [x_k, x_{k+1}], \ k = 0, 1, \ldots, M, \tag{2.13}$$

where M is the number of the singularities of the input function $g(x)$. This transformation will be called the modified Kress transformation. It is clear that Kress transformation is a special case of the modified Kress transformation, that is the equation (2.7) could be obtained from (2.13) by setting $M = 0$.

3. Product Integration Method

Product integration method is a powerful tool for numerical calculation of integrals whose integrands have singularities. We write the integral as

$$I(f) = \int_{-1}^{1} k(x)f(x)dx, \tag{3.1}$$

where $f(x)$ is assumed to be continuous, and whatever singularities or poor behaviour in the integrand are included in $k(x)$. The function $k(x)$ is assumed to be a real-valued absolutely integrable function, but needs not be continuous or of one sign. We then approximate $f(x)$ by an interpolating function $f_n(x)$, where

$$f(x_i) = f_n(x_i),\ i = 0, 1, \dots, n,$$

and then compute the integral

$$I_n(f) = \int_{-1}^{1} k(x)f_n(x)dx. \tag{3.2}$$

The type of approximation must be chosen so that the integral in (3.2) can be evaluated (either explicitly or by an efficient numerical technique).

Let $\mathbb{P}_n$ be the space of all polynomials of degree less than or equal to n, and let $\phi_0(x), \phi_1(x), \dots, \phi_n(x)$ be a basis for $\mathbb{P}_n$. The functions $\phi_0(x), \phi_1(x), \dots, \phi_n(x)$ will be called interpolating elements. In this paper, the interpolating function $f_n(x)$ will be assumed to be the interpolating polynomial

$$f_n(x) = L_n^f(x) = \sum_{j=0}^{n} \phi_j(x)f_n(x_j). \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain

$$I_n(f) = \sum_{j=0}^{n} \left(\int_{-1}^{1} k(x)\phi_j(x)dx \right) f(x_j).$$

Hence the product integration rule for $I(f)$ is given by

$$I_n(f) = \sum_{j=0}^{n} \omega_j^{(k)} f(x_j), \tag{3.4}$$

where

$$\omega_j^{(k)} = \int_{-1}^{1} k(x)\phi_j(x)dx, \quad j = 0, 1, \dots, n, \tag{3.5}$$

and where $\omega_j^{(k)}$ are the weights.

It is assumed that the integrals in (3.5) are assumed that they can be evaluated either explicitly or by an efficient numerical technique. Davis and Rabinowitz [23] prove that the replacement of the function $f(x)$ by interpolating polynomials is equivalent to the choice of the weights $\omega_j^{(k)}$, $j = 0, 1, \dots, n$ in (3.4) such that the rule (3.4) is exact when f is any polynomial of degree less than or equal to n.

3.1. Product integration with Gaussian abscissae and weights. Interpolating elements $\phi_j(x)$, $j = 0, 1, \dots, n$ for production integration with Gaussian abscissae and weights can be obtained as

$$\phi_j(x) = \omega_j \sum_{m=0}^{n} \frac{2m+1}{2} P_m(x_j)P_m(x), \tag{3.6}$$

where ω_j, $0 \le j \le n$, are the $(n+1)$-point Gauss-Legendre weights, and x_j, $0 \le j \le n$, are the zeros of the Legendre polynomial of degree $n+1$, P_{n+1}.

Substituting $f(y)$ in the integral in (1.1) from (3.3) and collocating at the points x_i, we obtain

$$f(x_i) - \sum_{j=0}^{n} f(x_j)\lambda \int_{-1}^{1} k(x_i, y)\phi_j(y)dy = g(x_i),\ i = 0, 1, \dots, n. \tag{3.7}$$

If we define

$$A_{ij} = \int_{-1}^{1} k(x_i, y)\phi_j(y)dy, \tag{3.8}$$

then the equation (3.7) can be written as the $(n+1) \times (n+1)$ linear system

$$(I - \lambda A)f_n = g_n, \tag{3.9}$$

where $f_n = (f(x_0), f(x_1), \dots, f(x_n))^T$, $g_n = (g(x_0), g(x_1), \dots, g(x_n))^T$ and A is the matrix whose $(i, j)^{th}$ element is given by (3.8).

Substituting (3.6) into (3.8), gives

$$A_{ij} = \omega_j \sum_{m=0}^{n} \frac{2m+1}{2} P_m(x_j)a_m(x_i), \tag{3.10}$$

where $a_m(x_i),\ 0 \le m,\ i \le n$ are defined by

$$a_m(x_i) = \int_{-1}^{1} k(x_i, y) P_m(y) dy, \tag{3.11}$$

which are given by the recurrence relation described in Baker [6].

3.2. Product integration with Curtis-Clenshaw points. Interpolating elements $\phi_j(x)$, $j = 0, 1, \ldots, n$ for production integration with Curtis-Clenshaw points can be obtained as

$$\phi_j(x) = \frac{2\gamma_j}{n} \sum_{j=0}^{n} \gamma_i T_i(x_j) T_i(x), \tag{3.12}$$

where $T_i(x)$ is the Chebyshev polynomial of the first kind defined by

$$T_i(\cos(\theta)) = \cos(i\theta), \tag{3.13}$$

and

$$\gamma_i = \begin{cases} 1/2, & i = 0 \text{ or } i = n, \\ 1, & i = 1, 2, \ldots, n-1, \end{cases} \tag{3.14}$$

and

$$x_i = \cos\left(\frac{i\pi}{n}\right),\ i = 0, 1, \ldots, n. \tag{3.15}$$

As in Section 3.1 the solution of (1.1) can be reduced to the system (3.9), with

$$A_{ij} = \frac{2\gamma_j}{n} \sum_{m=0}^{n} \gamma_m T_m(x_j) a_m(x_i), \tag{3.16}$$

where $a_m(x_i),\ 0 \le m,\ i \le n$ are defined by

$$a_m(x_i) = \int_{-1}^{1} k(x_i, y) T_m(y) dy, \tag{3.17}$$

which are given by recurrence relation described in Baker [6].

4. Smoothing the Equation

Introducing the change $x = w(t)$ into (1.1), where $w(t)$ is the Hermite, Kress or modified Kress transformation as in equations (2.6), (2.7), and (2.13) respectively, we get

$$f(w(t)) - \lambda \int_{-1}^{1} k(w(t), y) f(y) dy = g(w(t)), \quad -1 \le w(t) \le 1. \tag{4.1}$$

Setting $y = w(s)$ in (4.1), we obtain

$$f(w(t)) - \lambda \int_{-1}^{1} k(w(t), w(s)) f(w(s)) w'(s) ds = g(w(t)),\ -1 \le w(t) \le 1, \tag{4.2}$$

where $-1 = w^{-1}(-1) \le t \le w^{-1}(1) = 1$.

Multiplying both sides of (4.2) by $w'(t)$ and setting

$$\theta(t) = w'(t) f(w(t)), \quad \xi(t) = g(w(t)) w'(t), \tag{4.3}$$

we obtain

$$\theta(t) - \lambda \int_{-1}^{1} k(w(t), w(s)) \theta(s) w'(t) ds = \xi(t), \quad -1 \le w(t) \le 1. \tag{4.4}$$

For the case of Abel kernel, we set

$$\delta_\alpha(t,s) = \begin{cases} \left| \dfrac{w(t) - w(s)}{t - s} \right|^{-\alpha} w'(t), & t \neq s, \\ |w'(t)|^{-\alpha} w'(t), & t = s, \end{cases} \tag{4.5}$$

and rewrite (4.4) as

$$\theta(t) - \lambda \int_{-1}^{1} \delta_\alpha(t,s) |t - s|^{-\alpha} \theta(s) ds = \xi(t). \tag{4.6}$$

Using product integration method described in previous section, the solution of (4.6) can be converted to the solution of the following system

$$(I - \lambda A)\theta_n = \xi_n, \tag{4.7}$$

where $\theta_n = (\theta(x_0), \theta(x_1), \ldots, \theta(x_n))^T$, $\xi_n = (\xi(x_0), \xi(x_1), \ldots, \xi(x_n))^T$ and $A = (a_{ij})_{(n+1)\times(n+1)}$ is the matrix whose $(i,j)^{\text{th}}$ element is given by

$$a_{ij} = \delta_\alpha(x_i, x_j) \int_{-1}^{1} |x_i - s|^{-\alpha} \phi_j(s) ds. \tag{4.8}$$

The integral in (4.8) will be calculated as described in previous section.

For the case of logarithmic kernel, we set

$$\delta(t,s) = \begin{cases} \ln \left| \dfrac{w(t) - w(s)}{t - s} \right| w'(t), & t \neq s, \\ \ln |w'(t)| w'(t), & t = s. \end{cases} \tag{4.9}$$

Now, we know that

$$\begin{aligned}\ln\Big|w(t)-w(s)\Big| &= \ln\left|\frac{w(t)-w(s)}{t-s}(t-s)\right| \\ &= \ln\left|\frac{w(t)-w(s)}{t-s}\right| + \ln|t-s|.\end{aligned} \tag{4.10}$$

From (4.10) and (4.9) we can rewrite (4.4) as

$$\theta(t)-\lambda\left(\int_{-1}^{1}\delta(t,s)\theta(s)ds+\int_{-1}^{1}w'(t)\ln|t-s|\theta(s)ds\right)=\xi(t). \tag{4.11}$$

Using product integration method described in previous section, the solution of (4.11) can be converted to the solution of the following system

$$(I-\lambda A)\theta_n=\xi_n, \tag{4.12}$$

where $\theta_n=(\theta(x_0),\theta(x_1),\ldots,\theta(x_n))^T$, $\xi_n=(\xi(x_0),\xi(x_1),\ldots,\xi(x_n))^T$ and $A=(a_{ij})_{(n+1)\times(n+1)}$ is the matrix whose $(i,j)^{\text{th}}$ element is given by

$$a_{ij}=\delta(x_i,x_j)\int_{-1}^{1}\phi_j(s)ds+w'(x_i)\int_{-1}^{1}\ln|x_i-s|\phi_j(s)ds. \tag{4.13}$$

The integrals in (4.13) will be calculated as described in previous section.

5. Numerical Examples

We solve the equations (4.6) and (4.11) with $\lambda=\frac{1}{\pi}$, and $\alpha=\frac{1}{2}$.

We will use the following abbreviations: GM for Gauss method, CM for Clenshaw method, HT for the Hermite transformation, KT for the Kress transformation, and MKT for the modified Kress transformation.

Example 5.1. In this example we explain how the previous methods work. Solve (4.6) using GM with KT ($p=2$), recall that KT is a special case of MKT, that is when the input function $g=g(x)$ has no singularties, i.e. when $M=0$. In this example we make a comparison between the exact solution and the approximate solution of the original equation (1.1) and the transformed equation (4.6). Consider the kernel (1.2) with $\lambda=\frac{1}{\pi}$ and $\alpha=\frac{1}{2}$. Suppose that the exact solution is $f(x)=x^3$. Firstly, we determine the input function $g(x)$ as shown below.

Substituting $f(x)=x^3$ into (1.1) provides

$$g(x)=x^3-\frac{1}{\pi}I(x),$$

where
$$I(x)=\int_{-1}^{1}|x-y|^{-\frac{1}{2}}y^3dy.$$

Now
$$I(x)=I_1(x)+I_2(x),$$

where
$$\begin{cases} I_1(x)=\displaystyle\int_{-1}^{x}(x-y)^{-\frac{1}{2}}y^3dy,\\ I_2(x)=\displaystyle\int_{x}^{1}(y-x)^{-\frac{1}{2}}y^3dy.\end{cases}$$

For calculating $I_1(x)$ let $x-y=u^2$, we obtain
$$\begin{aligned}I_1(x)=&\int_0^{(1+x)^{\frac{1}{2}}}(x-u^2)^3du\\ =&2\big[-\frac{1}{7}x_1^7+\frac{3}{5}xx_1^5-x^2x_1^3+x^3x_1\big],\end{aligned}$$

where $x_1=(1+x)^{\frac{1}{2}}$. Using the same way we obtain
$$I_2(x)=2\big[\frac{1}{7}x_2^7+\frac{3}{5}xx_2^5x^2x_2^3+x^3x_2\big],$$

where $x_2=(1-x)^{\frac{1}{2}}$.

Then
$$g(x)=x^3-\frac{2}{\pi}\Big[\big[-\frac{1}{7}x_1^7+\frac{3}{5}xx_1^5-x^2x_1^3+x^3x_1\big]+\big[\frac{1}{7}x_2^7+\frac{3}{5}xx_2^5x^2x_2^3+x^3x_2\big]\Big],$$

where $x_1=(1+x)^{\frac{1}{2}}$ and $x_2=(1-x)^{\frac{1}{2}}$.

The solution of (4.6) refers to the solution of the linear algebraic system (4.7). For $n=4$, the system is
$$\Big(I-\frac{1}{\pi}A\Big)\theta_4=\xi_4,$$

where I is identity matrix of degree 5,
$$\theta_4=\begin{pmatrix}\theta(x_0)\\ \theta(x_1)\\ \theta(x_1)\\ \theta(x_3)\\ \theta(x_4)\end{pmatrix}=\begin{pmatrix}w'(x_0)f(w(x_0))\\ w'(x_1)f(w(x_1))\\ w'(x_2)f(w(x_2))\\ w'(x_3)f(w(x_3))\\ w'(x_4)f(w(x_4))\end{pmatrix}$$

is the approximate solution vector,

$$\xi_4 = \begin{pmatrix} \xi(x_0) \\ \xi(x_1) \\ \xi(x_1) \\ \xi(x_3) \\ \xi(x_4) \end{pmatrix} = \begin{pmatrix} w'(x_0)g(w(x_0)) \\ w'(x_1)g(w(x_1)) \\ w'(x_2)g(w(x_2)) \\ w'(x_3)g(w(x_3)) \\ w'(x_4)g(w(x_4)) \end{pmatrix}$$

is the input vector which is calculated from (4.3) and $A = (a_{ij})$ is the matrix of degree 5 which can be calculated from (4.8). The vector $x = \big(x_0, x_1, x_2, x_3, x_4\big)$ includes the zeros of Legendre polynomial of degree 5, $P_5(x)$, w is the Kress transformation (2.7), and w' is its first derivative (2.8).

Now, for $n = 4$, and $p = 2$, we obtain

$$\begin{aligned} x &= (-9.0618(-01), -5.3847(-01), 0, +5.3847(-01), +9.0618(-01)), \\ w &= (-9.9517(-01), -8.3487(-01), 0, +8.3487(-01), +9.9517(-01)), \\ w' &= (1.0784(-01), 8.5344(-01), 2.0000(+00), 8.5344(-01), 1.0784(-01)), \end{aligned}$$

and

$$A = \begin{pmatrix} 4.516(-01) & 1.370(-01) & 5.968(-02) & 3.884(-02) & 1.791(-02) \\ 5.333(-01) & 1.807(+00) & 5.623(-01) & 3.086(-01) & 1.512(-01) \\ 4.648(-01) & 1.107(+00) & 3.017(+00) & 1.107(+00) & 4.648(-01) \\ 1.512(-01) & 3.086(-01) & 5.623(-01) & 1.807(+00) & 5.333(-01) \\ 1.791(-02) & 3.884(-02) & 5.968(-02) & 1.370(-01) & 4.516(-01) \end{pmatrix}.$$

The approximate solution is

$$\theta_4 = (-1.0849(-01), -5.2413(-01), -2.4748(-16), 5.2413(-01), 1.0849(-01))^T$$

while the exact solution is

$$\theta = (-1.0629(-01), -4.9663(-01), 0, 4.9663(-01), 1.0629(-01))^T.$$

Finally, we found that $\|\theta - \theta_4\|_\infty = 2.7503(-02)$. In the next examples as long as the dimension of the solution vector is greater or equal to $n = 64$ the focus will be on the infinity error norm as a measure of the efficiency of our method.

Example 5.2. Solve (4.6) using GM with KT ($p = 2, 3$), exact solution is $f(x) = x^3$ ($g(x)$ as in Example 5.1), as shown in Table 1.

TABLE 1. Error norm of Example 5.2.

n	$\|\theta - \theta_n\|_\infty$	
	$p = 2$	$p = 3$
64	6.6385(−09)	6.8719(−12)
128	4.3428(−10)	5.8155(−14)
256	2.7777(−11)	6.3421(−15)

Example 5.3. Solve (4.6) using CM with HT $((\alpha_0 = \alpha_1 = 2), (\alpha_0 = \alpha_1 = 3))$, exact solution $f(x) = x^3$ ($g(x)$ as in Example 5.1), as shown in Table 2.

TABLE 2. Error norm of Example 5.3.

n	$\|\theta - \theta_n\|_\infty$	
	$\alpha_0 = \alpha_1 = 2$	$\alpha_0 = \alpha_1 = 3$
64	3.1397(−08)	2.5603(−11)
128	1.9595(−09)	2.0065(−13)
256	1.2243(−10)	2.4280(−14)

Example 5.4. Solve (4.11) using CM with KT ($p = 2, 3$), MKT ($p = 3$, $M = 1$), $g(x) = |x|$, consider the solution as $n = 256$ as a reference, as shown in Table 3.

TABLE 3. The values $|\theta_{256}(t) - \theta_n(t)|$ of Example 5.4.

t	KT($p = 2$)		KT($p = 3$)		MKT($p = 3$)	
	$\theta_{256}(t)$	$n = 128$	$\theta_{256}(t)$	$n = 128$	$\theta_{256}(t)$	$n = 128$
0.1	0.175900	2.8483(−04)	0.178485	2.8337(−04)	−0.012670	1.2624(−07)
0.2	0.427608	5.6714(−04)	0.447955	5.6523(−04)	−0.027860	1.6175(−07)
0.3	0.576496	9.5689(−04)	0.627090	9.5584(−04)	0.061948	1.8510(−07)
0.4	0.622291	7.7464(−04)	0.693030	7.7502(−04)	0.326010	1.0380(−07)
0.5	0.581377	7.1775(−04)	0.638539	7.1624(−04)	0.669071	1.0653(−07)
0.6	0.482421	1.9204(−04)	0.490363	1.9311(−04)	0.921528	4.8837(−08)
0.7	0.354348	3.1647(−04)	0.303164	3.1578(−04)	0.890582	6.1989(−08)
0.8	0.221197	1.0949(−04)	0.137306	1.0940(−04)	0.531261	2.8196(−08)
0.9	0.099896	1.3693(−04)	0.032957	1.3700(−04)	0.138940	2.7981(−08)

Example 5.5. Solve (4.11) using CM with HT $((\alpha_0 = \alpha_1 = 3), (\alpha_0 = \alpha_2 = 4, \alpha_1 = 9))$, $g(x) = |x|$, consider the solution as $n = 256$ as a reference, as shown in Table 4.

Example 5.6. Solve (4.6) using GM with HT $((\alpha_0 = \alpha_2 = 4, \alpha_1 = 9), (\alpha_0 = \alpha_1 = 3))$, $g(x) = \frac{x}{\sqrt{2-x}}$, consider the solution as $n = 256$ as a reference, as shown in Table 5.

TABLE 4. The values $|\theta_{256}(t) - \theta_n(t)|$ of Example 5.5.

t	$\alpha_0 = \alpha_1 = 3$			$\alpha_0 = \alpha_2 = 4,\ \alpha_1 = 9$		
	$\theta_{256}(t)$	$n = 64$	$n = 128$	$\theta_{256}(t)$	$n = 64$	$n = 128$
0.1	0.148755	3.6131(−04)	2.5165(−04)	−0.000001	8.3933(−11)	3.0582(−13)
0.2	0.389319	1.1967(−03)	4.9594(−04)	−0.000217	1.9900(−10)	4.7585(−13)
0.3	0.557379	2.1609(−03)	8.3815(−04)	−0.003656	3.1068(−10)	5.6272(−13)
0.4	0.640252	2.8805(−03)	6.7877(−04)	−0.019282	3.8254(−10)	2.9490(−13)
0.5	0.631078	1.7923(−03)	6.3234(−04)	−0.015020	2.8303(−10)	2.8337(−13)
0.6	0.537067	7.1180(−04)	1.6870(−04)	0.240863	8.8108(−11)	1.5016(−13)
0.7	0.381112	7.5315(−04)	2.7819(−04)	1.012056	1.2179(−10)	2.0184(−13)
0.8	0.203849	3.9378(−04)	9.6241(−05)	1.594829	5.1527(−11)	7.3719(−14)
0.9	0.058788	3.8316(−04)	1.2049(−04)	0.702266	5.7386(−11)	8.5820(−14)

TABLE 5. The values $|\theta_{256}(t) - \theta_n(t)|$ of Example 5.6.

t	$\alpha_0 = \alpha_2 = 4,\ \alpha_1 = 9$			$\alpha_0 = \alpha_1 = 3$		
	$\theta_{256}(t)$	$n = 64$	$n = 128$	$\theta_{256}(t)$	$n = 64$	$n = 128$
0.1	−0.000006	7.7574(−10)	2.5102(−12)	−0.138073	1.7839(−13)	1.5821(−14)
0.2	−0.001109	2.1302(−09)	2.2588(−12)	0.487376	3.6326(−13)	2.4980(−15)
0.3	−0.018904	1.0975(−09)	2.8227(−12)	0.955954	5.9974(−13)	7.7716(−16)
0.4	−0.111977	1.9168(−09)	2.9243(−13)	1.190959	7.2098(−13)	1.1102(−15)
0.5	−0.282775	6.4345(−10)	1.5307(−12)	1.177626	1.4744(−13)	7.5495(−15)
0.6	0.003712	8.8822(−10)	2.4633(−12)	0.963847	1.2749(−12)	4.5519(−15)
0.7	1.666101	1.5513(−09)	1.1520(−12)	0.642312	7.0266(−13)	5.6621(−15)
0.8	2.974454	4.5732(−10)	1.9993(−12)	0.319919	2.1594(−12)	6.2728(−15)
0.9	1.126803	1.0282(−09)	1.0045(−12)	0.086585	4.5817(−12)	6.5226(−16)

Example 5.7. Solve (4.6) using GM with KT ($p = 2, 3$), $g(x) = \frac{x}{\sqrt{2-x}}$, consider the solution as $n = 256$ as a reference, as shown in Table 6.

TABLE 6. The values $|\theta_{256}(t) - \theta_n(t)|$ of Example 5.7.

t	$p = 2$			$p = 3$		
	$\theta_{256}(t)$	$n = 64$	$n = 128$	$\theta_{256}(t)$	$n = 64$	$n = 128$
0.1	−0.101082	2.6118(−12)	2.2884(−14)	−0.099675	1.8499(−14)	3.0420(−14)
0.2	0.570833	5.9392(−12)	9.5812(−14)	0.605571	8.6042(−14)	1.5987(−14)
0.3	1.007205	7.7636(−12)	1.3411(−13)	1.105950	1.7031(−13)	5.3291(−15)
0.4	1.161463	8.5625(−12)	1.8363(−13)	1.297594	2.1028(−13)	3.7748(−15)
0.5	1.083235	1.0510(−11)	2.2671(−13)	1.179647	4.3077(−14)	4.4409(−15)
0.6	0.866899	1.4481(−11)	1.2468(−13)	0.860702	3.9790(−13)	6.2172(−15)
0.7	0.603024	6.6781(−12)	1.6043(−13)	0.497348	2.1794(−13)	3.7192(−15)
0.8	0.353775	1.5243(−11)	6.0285(−14)	0.210773	6.5448(−13)	4.3854(−15)
0.9	0.150129	2.3390(−11)	8.6597(−15)	0.048046	1.4015(−12)	2.9976(−15)

Example 5.8. Solve (4.11) using GM with KT ($p = 2, 3$), HT (($\alpha_0 = \alpha_1 = 2$), exact solution $f(x) = 1$, input function $g(x) = 1 - \frac{2}{\pi}[(1-x)^{\frac{1}{2}} + (1+x)^{\frac{1}{2}}]$, as shown in Table 7.

TABLE 7. Error norm of Example 5.8.

n	$\Vert\theta - \theta_n\Vert_\infty$		
	KT($p = 2$)	KT($p = 3$)	H($\alpha_0 = \alpha_1 = 2$)
64	3.6821(−12)	1.1102(−15)	3.3217(−11)
128	6.1696(−14)	1.9984(−15)	5.5560(−13)
256	5.1070(−15)	8.4377(−15)	8.9868(−15)

Example 5.9. Solve (4.11) using CM with KT ($p = 2, 3$), HT($\alpha_0 = \alpha_1 = 3$), $g(x) = x$, consider the solution as $n = 256$ as a reference, as shown in Table 8.

TABLE 8. The values $|\theta_{256}(t) - \theta_n(t)|$ of Example 5.9.

t	KT($p = 2$)		KT($p = 3$)		H($\alpha_0 = \alpha_1 = 3$)	
	$\theta_{256}(t)$	$n = 128$	$\theta_{256}(t)$	$n = 128$	$\theta_{256}(t)$	$n = 128$
0.1	0.229693	4.5991(−14)	0.232509	8.0214(−15)	0.204387	8.2989(−15)
0.2	0.416001	1.0081(−13)	0.434700	8.3267(−16)	0.383401	1.2212(−15)
0.3	0.529295	1.6098(−13)	0.574463	1.6653(−15)	0.513913	2.1094(−15)
0.4	0.560770	1.1535(−13)	0.623500	1.9984(−15)	0.577782	0
0.5	0.520730	1.6065(−13)	0.571301	5.4401(−15)	0.565390	6.1062(−15)
0.6	0.430972	1.2618(−13)	0.437798	4.2744(−15)	0.479744	2.7756(−15)
0.7	0.316094	2.0528(−13)	0.270348	1.8874(−15)	0.339945	6.9389(−15)
0.8	0.197201	1.6201(−13)	0.122372	1.3878(−16)	0.181726	5.6899(−15)
0.9	0.089044	3.2971(−13)	0.029357	6.5260(−15)	0.052393	2.4425(−15)

In these examples which involve solving equation (1.1), we used three transformations; Hermite (2.6), Kress (2.7), and modified Kress (2.13), to reduce (1.1) to an equivalent equation (4.4) which has smoother solution; then we solved the new equation using two methods; Gauss, Clenshaw methods.

We considered two cases for comparison efficiency of the transformations; firstly, for the case in which the input function is smooth on whole domain of integration; secondly, for the case when it has finite jumps of singular points.

We investigated the Kress and Hermit transformations with known exact solution (exact solution is $f(x) = x^3$ or $f(x) = 1$) as shown in Tables 1, 2, and 7. In the Kress transformation we get various grades of accuracy by various values of the parameter as well as in the Hermite transformation, so one can obtain same accuracy by suitable choice of the parameters.

For input function $g(x)$ which is smooth on whole domain of integration, and suitably chose parameters, there is no difference between the Hermite and kress transformations; refer to Tables 5, 6, and 8.

Using input function $g(x)$ which has finite singularities in the domain of integration (for example $g(x) = |x|$), the Hermite and Kress transformations give poor accuracy in spite of taking various values of the parameters; refer to Table 3 columns KT($p = 2$) and KT($p = 3$) for Kress, and Table 4 column $\alpha_0 = \alpha_1 = 3$ for Hermite. The reason is the input function $g(x) = |x|$ has singular point $x_1 = 0$. To overcome the problem we divide the domain of integration, $[-1, 1]$, into two subintervals $[-1, 0]$ and $[0, 1]$, i.e., choosing $M = 1$. For this case the accuracy can be obtained clearly in Table 4 column $\alpha_0 = \alpha_2 = 4$, $\alpha_1 = 9$ for Hermite, and some accuracy appear in Table 3 column MKT($p = 3$) for the modified Kress transformation.

In the case of $g(x) = |x|$, we find that the Hermite transformation gives the best accuracy compared to the modified Kress transformations; refer to Table 4 for the Hermite transformation and Table 3 for the modified Kress transformations. This is because the Hermite transformation gives more smoothing of the solution since the transformation vanishes up to eight derivatives at the singular point $x_1 = 0$ which is related to the choice $\alpha_1 = 9$, while the modified Kress transformation vanishes up to only two derivatives at the same singular point which is related to $p = 3$. Choosing $p > 3$ gives unsolvable system since the concentration of the nodes near the singular points is so high, increasing as n becomes large. Another reason is that the modified Kress transformation is rational compared to the polynomial nature of the Hermite transformation so that the calculations become more complicated.

6. Conclusion

It could be shown from the results of this study that for the case in which the input function of the weakly singular Fredholm integral equation of the second kind is smooth on whole the domain of integration, the product integration methods together with the Kress transformation showed comparable results as the product integration methods together with the Hermite transformation. For the case in which the input function has finite number of singularities on the domain of integration, the product integration methods together with the modified Kress transformation showed higher accuracy than the product integration methods together with the Kress transformation. Nevertheless the most accurate results were showed by the product integration methods together with the Hermite transformation.

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