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Research Article

Pseudo-differential operators associated with the gyrator transform

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Abstract

In this present work, a brief introduction to the gyrator transform and its fundamental properties are given. The gyrator transform of tempered distributions is being discussed. This article made further discussion on the boundedness properties of pseudo-differential operators associated with the gyrator transform on Schwartz space as well as on Sobolev space.

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1. Introduction

The Fourier transform is one of the oldest and frequently used integral transform because of its wide range of applications in applied mathematics, physics, signal processing, image processing, and engineering. The fractional Fourier transform, which is a generalization of the Fourier transform, has received great importance in the last 30 years or so because of its wide applications in optics, signal recovery, and image analysis [7, 9]. It has been also applied in many branches of pure mathematics such as wavelets [2, 14], pseudo-differential operator [10, 11].

Recall the classical Fourier transform of a function f is denoted by $\hat{f}$, is defined as

$$\hat{f}(w) = \mathcal{F}(f)(w) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(t) e^{-i\langle t,w\rangle} dt, \quad w \in \mathbb{R}^n, \tag{1.1}$$

so that its inverse is given by

$$f(t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{f}(w) e^{i\langle t,w\rangle} dw, \quad t \in \mathbb{R}^n, \tag{1.2}$$

provided the integrals exist.

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The fractional Fourier transform $\mathcal{F}^\alpha$ of a function $f(t)$, is defined as [1, 7–9, 14],

$$(\mathcal{F}^\alpha f)(w) = \int_{\mathbb{R}} K^\alpha(t,w) f(t)dt, \quad w \in \mathbb{R}, \tag{1.3}$$

where the kernel $K^\alpha(t,w)$ is given by

$$K^\alpha(t,w) = \begin{cases} \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{\frac{i}{2}(t^2+w^2)\cot\alpha - itw\csc\alpha}, & \alpha \neq n\pi, \\ \delta(t-w), & \alpha = 2n\pi, \\ \delta(t+w), & \alpha = (2n-1)\pi, n \in \mathbb{Z}. \end{cases} \tag{1.4}$$

Here and in what follows we always assume that $w, w_1, w_2, t, t_1, t_2, x_1, x_2 \in \mathbb{R}$.

The inverse fractional Fourier transform of (1.3) with respect to angle α is the fractional Fourier transform with angle $-\alpha$, is given by

$$f(t) = \int_{\mathbb{R}} (\mathcal{F}^\alpha f)(w) K^{-\alpha}(t,w)dw. \tag{1.5}$$

The $n-$dimensional fractional Fourier transform is defined by taking the tensor product of n copies of the one dimensional fractional Fourier transform [9, p. 175]. In particular, in two dimensions

$$\begin{aligned} &(\mathcal{F}^{\alpha_1,\alpha_2} f(t_1,t_2))(w_1,w_2) \\ = &\int_{\mathbb{R}^2} K^{\alpha_1}(t_1,w_1)K^{\alpha_2}(t_2,w_2)f(t_1,t_2)dt_1dt_2, \end{aligned} \tag{1.6}$$

where $K^{\alpha_1}(t_1,w_1)$ and $K^{\alpha_2}(t_2,w_2)$ are the kernels of the one dimensional fractional Fourier transform given by (1.4).

Pseudo-differential operators are a generalizations of linear partial differential operators. Pseudo-differential operators acts as an pivotal role in the area of partial differential equations. Various properties of pseudo-differential operators are investigated by Wong [18], Zaidman [19], Hörmander [4], Friedman [3], and Rodino [15] by exploiting the theory of Fourier transform. Later on a series of papers [10–13] is devoted to the study of pseudo-differential operators involving the fractional Fourier transform.

In 2000, Simon [17] introduced the gyrator transform named as 'cross gyrator' and studied its basic properties. Moreover, [5, 16] discussed the gyrator transform in detail and obtained certain fruitful results. Throughout this article, we assume that $\alpha \neq n\pi, n \in \mathbb{Z}$. The gyrator transform of $f(t_1,t_2)$ is defined by

$$(\mathcal{R}^\alpha f)(w_1,w_2) = \int_{\mathbb{R}^2} G_\alpha(t_1,t_2,w_1,w_2)f(t_1,t_2)dt_1dt_2, \tag{1.7}$$

where the kernel $G_\alpha(t_1,t_2,w_1,w_2)$ is defined by

$$\begin{aligned} &G_\alpha(t_1,t_2,w_1,w_2) \\ = &\frac{1}{2\pi|\sin\alpha|}\exp\left[i(t_1t_2+w_1w_2)\cot\alpha - i(t_2w_1+t_1w_2)\csc\alpha\right]. \end{aligned} \tag{1.8}$$

The inverse of (1.7) is given by [5, 16, 17]

$$f(t_1,t_2) = \int_{\mathbb{R}^2} \overline{G_\alpha(t_1,t_2,w_1,w_2)}(\mathcal{R}^\alpha f)(w_1,w_2)dw_1dw_2. \tag{1.9}$$

This present study is motivated by the work of [5, 16, 17] and [10–13, 18]. We have investigated the mapping properties of pseudo-differential operators involving the gyrator transform and derived certain boundedness results.

2. The gyrator transform

In this section we recall the definition of Schwartz space $S(\mathbb{R}^2)$ and derive certain important properties of the gyrator transform.

Definition 2.1. An infinitely differentiable complex valued function $f(t_1, t_2)$ on $\mathbb{R}^2$ is said to be a member of $S(\mathbb{R}^2)$ if and only if for every choice of m, n, p, and $q \in \mathbb{N}_0$, it satisfies

$$\sup_{(t_1,t_2)\in\mathbb{R}^2} \left| t_1^m t_2^n \frac{\partial^p}{\partial t_1^p} \frac{\partial^q}{\partial t_2^q} f(t_1, t_2) \right| < \infty. \tag{2.1}$$

Remark 2.2. If $\phi(t_1, t_2)$ is of polynomial growth and locally integrable function on $\mathbb{R}^2$, then ϕ generates a distribution in $S'(\mathbb{R}^2)$ as follows:

$$\langle \phi, f \rangle = \int_{\mathbb{R}^2} \phi(t_1, t_2) f(t_1, t_2) dt_1 dt_2, \text{ for all } \; f \in S(\mathbb{R}^2).$$

Definition 2.3. The space $S_\alpha(\mathbb{R}^2)$ is defined as the collection of all complex valued smooth functions $f(t_1, t_2) \in \mathbb{R}^2$ which for every m, n, p and q of non-negative integers it satisfies

$$\sup_{(t_1,t_2)\in\mathbb{R}^2} \left| t_1^m t_2^n \Delta^{p,q}_{t_1,t_2,\alpha} f(t_1, t_2) \right| < \infty, \tag{2.2}$$

where $\Delta^{p,q}_{t_1,t_2,\alpha} = \left(\frac{\partial}{\partial t_1} - it_2 \cot\alpha\right)^p \left(\frac{\partial}{\partial t_2} - it_1 \cot\alpha\right)^q$.

Proposition 2.4. *Let $G_\alpha(t_1, t_2, w_1, w_2)$ be the kernel of the gyrator transform and $\Delta^{p,q}_{t_1,t_2,\alpha}$ $= \left(\frac{\partial}{\partial t_1} - it_2 \cot\alpha\right)^p \left(\frac{\partial}{\partial t_2} - it_1 \cot\alpha\right)^q$. Then, we have*

$$\Delta^{p,q}_{t_1,t_2,\alpha} G_\alpha(t_1, t_2, w_1, w_2) = (-i \csc\alpha)^{(p+q)} w_1^q w_2^p G_\alpha(t_1, t_2, w_1, w_2).$$

Proof. See [6, Proposition 2.3(i)]. □

Proposition 2.5. *Let $f(t_1, t_2) \in S(\mathbb{R}^2)$, and let r, p, and q be any non-negative integers. Then we have*

(i) $\int_{\mathbb{R}^2} \Delta^{p,q}_{t_1,t_2,\alpha} G_\alpha(t_1, t_2, w_1, w_2) f(t_1, t_2) dt_1 dt_2$
$= \int_{\mathbb{R}^2} G_\alpha(t_1, t_2, w_1, w_2) \Delta^{*\,p,q}_{t_1,t_2,\alpha} f(t_1, t_2) dt_1 dt_2$,

(ii) $\left(\mathcal{R}^\alpha \Delta^{*\,p,q}_{t_1,t_2,\alpha} f(t_1, t_2)\right)(w_1, w_2)$
$= (-i \csc\alpha)^{(p+q)}\, w_1^q w_2^p \left(\mathcal{R}^\alpha f(t_1, t_2)\right)(w_1, w_2)$,

(iii) $\Delta^{p,q}_{w_1,w_2,\alpha} \left(\mathcal{R}^\alpha f(t_1, t_2)\right)(w_1, w_2) = (-i \csc\alpha)^{(p+q)} \left(\mathcal{R}^\alpha t_1^q t_2^p f(t_1, t_2)\right)(w_1, w_2)$,

(iv) $\mathcal{R}^\alpha : S(\mathbb{R}^2) \to S_\alpha(\mathbb{R}^2)$, *is linear and continuous mapping,*

where $\Delta^{*\,p,q}_{t_1,t_2,\alpha} = (-1)^{p+q} \left(\frac{\partial}{\partial t_1} + it_2 \cot\alpha\right)^p \left(\frac{\partial}{\partial t_2} + it_1 \cot\alpha\right)^q$.

Proof. (i) See [6, Proposition 2.4].
(ii) See [6, Theorem 2.5(i)].
(iii) We know that,

$$\begin{aligned}
&\Delta^{p,q}_{w_1,w_2,\alpha} \left(\mathcal{R}^\alpha f(t_1, t_2)\right)(w_1, w_2) \\
&= \int_{\mathbb{R}^2} \Delta^{p,q}_{w_1,w_2,\alpha} G_\alpha(t_1, t_2, w_1, w_2) f(t_1, t_2) dt_1 dt_2 \\
&= (-i \csc\alpha)^{(p+q)} \int_{\mathbb{R}^2} t_1^q t_2^p G_\alpha(t_1, t_2, w_1, w_2) f(t_1, t_2) dt_1 dt_2 \\
&= (-i \csc\alpha)^{(p+q)} \left(\mathcal{R}^\alpha t_1^q t_2^p f(t_1, t_2)\right)(w_1, w_2).
\end{aligned}$$

In this way, we obtain the desired result.
(iv) Linearity of $\mathcal{R}^\alpha$ is obvious.

Let $\{f_j\}_{j\in\mathbb{N}_0}$ be a sequence taken from $S(\mathbb{R}^2)$. Consider,

$$\sup_{(w_1,w_2)\in\mathbb{R}^2} |w_1^r w_2^s \left((\Delta^{p,q}_{w_1,w_2,\alpha})\mathcal{R}^\alpha f_j(t_1,t_2)\right)(w_1,w_2)|$$
$$= \sup_{(w_1,w_2)\in\mathbb{R}^2} |w_1^r w_2^s \mathcal{R}^\alpha \left((-i\csc\alpha)^{(p+q)} t_1^q t_2^p f_j(t_1,t_2)\right)(w_1,w_2)|.$$

Since, $\{f_j\}_{j\in\mathbb{N}_0} \in S(\mathbb{R}^2)$ implies $(-i\csc\alpha)^{(p+q)} t_1^q t_2^p f_j(t_1,t_2) \in S(\mathbb{R}^2)$.
Hence, $\sup_{(w_1,w_2)\in\mathbb{R}^2} |w_1^r w_2^s \left(\Delta^{p,q}_{w_1,w_2,\alpha}\mathcal{R}^\alpha f_j(t_1,t_2)\right)(w_1,w_2)| \longrightarrow 0$, if $\{f_j\} \longrightarrow 0$ in $S(\mathbb{R}^2)$.
Which shows the continuity of $\mathcal{R}^\alpha$. □

3. The gyrator transform of tempered distribution

In this section, we obtain the relation between the gyrator transform and the two dimensional fractional Fourier transform. We notice that the results of this section can be found in [5]. However, for the sake of completeness, and since we use a different technique, we decided to present the proofs here below. Moreover, we derive the continuity of the gyrator transform in Schwartz type space.

Lemma 3.1. *If $\alpha \neq n\pi, n \in \mathbb{Z}$. Then*

$$(\mathcal{R}^\alpha f)(w_1,w_2) = \left[\mathcal{F}^{\alpha,-\alpha} f\left(\frac{x_1-x_2}{\sqrt{2}}, \frac{x_1+x_2}{\sqrt{2}}\right)\right]\left(\frac{w_1+w_2}{\sqrt{2}}, \frac{-w_1+w_2}{\sqrt{2}}\right).$$

Proof. In viewing (1.6), we have

$$\begin{aligned}
&\left[\mathcal{F}^{\alpha,-\alpha} f\Big(\frac{x_1-x_2}{\sqrt{2}}, \frac{x_1+x_2}{\sqrt{2}}\Big)\right]\Big(\frac{w_1+w_2}{\sqrt{2}}, \frac{-w_1+w_2}{\sqrt{2}}\Big)\\
&= \int_{\mathbb{R}^2} K^\alpha\Big(x_1, \frac{w_1+w_2}{\sqrt{2}}\Big) K^{-\alpha}\Big(x_2, \frac{-w_1+w_2}{\sqrt{2}}\Big) f\Big(\frac{x_1-x_2}{\sqrt{2}}, \frac{x_1+x_2}{\sqrt{2}}\Big) dx_1 dx_2\\
&= \frac{1}{2\pi|\sin\alpha|}\int_{\mathbb{R}^2} e^{\frac{i}{2}\left[x_1^2 + \frac{(w_1+w_2)^2}{2}\right]\cot\alpha - \frac{i}{\sqrt{2}}x_1(w_1+w_2)\csc\alpha}\\
&\quad\times e^{-\frac{i}{2}\left[x_2^2 + \frac{(-w_1+w_2)^2}{2}\right]\cot\alpha + \frac{i}{\sqrt{2}}x_2(-w_1+w_2)\csc\alpha} f\Big(\frac{x_1-x_2}{\sqrt{2}}, \frac{x_1+x_2}{\sqrt{2}}\Big) dx_1 dx_2\\
&= \frac{1}{2\pi|\sin\alpha|}\int_{\mathbb{R}^2} e^{\frac{i}{2}[x_1^2 - x_2^2 + 2w_1w_2]\cot\alpha} e^{\frac{i}{\sqrt{2}}[-x_1w_1 - x_1w_2 - x_2w_1 + x_2w_2]\csc\alpha}\\
&\quad\times f\Big(\frac{x_1-x_2}{\sqrt{2}}, \frac{x_1+x_2}{\sqrt{2}}\Big) dx_1 dx_2.
\end{aligned}$$

Putting $\frac{x_1-x_2}{\sqrt{2}} = t_1$ and $\frac{x_1+x_2}{\sqrt{2}} = t_2$, the above expression becomes

$$\begin{aligned}
&= \frac{1}{2\pi|\sin\alpha|}\int_{\mathbb{R}^2} e^{\frac{i}{2}\left[\frac{(t_1+t_2)^2}{2} - \frac{(-t_1+t_2)^2}{2} + 2w_1w_2\right]\cot\alpha}\\
&\quad\times e^{\frac{i}{\sqrt{2}}\left[-\frac{(t_1+t_2)}{\sqrt{2}}w_1 - \frac{(t_1+t_2)}{\sqrt{2}}w_2 - \frac{(-t_1+t_2)}{\sqrt{2}}w_1 + \frac{(-t_1+t_2)}{\sqrt{2}}w_2\right]\csc\alpha}\\
&\quad\times f(t_1,t_2)dt_1dt_2\\
&= \frac{1}{2\pi|\sin\alpha|}\int_{\mathbb{R}^2} e^{i(t_1t_2+w_1w_2)\cot\alpha - i(t_1w_2+t_2w_1)\csc\alpha} f(t_1,t_2)dt_1dt_2\\
&= (\mathcal{R}^\alpha f)(w_1,w_2).
\end{aligned}$$

Which completes the proof. □

Theorem 3.2. *The gyrator transform is a continuous linear map of $S(\mathbb{R}^2)$ onto itself.*

Proof. From (1.6), we observe that

$$\begin{aligned}
&(\mathcal{R}^\alpha f)(w_1,w_2)\\
&=\frac{1}{2\pi|\sin\alpha|}\int_{\mathbb{R}^2} e^{i(x_1x_2+w_1w_2)\cot\alpha-i(x_1w_2+x_2w_1)\csc\alpha} f(x_1,x_2)dx_1dx_2\\
&=\frac{e^{iw_1w_2\cot\alpha}}{2\pi|\sin\alpha|}\int_{\mathbb{R}^2} e^{-i(x_1w_2+x_2w_1)\csc\alpha} e^{ix_1x_2\cot\alpha} f(x_1,x_2)dx_1dx_2\\
&=\frac{e^{iw_1w_2\cot\alpha}}{2\pi|\sin\alpha|}\mathcal{F}[e^{ix_1x_2\cot\alpha}f(x_1,x_2)](w_2\csc\alpha,w_1\csc\alpha)\\
&=e^{iw_1w_2\cot\alpha}\Phi_\alpha(w_1,w_2),
\end{aligned}$$

where $\Phi_\alpha(w_1,w_2)=\frac{1}{2\pi|\sin\alpha|}\mathcal{F}[e^{ix_1x_2\cot\alpha}f(x_1,x_2)](w_2\csc\alpha,w_1\csc\alpha)$.
Now, for any $f\in S(\mathbb{R}^2)$, we need to prove that $(\mathcal{R}^\alpha f)(w_1,w_2)\in S(\mathbb{R}^2)$. For which, we have to show

$$\sup_{(w_1,w_2)\in\mathbb{R}^2}\left|w_1^m w_2^n\frac{\partial^p}{\partial w_1^p}\frac{\partial^q}{\partial w_2^q}(\mathcal{R}^\alpha f)(w_1,w_2)\right|<\infty,$$

i.e,

$$\sup_{(w_1,w_2)\in\mathbb{R}^2}\left|w_1^m w_2^n\frac{\partial^p}{\partial w_1^p}\frac{\partial^q}{\partial w_2^q}e^{iw_1w_2\cot\alpha}\Phi_\alpha(w_1,w_2)\right|<\infty,$$

for all $m,n,p,q\in\mathbb{N}_0$.
Now, consider

$$\begin{aligned}
&\sup_{(w_1,w_2)\in\mathbb{R}^2}\left|w_1^m w_2^n\frac{\partial^p}{\partial w_1^p}\frac{\partial^q}{\partial w_2^q}e^{iw_1w_2\cot\alpha}\Phi_\alpha(w_1,w_2)\right|\\
=&\sup_{(w_1,w_2)\in\mathbb{R}^2}\left|w_1^m w_2^n\frac{\partial^p}{\partial w_1^p}\sum_{\beta=0}^{q}\binom{q}{\beta}\frac{\partial^\beta}{\partial w_2^\beta}e^{iw_1w_2\cot\alpha}\frac{\partial^{q-\beta}}{\partial w_2^{q-\beta}}\Phi_\alpha(w_1,w_2)\right|\\
=&\sup_{(w_1,w_2)\in\mathbb{R}^2}\left|w_1^m w_2^n\frac{\partial^p}{\partial w_1^p}\sum_{\beta=0}^{q}\binom{q}{\beta}(iw_1\cot\alpha)^\beta e^{iw_1w_2\cot\alpha}\frac{\partial^{q-\beta}}{\partial w_2^{q-\beta}}\Phi_\alpha(w_1,w_2)\right|\\
=&\sup_{(w_1,w_2)\in\mathbb{R}^2}\Bigg|w_1^m w_2^n\sum_{\alpha=0}^{p}\binom{p}{\alpha}\sum_{\beta=0}^{q}\binom{q}{\beta}\frac{\partial^\alpha}{\partial w_1^\alpha}\{(iw_1\cot\alpha)^\beta e^{iw_1w_2\cot\alpha}\}\\
&\times\frac{\partial^{p-\alpha}}{\partial w_1^{p-\alpha}}\frac{\partial^{q-\beta}}{\partial w_2^{q-\beta}}\Phi_\alpha(w_1,w_2)\Bigg|\\
=&\sup_{(w_1,w_2)\in\mathbb{R}^2}\Bigg|w_1^m w_2^n\sum_{\alpha=0}^{p}\binom{p}{\alpha}\sum_{\beta=0}^{q}\binom{q}{\beta}(i\cot\alpha)^\beta\sum_{\alpha'=0}^{\alpha}\binom{\alpha}{\alpha'}\frac{\partial^{\alpha'}}{\partial w_1^{\alpha'}}w_1^\beta\\
&\times\frac{\partial^{\alpha-\alpha'}}{\partial w_1^{\alpha-\alpha'}}e^{iw_1w_2\cot\alpha}\frac{\partial^{p-\alpha}}{\partial w_1^{p-\alpha}}\frac{\partial^{q-\beta}}{\partial w_2^{q-\beta}}\Phi_\alpha(w_1,w_2)\Bigg|\\
=&\sup_{(w_1,w_2)\in\mathbb{R}^2}\Bigg|w_1^m w_2^n\sum_{\alpha=0}^{p}\binom{p}{\alpha}\sum_{\beta=0}^{q}\binom{q}{\beta}\sum_{\alpha'=0}^{\alpha}\binom{\alpha}{\alpha'}(i\cot\alpha)^{\alpha-\alpha'+\beta}\mathcal{A}w_1^{\beta-\alpha'}\\
&\times w_2^{\alpha-\alpha'}e^{iw_1w_2\cot\alpha}\frac{\partial^{p-\alpha}}{\partial w_1^{p-\alpha}}\frac{\partial^{q-\beta}}{\partial w_2^{q-\beta}}\Phi_\alpha(w_1,w_2)\Bigg|\\
=&\sup_{(w_1,w_2)\in\mathbb{R}^2}\Bigg|\sum_{\alpha=0}^{p}\binom{p}{\alpha}\sum_{\beta=0}^{q}\binom{q}{\beta}\sum_{\alpha'=0}^{\alpha}\binom{\alpha}{\alpha'}(i\cot\alpha)^{\alpha-\alpha'+\beta}\mathcal{A}w_1^{\beta-\alpha'+m}
\end{aligned}$$

$$
\begin{aligned}
&\times w_2^{\alpha-\alpha'+n} e^{iw_1w_2\cot\alpha}\frac{\partial^{p-\alpha}}{\partial w_1^{p-\alpha}}\frac{\partial^{q-\beta}}{\partial w_2^{q-\beta}}\Phi_\alpha(w_1,w_2)\Big|\\
\leq\ & \sum_{\alpha=0}^{p}\binom{p}{\alpha}\sum_{\beta=0}^{q}\binom{q}{\beta}\sum_{\alpha'=0}^{\alpha}\binom{\alpha}{\alpha'}(i\cot\alpha)^{\alpha-\alpha'+\beta}\mathcal{A}\\
&\times \sup_{(w_1,w_2)\in\mathbb{R}^2}\Big|w_1^{\beta-\alpha'+m}w_2^{\alpha-\alpha'+n}\frac{\partial^{p-\alpha}}{\partial w_1^{p-\alpha}}\frac{\partial^{q-\beta}}{\partial w_2^{q-\beta}}\Phi_\alpha(w_1,w_2)\Big| < \infty,
\end{aligned}
$$

as $\Phi_\alpha(w_1,w_2)\in S(\mathbb{R}^2)$. Thus, $(\mathcal{R}^\alpha f)(w_1,w_2)\in S(\mathbb{R}^2)$ for all $f\in S(\mathbb{R}^2)$.
To show the continuity. Assume that $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of functions in $S(\mathbb{R}^2)$ converging to zero in $S(\mathbb{R}^2)$ as $j\to\infty$, then exploiting the fact that the Fourier transform as well as the fractional Fourier transforms are continuous linear map on $S(\mathbb{R}^2)$, we see that $\{\mathcal{R}^\alpha f_j\}\to 0$ as $j\to\infty$.
Which shows the continuity of the gyrator transform. □

Remark 3.3. The inverse of the gyrator transform is a continuous linear map from $S(\mathbb{R}^2)$ onto itself.

Definition 3.4. The generalized gyrator transform $\mathcal{R}^\alpha\phi$ of $\phi\in S'(\mathbb{R}^2)$ is defined by

$$
\langle\mathcal{R}^\alpha\phi, f\rangle = \langle\phi,\mathcal{R}^\alpha f\rangle,\ \text{ where } f\in S(\mathbb{R}^2). \tag{3.1}
$$

The generalized inverse gyrator transform $\mathcal{R}^{-\alpha}\phi$ of $\phi\in S'(\mathbb{R}^2)$ is defined by

$$
\langle\mathcal{R}^{-\alpha}\phi, f\rangle = \langle\phi,\mathcal{R}^{-\alpha} f\rangle,\ \text{ where } f\in S(\mathbb{R}^2). \tag{3.2}
$$

Theorem 3.5. *The generalized gyrator transform is a continuous linear map from $S'(\mathbb{R}^2)$ onto itself.*

Proof. By the previous Theorem 3.2, we see that $\mathcal{R}^\alpha f\in S(\mathbb{R}^2)$, whenever $f\in S(\mathbb{R}^2)$. Hence the right side of (3.1) is well defined. Assume that $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in $S(\mathbb{R}^2)$ converges to 0, then by the continuity of the gyrator transform $\{\mathcal{R}^\alpha\phi_n\}\to 0$ as $n\to\infty$. So the right side of (3.1) converges to 0, which implies that $\{\langle\mathcal{R}^\alpha f,\phi_n\rangle\}\to 0$ as $n\to\infty$. Thus $\mathcal{R}^\alpha\phi$ is continuous on $S'(\mathbb{R}^2)$.
In the similar way we can conclude that the generalized inverse gyrator transform is also continuous on $S'(\mathbb{R}^2)$. □

Theorem 3.6. *(Parseval's identity) If $f,g\in S(\mathbb{R}^2)$, then we have*

$$
\langle(\mathcal{R}^\alpha f)(w_1,w_2),(\mathcal{R}^\alpha g)(w_1,w_2)\rangle = \langle f(w_1,w_2), g(w_1,w_2)\rangle. \tag{3.3}
$$

Proof. See [16]. □

4. Pseudo-differential operator

This present section is dedicated for deriving the boundedness properties of pseudo-differential operator associated with the gyrator transform for the symbol class $S_l^{m_1,m_2}$ on $S(\mathbb{R}^2)$.

Definition 4.1. Let $m_1,m_2\in\mathbb{R}$. We define symbol class $S_l^{m_1,m_2}$ to be the set of all functions $\nu(t_1,t_2,w_1,w_2)$ in $C^\infty(\mathbb{R}^4)$ such that for any non-negative integers s,m,n,p and q, there exists a positive constant $\mathcal{C}$ depending on s,m,n,p and q only, such that

$$
\begin{aligned}
&|(1+|t_2\cot\alpha|)^s\frac{\partial^m}{\partial t_1^m}\frac{\partial^n}{\partial t_2^n}\frac{\partial^p}{\partial w_1^p}\frac{\partial^q}{\partial w_2^q}\nu(t_1,t_2,w_1,w_2)|\\
&\leq \mathcal{C}(1+|w_1\cot\alpha|)^{m_1-|p|}(1+|w_2\cot\alpha|)^{m_2-|q|},
\end{aligned} \tag{4.1}
$$

for all $t_1,t_2,w_1,w_2\in\mathbb{R}$.

Definition 4.2. Let $\nu(t_1,t_2,w_1,w_2)$ be a symbol satisfying (4.1). Then the pseudo-differential operator $\mathcal{A}_{\nu,\alpha}$ associated with the gyrator transform is defined by

$$\begin{aligned}&\mathcal{A}_{\nu,\alpha}f(t_1,t_2)\\ =&\int_{\mathbb{R}^2}\overline{G_\alpha(t_1,t_2,w_1,w_2)}\nu(t_1,t_2,w_1,w_2)(\mathcal{R}^\alpha f)(w_1,w_2)dw_1dw_2,\end{aligned}$$

where $(\mathcal{R}^\alpha f)(w_1,w_2)$ is defined in (1.7).

Theorem 4.3. *Let $\nu(t_1,t_2,w_1,w_2)$ be a symbol taken from $S_1^{m_1,m_2}, m_1,m_2<-1$. Then $\mathcal{A}_{\nu,\alpha}$ maps the Schwartz space $S(\mathbb{R}^2)$ into itself.*

Proof. Let $f\in S(\mathbb{R}^2)$. Then for any non-negative integers m,n,p and $q\in\mathbb{N}_0$, we need to prove that

$$\sup_{t_1,t_2\in\mathbb{R}}|t_1^mt_2^n\frac{\partial^p}{\partial t_1^p}\frac{\partial^q}{\partial t_2^q}\mathcal{A}_{\nu,\alpha}f(t_1,t_2)|<\infty.$$

Let $B_\alpha=\frac{1}{2\pi|\sin\alpha|}$, then from Definition 4.2 we have

$$\begin{aligned}&t_1^mt_2^nB_\alpha\frac{\partial^p}{\partial t_1^p}\frac{\partial^q}{\partial t_2^q}\int_{\mathbb{R}^2}e^{-i(t_1t_2+w_1w_2)\cot\alpha+i(w_1t_2+w_2t_1)\csc\alpha}\nu(t_1,t_2,w_1,w_2)\\&\times(\mathcal{R}^\alpha f)(w_1,w_2)dw_1dw_2\\ =&\ t_1^mt_2^nB_\alpha\frac{\partial^p}{\partial t_1^p}\int_{\mathbb{R}^2}\sum_{\beta=0}^{q}\binom{q}{\beta}\frac{\partial^\beta}{\partial t_2^\beta}\Big(e^{-i(t_1t_2+w_1w_2)\cot\alpha+i(w_1t_2+w_2t_1)\csc\alpha}\Big)\\&\times\Big(\frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}}\nu(t_1,t_2,w_1,w_2)\Big)(\mathcal{R}^\alpha f)(w_1,w_2)dw_1dw_2\\ =&\ t_1^mt_2^nB_\alpha\frac{\partial^p}{\partial t_1^p}\int_{\mathbb{R}^2}\sum_{\beta=0}^{q}\binom{q}{\beta}\sum_{\beta'=0}^{\beta}\binom{\beta}{\beta'}\frac{\partial^{\beta'}}{\partial t_2^{\beta'}}(e^{-i(t_1t_2+w_1w_2)\cot\alpha})\\&\times\frac{\partial^{\beta-\beta'}}{\partial t_2^{\beta-\beta'}}(e^{i(w_1t_2+w_2t_1)\csc\alpha})\Big(\frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}}\nu(t_1,t_2,w_1,w_2)\Big)(\mathcal{R}^\alpha f)(w_1,w_2)dw_1dw_2\\ =&\ t_1^mt_2^nB_\alpha\frac{\partial^p}{\partial t_1^p}\int_{\mathbb{R}^2}\sum_{\beta=0}^{q}\binom{q}{\beta}\sum_{\beta'=0}^{\beta}\binom{\beta}{\beta'}(-it_1\cot\alpha)^{\beta'}e^{-i(t_1t_2+w_1w_2)\cot\alpha}\\&\times(iw_1\csc\alpha)^{\beta-\beta'}e^{i(w_1t_2+w_2t_1)\csc\alpha}\frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}}\Big(\nu(t_1,t_2,w_1,w_2)\Big)(\mathcal{R}^\alpha f)(w_1,w_2)dw_1dw_2\\ =&\ t_1^mB_\alpha\frac{\partial^p}{\partial t_1^p}\int_{\mathbb{R}^2}\sum_{\beta=0}^{q}\binom{q}{\beta}\sum_{\beta'=0}^{\beta}\binom{\beta}{\beta'}(-it_1\cot\alpha)^{\beta'}e^{-i(t_1t_2+w_1w_2)\cot\alpha}(i\csc\alpha)^{\beta-\beta'-n}\\&\times\frac{\partial^n}{\partial w_1^n}e^{i(w_1t_2+w_2t_1)\csc\alpha}\Big(\frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}}\nu(t_1,t_2,w_1,w_2)\Big){w_1}^{\beta-\beta'}(\mathcal{R}^\alpha f)(w_1,w_2)dw_1dw_2.\end{aligned}$$

As $\frac{\partial^n}{\partial w_1^n}e^{i(w_1t_2+w_2t_1)\csc\alpha}=(it_2\csc\alpha)^ne^{i(w_1t_2+w_2t_1)\csc\alpha}$ and $\langle D^nf,g\rangle=(-1)^n\langle f,D^ng\rangle$ for all $f,g\in S(\mathbb{R}^2)$.

Then, the above expression becomes

$$\begin{aligned}
&= (-1)^n t_1^m B_\alpha \frac{\partial^p}{\partial t_1^p} \int_{\mathbb{R}^2} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (-it_1 \cot\alpha)^{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \\
&\quad \times e^{i(w_1t_2+w_2t_1)\csc\alpha} \frac{\partial^n}{\partial w_1^n} \Big[\{ e^{-i(t_1t_2+w_1w_2)\cot\alpha} \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \} \\
&\quad \times \{ {w_1}^{\beta-\beta'} (\mathcal{R}^\alpha f)(w_1,w_2) \} \Big] dw_1 dw_2 \\
&= (-1)^n t_1^m B_\alpha \frac{\partial^p}{\partial t_1^p} \int_{\mathbb{R}^2} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (-it_1 \cot\alpha)^{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \\
&\quad \times e^{i(w_1t_2+w_2t_1)\csc\alpha} \sum_{\mu=0}^{n} \binom{n}{\mu} \frac{\partial^\mu}{\partial w_1^\mu} \{ e^{-i(t_1t_2+w_1w_2)\cot\alpha} \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \} \\
&\quad \times \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} \{ {w_1}^{\beta-\beta'} (\mathcal{R}^\alpha f)(w_1,w_2) \} dw_1 dw_2 \\
&= (-1)^n t_1^m B_\alpha \frac{\partial^p}{\partial t_1^p} \int_{\mathbb{R}^2} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (-it_1 \cot\alpha)^{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \\
&\quad \times e^{i(w_1t_2+w_2t_1)\csc\alpha} \sum_{\mu=0}^{n} \binom{n}{\mu} \sum_{\mu'=0}^{\mu} \binom{\mu}{\mu'} (-iw_2\cot\alpha)^{\mu'} e^{-i(t_1t_2+w_1w_2)\cot\alpha} \\
&\quad \times \frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}} \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} \{ {w_1}^{\beta-\beta'} (\mathcal{R}^\alpha f)(w_1,w_2) \} dw_1 dw_2 \\
&= (-1)^n B_\alpha t_1^m \int_{\mathbb{R}^2} \sum_{\gamma=0}^{p} \binom{p}{\gamma} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \frac{\partial^\gamma}{\partial t_1^\gamma} \{ t_1^{\beta'} \\
&\quad \times e^{i(w_1t_2+w_2t_1)\csc\alpha} \} \frac{\partial^{p-\gamma}}{\partial t_1^{p-\gamma}} \Big[\sum_{\mu=0}^{n} \binom{n}{\mu} \sum_{\mu'=0}^{\mu} \binom{\mu}{\mu'} (-i\cot\alpha)^{\beta'+\mu'} e^{-i(t_1t_2+w_1w_2)\cot\alpha} \\
&\quad \times \frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}} \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \} \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} \{ {w_1}^{\beta-\beta'} w_2^{\mu'} (\mathcal{R}^\alpha f)(w_1,w_2) \} \Big] dw_1 dw_2 \\
&= (-1)^n B_\alpha t_1^m \int_{\mathbb{R}^2} \sum_{\gamma=0}^{p} \binom{p}{\gamma} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \sum_{\gamma'=0}^{\gamma} \binom{\gamma}{\gamma'} \frac{\partial^{\gamma'}}{\partial t_1^{\gamma'}} t_1^{\beta'} \\
&\quad \times \frac{\partial^{\gamma-\gamma'}}{\partial t_1^{\gamma-\gamma'}} \Big[e^{i(w_1t_2+w_2t_1)\csc\alpha} \Big] \sum_{r=0}^{p-\gamma} \binom{p-\gamma}{r} \sum_{\mu=0}^{n} \binom{n}{\mu} \sum_{\mu'=0}^{\mu} \binom{\mu}{\mu'} \{ (-i\cot\alpha)^{\beta'+\mu'} \\
&\quad \times \frac{\partial^r}{\partial t_1^r} (e^{-i(t_1t_2+w_1w_2)\cot\alpha}) \frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}} \frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}} \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \} \\
&\quad \times \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} \{ {w_1}^{\beta-\beta'} w_2^{\mu'} (\mathcal{R}^\alpha f)(w_1,w_2) \} dw_1 dw_2 \\
&= (-1)^n B_\alpha t_1^m \int_{\mathbb{R}^2} \sum_{\gamma=0}^{p} \binom{p}{\gamma} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \sum_{\gamma'=0}^{\gamma} \binom{\gamma}{\gamma'} \mathcal{A} t_1^{\beta'-\gamma'} \\
&\quad \times (iw_2\csc\alpha)^{\gamma-\gamma'} e^{i(w_1t_2+w_2t_1)\csc\alpha} \sum_{r=0}^{p-\gamma} \binom{p-\gamma}{r} \sum_{\mu=0}^{n} \binom{n}{\mu} \sum_{\mu'=0}^{\mu} \binom{\mu}{\mu'} (-i\cot\alpha)^{\beta'+\mu'} \\
&\quad \times (-it_2\cot\alpha)^r e^{-i(t_1t_2+w_1w_2)\cot\alpha} \frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}} \frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}} \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \\
&\quad \times \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} \{ {w_1}^{\beta-\beta'} w_2^{\mu'} (\mathcal{R}^\alpha f)(w_1,w_2) \} dw_1 dw_2
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n B_\alpha \int_{\mathbb{R}^2} \sum_{\gamma=0}^{p} \binom{p}{\gamma} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \sum_{\gamma'=0}^{\gamma} \binom{\gamma}{\gamma'} \mathcal{A} t_1^{\beta'-\gamma'+m} \\
&\quad \times (i\csc\alpha)^{\gamma-\gamma'} e^{i(w_1t_2+w_2t_1)\csc\alpha} \sum_{r=0}^{p-\gamma} \binom{p-\gamma}{r} \sum_{\mu=0}^{n} \binom{n}{\mu} \sum_{\mu'=0}^{\mu} \binom{\mu}{\mu'} (-i\cot\alpha)^{\beta'+\mu'} \\
&\quad \times (-it_2\cot\alpha)^r e^{-i(t_1t_2+w_1w_2)\cot\alpha} \frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}} \frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}} \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \\
&\quad \times \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} \{ {w_1}^{\beta-\beta'} w_2^{\mu'+\gamma-\gamma'} (\mathfrak{R}^\alpha f)(w_1,w_2) \} dw_1 dw_2 \\
&= (-1)^n B_\alpha \int_{\mathbb{R}^2} \sum_{\gamma=0}^{p} \binom{p}{\gamma} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \sum_{\gamma'=0}^{\gamma} \binom{\gamma}{\gamma'} \mathcal{A} \\
&\quad \times (i\csc\alpha)^{\gamma-\beta'-m} \frac{\partial^{\beta'-\gamma'+m}}{\partial w_2^{\beta'-\gamma'+m}} (e^{i(w_1t_2+w_2t_1)\csc\alpha}) \sum_{r=0}^{p-\gamma} \binom{p-\gamma}{r} \sum_{\mu=0}^{n} \binom{n}{\mu} \sum_{\mu'=0}^{\mu} \binom{\mu}{\mu'} \\
&\quad \times (-it_2\cot\alpha)^r e^{-i(t_1t_2+w_1w_2)\cot\alpha} \frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}} \frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}} \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \\
&\quad \times (-i\cot\alpha)^{\beta'+\mu'} \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} \{ {w_1}^{\beta-\beta'} w_2^{\mu'+\gamma-\gamma'} (\mathfrak{R}^\alpha f)(w_1,w_2) \} dw_1 dw_2.
\end{aligned}
$$

Again, using $\langle D^n f, g\rangle = (-1)^n \langle f, D^n g\rangle$, for all $f, g \in S(\mathbb{R}^2)$, the above expression becomes

$$
\begin{aligned}
&= (-1)^{n+\beta'-\gamma'+m} B_\alpha \int_{\mathbb{R}^2} \sum_{\gamma=0}^{p} \binom{p}{\gamma} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \sum_{\gamma'=0}^{\gamma} \binom{\gamma}{\gamma'} \\
&\quad \times \mathcal{A} (i\csc\alpha)^{\gamma-\beta'-m} e^{i(w_1t_2+w_2t_1)\csc\alpha} \sum_{r=0}^{p-\gamma} \binom{p-\gamma}{r} \sum_{\mu=0}^{n} \binom{n}{\mu} \sum_{\mu'=0}^{\mu} \binom{\mu}{\mu'} \\
&\quad \times (-i\cot\alpha)^{\beta'+\mu'} (-it_2\cot\alpha)^r \frac{\partial^{\beta'-\gamma'+m}}{\partial w_2^{\beta'-\gamma'+m}} \Big[e^{-i(t_1t_2+w_1w_2)\cot\alpha} \frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}} \frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}} \\
&\quad \times \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} \{ {w_1}^{\beta-\beta'} w_2^{\mu'+\gamma-\gamma'} (\mathfrak{R}^\alpha f)(w_1,w_2) \} \Big] dw_1 dw_2 \\
&= (-1)^{n+\beta'-\gamma'+m} B_\alpha \int_{\mathbb{R}^2} \sum_{\gamma=0}^{p} \binom{p}{\gamma} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \sum_{\gamma'=0}^{\gamma} \binom{\gamma}{\gamma'} \mathcal{A} \\
&\quad \times (i\csc\alpha)^{\gamma-\beta'-m} e^{i(w_1t_2+w_2t_1)\csc\alpha} \sum_{r=0}^{p-\gamma} \binom{p-\gamma}{r} \sum_{\mu=0}^{n} \binom{n}{\mu} \sum_{\mu'=0}^{\mu} \binom{\mu}{\mu'} (-i\cot\alpha)^{\beta'+\mu'} \\
&\quad \times (-it_2\cot\alpha)^r \sum_{\eta=0}^{\beta'-\gamma'+m} \binom{\beta'-\gamma'+m}{\eta} \frac{\partial^\eta}{\partial w_2^\eta} \Big[e^{-i(t_1t_2+w_1w_2)\cot\alpha} \frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}} \frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}} \\
&\quad \times \frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}} \nu(t_1,t_2,w_1,w_2) \Big] \frac{\partial^{\beta'-\gamma'+m-\eta}}{\partial w_2^{\beta'-\gamma'+m-\eta}} \{ \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} {w_1}^{\beta-\beta'} w_2^{\mu'+\gamma-\gamma'} (\mathfrak{R}^\alpha f)(w_1,w_2) \} dw_1 dw_2 \\
&= (-1)^{n+\beta'-\gamma'+m} B_\alpha \int_{\mathbb{R}^2} \sum_{\gamma=0}^{p} \binom{p}{\gamma} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta}{\beta'} (i\csc\alpha)^{\beta-\beta'-n} \sum_{\gamma'=0}^{\gamma} \binom{\gamma}{\gamma'} \mathcal{A}
\end{aligned}
$$

$$\times(i\csc\alpha)^{\gamma-\beta'-m}e^{i(w_1t_2+w_2t_1)\csc\alpha}\sum_{r=0}^{p-\gamma}\binom{p-\gamma}{r}\sum_{\mu=0}^{n}\binom{n}{\mu}\sum_{\mu'=0}^{\mu}\binom{\mu}{\mu'}(-i\cot\alpha)^{\beta'+\mu'}$$
$$\times(-it_2\cot\alpha)^r\sum_{\eta=0}^{\beta'-\gamma'+m}\binom{\beta'-\gamma'+m}{\eta}\sum_{\eta'=0}^{\eta}\binom{\eta}{\eta'}\frac{\partial^{\eta'}}{\partial w_2^{\eta'}}\{e^{-i(t_1t_2+w_1w_2)\cot\alpha}\}\frac{\partial^{\eta-\eta'}}{\partial w_2^{\eta-\eta'}}$$
$$\times\Big(\frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}}\frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}}\frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}}\nu(t_1,t_2,w_1,w_2)\Big)\frac{\partial^{\beta'-\gamma'+m-\eta}}{\partial w_2^{\beta'-\gamma'+m-\eta}}\frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}}{w_1}^{\beta-\beta'}w_2^{\mu'+\gamma-\gamma'}$$
$$\times(\mathcal{R}^\alpha f)(w_1,w_2)dw_1dw_2$$
$$=\ (-1)^{n+\beta'-\gamma'+m}B_\alpha\int_{\mathbb{R}^2}\sum_{\gamma=0}^{p}\binom{p}{\gamma}\sum_{\beta=0}^{q}\binom{q}{\beta}\sum_{\beta'=0}^{\beta}\binom{\beta}{\beta'}(i\csc\alpha)^{\beta-\beta'-n}\sum_{\gamma'=0}^{\gamma}\binom{\gamma}{\gamma'}$$
$$\times\mathcal{A}(i\csc\alpha)^{\gamma-\beta'-m}e^{i(w_1t_2+w_2t_1)\csc\alpha}\sum_{r=0}^{p-\gamma}\binom{p-\gamma}{r}\sum_{\mu=0}^{n}\binom{n}{\mu}\sum_{\mu'=0}^{\mu}\binom{\mu}{\mu'}(-i\cot\alpha)^{\beta'+\mu'}$$
$$\times(-it_2\cot\alpha)^r\sum_{\eta=0}^{\beta'-\gamma'+m}\binom{\beta'-\gamma'+m}{\eta}\sum_{\eta'=0}^{\eta}\binom{\eta}{\eta'}(-iw_1\cot\alpha)^{\eta'}e^{-i(t_1t_2+w_1w_2)\cot\alpha}$$
$$\times\frac{\partial^{\eta-\eta'}}{\partial w_2^{\eta-\eta'}}\frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}}\frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}}\frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}}\nu(t_1,t_2,w_1,w_2)\{\frac{\partial^{\beta'-\gamma'+m-\eta}}{\partial w_2^{\beta'-\gamma'+m-\eta}}\frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}}{w_1}^{\beta-\beta'}$$
$$\times w_2^{\mu'+\gamma-\gamma'}(\mathcal{R}^\alpha f)(w_1,w_2)\}dw_1dw_2.$$

Therefore,

$$\Big|t_1^mt_2^n\frac{\partial^p}{\partial t_1^p}\frac{\partial^q}{\partial t_2^q}\mathcal{A}_{\nu,\alpha}f(t_1,t_2)\Big|$$
$$\leq\ |B_\alpha|\sum_{\gamma=0}^{p}\binom{p}{\gamma}\sum_{\beta=0}^{q}\binom{q}{\beta}\sum_{\beta'=0}^{\beta}\binom{\beta}{\beta'}|\csc\alpha|^{\beta-\beta'-n}\sum_{\gamma'=0}^{\gamma}\binom{\gamma}{\gamma'}\mathcal{A}|\csc\alpha|^{\gamma-\beta'-m}$$
$$\times\sum_{r=0}^{p-\gamma}\binom{p-\gamma}{r}\sum_{\mu=0}^{n}\binom{n}{\mu}\sum_{\mu'=0}^{\mu}\binom{\mu}{\mu'}|\cot\alpha|^{\beta'+\mu'}|t_2\cot\alpha|^r\sum_{\eta=0}^{\beta'-\gamma'+m}\binom{\beta'-\gamma'+m}{\eta}$$
$$\times\sum_{\eta'=0}^{\eta}\binom{\eta}{\eta'}\int_{\mathbb{R}^2}|w_1\cot\alpha|^{\eta'}|\frac{\partial^{\eta-\eta'}}{\partial w_2^{\eta-\eta'}}\frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}}\frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}}\frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}}\nu(t_1,t_2,w_1,w_2)|$$
$$\times|\frac{\partial^{\beta'-\gamma'+m-\eta}}{\partial w_2^{\beta'-\gamma'+m-\eta}}\frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}}(\mathcal{R}_1^\alpha f)(w_1,w_2)|dw_1dw_2,$$

where $\mathcal{R}_1^\alpha={w_1}^{\beta-\beta'}w_2^{\mu'+\gamma-\gamma'}(\mathcal{R}^\alpha f)(w_1,w_2)$. Then, the above expression becomes

$$\leq\ |B_\alpha|\sum_{\gamma=0}^{p}\binom{p}{\gamma}\sum_{\beta=0}^{q}\binom{q}{\beta}\sum_{\beta'=0}^{\beta}\binom{\beta}{\beta'}|\csc\alpha|^{\beta-\beta'-n}\sum_{\gamma'=0}^{\gamma}\binom{\gamma}{\gamma'}\mathcal{A}|\csc\alpha|^{\gamma-\beta'-m}$$
$$\times\sum_{r=0}^{p-\gamma}\binom{p-\gamma}{r}\sum_{\mu=0}^{n}\binom{n}{\mu}\sum_{\mu'=0}^{\mu}\binom{\mu}{\mu'}|\cot\alpha|^{\beta'+\mu'}\sum_{\eta=0}^{\beta'-\gamma'+m}\binom{\beta'-\gamma'+m}{\eta}$$
$$\times\sum_{\eta'=0}^{\eta}\binom{\eta}{\eta'}\int_{\mathbb{R}^2}(1+|w_1\cot\alpha|)^{\eta'}|(1+|t_2\cot\alpha|)^p|\frac{\partial^{\eta-\eta'}}{\partial w_2^{\eta-\eta'}}\frac{\partial^{p-\gamma-r}}{\partial t_1^{p-\gamma-r}}\frac{\partial^{\mu-\mu'}}{\partial w_1^{\mu-\mu'}}$$
$$\times\frac{\partial^{q-\beta}}{\partial t_2^{q-\beta}}\nu(t_1,t_2,w_1,w_2)|\Big\}|\frac{\partial^{\beta'-\gamma'+m-\eta}}{\partial w_2^{\beta'-\gamma'+m-\eta}}\frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}}(\mathcal{R}_1^\alpha f)(w_1,w_2)|dw_1dw_2.$$

Using the condition (4.1), we get,

$$\begin{aligned}
\leq \ & |B_\alpha| \sum_{\gamma=0}^{p} \binom{p}{\gamma} \sum_{\beta=0}^{q} \binom{q}{\beta} \sum_{\beta'=0}^{\beta} \binom{\beta'}{\beta} |\csc\alpha|^{\beta-\beta'-n} \sum_{\gamma'=0}^{\gamma} \binom{\gamma}{\gamma'} \mathcal{A} |\csc\alpha|^{\gamma-\beta'-m} \\
& \times \sum_{r=0}^{p-\gamma} \binom{p-\gamma}{r} \sum_{\mu=0}^{n} \binom{n}{\mu} \sum_{\mu'=0}^{\mu} \binom{\mu}{\mu'} |\cot\alpha|^{\beta'+\mu'} \sum_{\eta=0}^{\beta'-\gamma'+m} \binom{\beta'-\gamma'+m}{\eta} \\
& \times \sum_{\eta'=0}^{\eta} \binom{\eta}{\eta'} \mathcal{C} \int_{\mathbb{R}^2} (1+|w_1\cot\alpha|)^{m_1-|\mu-\mu'|+\eta'} (1+|w_2\cot\alpha|)^{m_2-|\eta-\eta'|} \\
& \times \Big| \frac{\partial^{\beta'-\gamma'+m-\eta}}{\partial w_2^{\beta'-\gamma'+m-\eta}} \frac{\partial^{n-\mu}}{\partial w_1^{n-\mu}} (\mathcal{R}_1^\alpha f)(w_1,w_2) \Big| dw_1 dw_2,
\end{aligned}$$

where, $\mathcal{C}$ depends only on $p, q, r, \mu, \mu', \eta, \eta', \beta, \beta'$.
Since, $\mathcal{R}_1^\alpha f(w_1, w_2) \in S(\mathbb{R}^2)$, the last integral is convergent.
Hence $\sup_{(t_1,t_2)\in\mathbb{R}^2} |t_1^m t_2^n \frac{\partial^p}{\partial t_1^p} \frac{\partial^q}{\partial t_2^q} \mathcal{A}_{\nu,\alpha} f(t_1,t_2)| < \infty$.
Which completes the proof. □

5. Boundedness of pseudo-differential operator

In this section we assume that $\nu(x_1, x_2, w_1, w_2)$ satisfies the following condition instead of (4.1). For given $m \in \mathbb{R}$, assume that

$$\begin{aligned}
& |(1+\sqrt{x_1^2+x_2^2}\csc\alpha)^n D_{x_1}^k D_{x_2}^l D_{w_1}^p D_{w_2}^q \nu(x_1,x_2,w_1,w_2)| \\
& \leq \mathcal{C}(1+\sqrt{w_1^2+w_2^2}\csc\alpha)^{m-(p+q)},
\end{aligned} \tag{5.1}$$

for all $x_1, x_2, w_1, w_2 \in \mathbb{R}$ and, for all $n, k, l, p, q \in \mathbb{N}_0$. The class of all such symbols is denoted by S^m, where $D_x = \frac{\partial}{\partial x}$.

Definition 5.1. A tempered distribution ψ belongs to generalized Sobolev space $H_\alpha^s(\mathbb{R}^2)$ and $s \in \mathbb{R}$, if it's gyrator transform $\mathcal{R}^\alpha\psi$, satisfies

$$||\psi||_{H_\alpha^s} = \Big(\int_{\mathbb{R}^2} |(1+(t_1^2+t_2^2)\csc^2\alpha)^{\frac{s}{2}} (\mathcal{R}^\alpha\psi)(t_1,t_2)|^2 dt_1 dt_2 \Big)^{\frac{1}{2}} < \infty, \tag{5.2}$$

for all $\psi \in S'(\mathbb{R}^2)$. The $H_\alpha^s(\mathbb{R}^2)$ space is complete with respect to the norm $\|\psi\|_{H_\alpha^s}$.

Proposition 5.2. *Let $f(t_1, t_2) \in S(\mathbb{R}^2)$ and for any $k \in \mathbb{N}_0$, we have*

$$\begin{aligned}
(i) \quad & \int_{\mathbb{R}^2} (\Lambda_{t_1,t_2})^k G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) dt_1 dt_2 \\
= \quad & \int_{\mathbb{R}^2} G_\alpha(t_1,t_2,w_1,w_2) (\Lambda'_{t_1,t_2})^k f(t_1,t_2) dt_1 dt_2, \\
(ii) \quad & \Big(\mathcal{R}^\alpha (\Lambda'_{t_1,t_2})^k f(t_1,t_2)\Big)(w_1,w_2) = (-i(w_1+w_2)\csc\alpha)^k \left(\mathcal{R}^\alpha f(t_1,t_2)\right)(w_1,w_2),
\end{aligned}$$

where $\Lambda_{t_1,t_2} = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + i(t_1+t_2)\cot\alpha$, *and* $\Lambda'_{t_1,t_2} = -(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} - i(t_1+t_2)\cot\alpha)$.

Proof. (i) For $k = 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Lambda_{t_1,t_2} G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) dt_1 dt_2 \\
&= \int_{\mathbb{R}^2} \Big(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} - i(t_1+t_2)\cot\alpha \Big) G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) dt_1 dt_2 \\
&= \int_{\mathbb{R}^2} \Big(\frac{\partial}{\partial t_1} G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) + \frac{\partial}{\partial t_2} G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) \Big) \\
&\quad - i(t_1+t_2)\cot\alpha G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) dt_1 dt_2 \\
&= \int_{\mathbb{R}^2} \Big(- G_\alpha(t_1,t_2,w_1,w_2) \frac{\partial}{\partial t_1} f(t_1,t_2) - G_\alpha(t_1,t_2,w_1,w_2) \frac{\partial}{\partial t_2} f(t_1,t_2) \\
&\quad - i(t_1+t_2)\cot\alpha G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) \Big) dt_1 dt_2 \\
&= \int_{\mathbb{R}^2} G_\alpha(t_1,t_2,w_1,w_2) \Lambda'_{t_1,t_2} f(t_1,t_2) dt_1 dt_2.
\end{aligned}$$

Continuing in similar manner, we get the required result.
(ii) As we know that

$$\Big(\mathcal{R}^\alpha (\Lambda'_{t_1,t_2})^k f(t_1,t_2) \Big)(w_1,w_2) = \int_{\mathbb{R}^2} G_\alpha(t_1,t_2,w_1,w_2) (\Lambda'_{t_1,t_2})^k f(t_1,t_2) dt_1 dt_2.$$

Exploiting the previous result of (i), for $k = 1$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} G_\alpha(t_1,t_2,w_1,w_2) \Lambda'_{t_1,t_2} f(t_1,t_2) dt_1 dt_2 \\
&= \int_{\mathbb{R}^2} \Lambda_{t_1,t_2} G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) dt_1 dt_2 \\
&= \int_{\mathbb{R}^2} (-i(w_1+w_2)\csc\alpha) G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) dt_1 dt_2 \\
&= (-i(w_1+w_2)\csc\alpha) \int_{\mathbb{R}^2} G_\alpha(t_1,t_2,w_1,w_2) f(t_1,t_2) dt_1 dt_2 \\
&= (-i(w_1+w_2)\csc\alpha) \left(\mathcal{R}^\alpha f(t_1,t_2) \right)(w_1,w_2).
\end{aligned}$$

Continuing in similar manner, we get the required result. □

Lemma 5.3. *For any symbol* $\nu(x_1,x_2,w_1,w_2)$ *belongs to* $S^m, m \in \mathbb{R}$ *and* $1 < k \in \mathbb{N}$, *there exists a positive constant* $\mathcal{C}_m$ *such that*

$$|(\mathcal{R}^\alpha \nu(x_1,x_2,w_1,w_2))(t_1,t_2,w_1,w_2)| \le \mathcal{C}_m \Big(1 + \sqrt{w_1^2 + w_2^2}\csc\alpha\Big)^m \Big(1 + (t_1+t_2)^2 \csc^2\alpha\Big)^{-\frac{k}{2}}.$$

Proof. Exploiting (1.7), we see that

$$(\mathcal{R}^\alpha \nu(x_1,x_2,w_1,w_2))(t_1,t_2,w_1,w_2) = \int_{\mathbb{R}^2} G_\alpha(x_1,x_2,t_1,t_2) \nu(x_1,x_2,w_1,w_2) dx_1 dx_2.$$

Following Proposition 5.2, we have

$$\begin{aligned}
&\Big(\mathcal{R}^\alpha (1 - \Lambda'_{x_1,x_2})^k f(x_1,x_2) \Big)(w_1,w_2) \\
&= \Big(1 + i(w_1+w_2)\csc\alpha \Big)^k \mathcal{R}^\alpha f(x_1,x_2)(w_1,w_2),
\end{aligned}$$

so that

$$\begin{aligned}
&\Big(1 + i(t_1+t_2)\csc\alpha \Big)^k \mathcal{R}^\alpha \nu(t_1,t_2,w_1,w_2) \\
= \quad &\int_{\mathbb{R}^2} G_\alpha(x_1,x_2,t_1,t_2)(1 - \Lambda'_{x_1,x_2})^k \nu(x_1,x_2,w_1,w_2) dx_1 dx_2. \qquad (5.3)
\end{aligned}$$

Now $(1-\Lambda'_{x_1,x_2})^k = \sum_{r=0}^{k}\binom{k}{r}(-1)^r(\Lambda'_{x_1,x_2})^r$, then we have

$$\begin{aligned}
&(1-\Lambda'_{x_1,x_2})^k \nu(x_1,x_2,w_1,w_2)\\
&=\sum_{r=0}^{k}\binom{k}{r}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+i(x_1+x_2)\cot\alpha\right)^r \nu(x_1,x_2,w_1,w_2)\\
&=\sum_{r=0}^{k}\binom{k}{r}P(x_1,x_2,l_1,l_2)D_{x_1}^{l_1}D_{x_2}^{l_2}\nu(x_1,x_2,w_1,w_2)
\end{aligned}$$

where $P(x_1,x_2,l_1,l_2)=\sum_{l_1+l_2\le r} C_{l_1,l_2}x_1^{l_1}x_2^{l_2}$. Then the above expression becomes

$$=\sum_{r=0}^{k}\binom{k}{r}\sum_{l_1+l_2\le r} C_{l_1,l_2}x_1^{l_1}x_2^{l_2}D_{x_1}^{l_1}D_{x_2}^{l_2}\nu(x_1,x_2,w_1,w_2).$$

Hence using (5.1), the above expression (5.3) can be re-written as

$$\begin{aligned}
&|\,(1+i(t_1+t_2)\csc\alpha)\,|^k|\mathcal{R}^\alpha \nu(t_1,t_2,w_1,w_2)|\\
&=|B_\alpha|\int_{\mathbb{R}^2}\sum_{r=0}^{k}\binom{k}{r}\sum_{l_1+l_2\le r} C_{l_1,l_2}x_1^{l_1}x_2^{l_2}D_{x_1}^{l_1}D_{x_2}^{l_2}\nu(x_1,x_2,w_1,w_2)dx_1dx_2\\
&\le |B_\alpha|\mathcal{D}(1+\sqrt{w_1^2+w_2^2}\csc\alpha)^m\int_{\mathbb{R}^2}(1+\sqrt{x_1^2+x_2^2}\csc\alpha)^{-k}dx_1dx_2.
\end{aligned}$$

Since the integral is convergent for large value of k, there exists a constant $\mathcal{C}_m>0$ depending on k,α,l,r,m. Thus from the above inequality we obtain,

$$|(\mathcal{R}^\alpha \nu(x_1,x_2,w_1,w_2))(t_1,t_2,w_1,w_2)|\le \mathcal{C}_m(1+\sqrt{w_1^2+w_2^2}\csc\alpha)^m(1+(t_1+t_2)^2\csc^2\alpha)^{-\frac{k}{2}}.$$

□

Proposition 5.4. *For any symbol $\nu(t_1,t_2,w_1,w_2)\in S^m, m\in\mathbb{R}$, the associated operator $\mathcal{A}_{\nu,\alpha}\psi$ admits the representation*

$$\begin{aligned}
&\mathcal{A}_{\nu,\alpha}\psi(x_1,x_2)\\
&=\frac{e^{ix_1x_2\cot\alpha}}{B_\alpha}\int_{\mathbb{R}^4}\overline{G_\alpha(x_1,x_2,t_1,t_2)}e^{-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}\\
&\quad\times\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)\mathcal{R}^\alpha\psi(w_1,w_2)dw_1dw_2dt_1dt_2,
\end{aligned}$$

provided all involved integrals are convergent.

Proof. In order to establish it's validity. We first note that

$(1+\sqrt{(w_1^2+w_2^2)}\csc\alpha)^m\le\mathcal{L}_m(1+(w_1^2+w_2^2)\csc^2\alpha)^{\frac{m}{2}}$, where $m\in\mathbb{R}$ *and* $\mathcal{L}_m=max(1,2^{\frac{m}{2}})$.

Hence, by Lemma 5.3, we obtain

$$|\mathcal{R}^\alpha\nu(t_1,t_2,w_1,w_2)|\le\mathcal{C}_m\mathcal{L}_m(1+(w_1^2+w_2^2)\csc^2\alpha)^{\frac{m}{2}}(1+(t_1+t_2)^2\csc^2\alpha)^{-\frac{k}{2}}.$$

Since $(\mathcal{R}^\alpha\psi)(w_1,w_2)\in S(\mathbb{R}^2)$, exploiting the above inequality we have

$$\begin{aligned}
&\left|\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)(\mathcal{R}^\alpha\psi)(w_1,w_2)\right|\\
\le\ &\mathcal{C}(1+(w_1^2+w_2^2)\csc^2\alpha)^{\frac{m}{2}-q}(1+((t_1-w_1)+(t_2-w_2))^2\csc^2\alpha)^{-\frac{k}{2}}\\
\le\ &\mathcal{C}(1+(w_1^2+w_2^2)\csc^2\alpha)^{\frac{m}{2}-q}(1+((t_1-w_1)^2+(t_2-w_2)^2)\csc^2\alpha)^{-\frac{k}{2}}.
\end{aligned}$$

where q and k can be chosen sufficiently large natural numbers. Hence,

$$\begin{aligned}&\int_{\mathbb{R}^2}|\mathcal{R}^\alpha \nu(t_1-w_1,t_2-w_2,w_1,w_2)(\mathcal{R}^\alpha\psi)(w_1,w_2)|dw_1dw_2\\&\leq \mathcal{C}\int_{\mathbb{R}^2}(1+(w_1^2+w_2^2)\csc^2\alpha)^{\frac{m}{2}-q}(1+((t_1-w_1)^2+(t_2-w_2)^2)\csc^2\alpha)^{-\frac{k}{2}}dw_1dw_2\\&<\infty.\end{aligned}$$

For large q and k, the right hand side is a convolution product of two integrable functions on $\mathbb{R}^2$. Therefore, the function

$$\begin{aligned}&g_\alpha(t_1,t_2)\\&=\int_{\mathbb{R}^2}e^{-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)(\mathcal{R}^\alpha\psi)(w_1,w_2)dw_1dw_2\end{aligned}$$

is in $L^1(\mathbb{R}^2)$. Now we compute it's inverse gyrator transform

$$\begin{aligned}&\int_{\mathbb{R}^2}\overline{G_\alpha(x_1,x_2,t_1,t_2)}\int_{\mathbb{R}^2}e^{-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)\\&\quad\times(\mathcal{R}^\alpha\psi)(w_1,w_2)dw_1dw_2dt_1dt_2\\&=B_\alpha\int_{\mathbb{R}^4}e^{-i(x_1x_2+t_1t_2)\cot\alpha+i(x_1t_2+x_2t_1)\csc\alpha}e^{-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}(\mathcal{R}^\alpha\psi)(w_1,w_2)\\&\quad\times\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)dw_1dw_2dt_1dt_2\\&=B_\alpha\int_{\mathbb{R}^4}e^{-i[x_1x_2+(t_1-w_1)(t_2-w_2)]\cot\alpha+i[x_1(t_2-w_2)+x_2(t_1-w_1)]\csc\alpha}e^{-i(t_1w_2+t_2w_1-w_1w_2)\cot\alpha}\\&\quad\times e^{i(x_1w_2+x_2w_1)\csc\alpha}e^{-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)\\&\quad\times(\mathcal{R}^\alpha\psi)(w_1,w_2)dw_1dw_2dt_1dt_2\\&=\int_{\mathbb{R}^4}e^{-i(t_1w_2+t_2w_1-w_1w_2)\cot\alpha+i(x_1w_2+x_2w_1)\csc\alpha-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}(\mathcal{R}^\alpha\psi)(w_1,w_2)\\&\quad\times B_\alpha e^{-i[x_1x_2+\tau_1\tau_2]\cot\alpha+i[x_1\tau_2+x_2\tau_1]\csc\alpha}\mathcal{R}^\alpha\nu(\tau_1,\tau_2,w_1,w_2)d\tau_1d\tau_2dw_1dw_2\\&=\int_{\mathbb{R}^2}e^{-i(t_1w_2+t_2w_1-w_1w_2)\cot\alpha+i(x_1w_2+x_2w_1)\csc\alpha-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}(\mathcal{R}^\alpha\psi)(w_1,w_2)\\&\quad\times\nu(x_1,x_2,w_1,w_2)dw_1dw_2\\&=e^{-iw_1w_2\cot\alpha+i(x_1w_2+x_2w_1)\csc\alpha}(\mathcal{R}^\alpha\psi)(w_1,w_2)\nu(x_1,x_2,w_1,w_2)dw_1dw_2\\&=\frac{e^{ix_1x_2\cot\alpha}}{B_\alpha}\int_{\mathbb{R}^2}\overline{G_\alpha(x_1,x_2,w_1,w_2)}(\mathcal{R}^\alpha\psi)(w_1,w_2)\nu(x_1,x_2,w_1,w_2)dw_1dw_2.\end{aligned}$$

This establishes the required Proposition. □

Lemma 5.5. ***(Peetre)*** *For any real number s and for all $(t_1,t_2),(w_1,w_2)\in\mathbb{R}^2$, the following inequality*

$$\frac{(1+(t_1^2+t_2^2)\csc^2\alpha)^s}{(1+(w_1^2+w_2^2)\csc^2\alpha)^s}\leq 2^{|s|}(1+((t_1-w_1)^2+(t_2-w_2)^2)\csc^2\alpha)^{|s|}$$

holds.

Proof. See [19, p. 97]. □

Theorem 5.6. *Let $\nu(x_1,x_2,t_1,t_2)\in S^m, m\in\mathbb{R}$ and $(\mathcal{A}_{\nu,\alpha}\psi)$ be the associated operator. Then the following estimate holds true:*

$$\|(\mathcal{A}_{\nu,\alpha}\psi)\|_{H^s_\alpha}\leq\mathcal{C}\|\psi\|_{H^{s+m}_\alpha},\tag{5.4}$$

for all $s\in\mathbb{R}$ and for all $\psi\in S(\mathbb{R}^2)$ for a certain constant $\mathcal{C}=\mathcal{C}(\alpha)$.

Proof. From Proposition 5.4, we have seen that the tempered distribution $\left(e^{-ix_1x_2\cot\alpha}\mathcal{A}_{\nu,\alpha}\psi\right)$ has the gyrator transform equal to

$$\frac{1}{B_\alpha}\int_{\mathbb{R}^2} e^{-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)(\mathcal{R}^\alpha\psi)(w_1,w_2)dw_1dw_2.$$

Set

$$\begin{aligned}
&\mathcal{U}_\alpha^s(t_1,t_2)\\
=&\frac{(1+(t_1^2+t_2^2)\csc^2\alpha)^{\frac{s}{2}}}{B_\alpha}\int_{\mathbb{R}^2} e^{-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)\\
&\times\mathcal{R}^\alpha\psi(w_1,w_2)dw_1dw_2\\
=&\frac{1}{B_\alpha}\int_{\mathbb{R}^2}\frac{(1+(t_1^2+t_2^2)\csc^2\alpha)^{\frac{s}{2}}}{(1+(w_1^2+w_2^2)\csc^2\alpha)^{\frac{s}{2}}}e^{-i(2w_1w_2-t_1w_2-t_2w_1)\cot\alpha}(1+(w_1^2+w_2^2)\csc^2\alpha)^{\frac{s}{2}}\\
&\times\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)\mathcal{R}^\alpha\psi(w_1,w_2)dw_1dw_2.
\end{aligned}$$

Using Lemma 5.5, we have

$$\begin{aligned}
&|\mathcal{U}_\alpha^s(t_1,t_2)|\\
\leq&\frac{2^{\frac{|s|}{2}+2}}{B_\alpha}\int_{\mathbb{R}^2}(1+(w_1^2+w_2^2)\csc^2\alpha)^{\frac{s}{2}}(1+((t_1-w_1)^2+(t_2-w_2)^2)\csc^2\alpha)^{\frac{s}{2}}\\
&\times|\mathcal{R}^\alpha\nu(t_1-w_1,t_2-w_2,w_1,w_2)|\mathcal{R}^\alpha\psi(w_1,w_2)dw_1dw_2.
\end{aligned}$$

Exploiting Lemma 5.3, we get,

$$\begin{aligned}
&|\mathcal{U}_\alpha^s(t_1,t_2)|\\
\leq&\frac{2^{\frac{|s|}{2}+2}}{B_\alpha}\mathcal{C}_m\int_{\mathbb{R}^2}(1+(w_1^2+w_2^2)\csc^2\alpha)^{\frac{m+s}{2}}\mathcal{R}^\alpha\psi(w_1,w_2)\\
&\times(1+((t_1-w_1)^2+(t_2-w_2)^2)\csc^2\alpha)^{\frac{s-k}{2}}dw_1dw_2\\
=&\int_{\mathbb{R}^2} f_\alpha(w_1,w_2)h_\alpha(t_1-w_1,t_2-w_2)dw_1dw_2\\
=&(f_\alpha * h_\alpha)(t_1,t_2).
\end{aligned}$$

If k is sufficiently large, $h_\alpha\in L^1(\mathbb{R}^2)$. Also, since $\mathcal{R}^\alpha\psi\in S(\mathbb{R}^2), f_\alpha(w_1,w_2)\in L^2(\mathbb{R}^2)$. Then, $(f_\alpha * h_\alpha)(t_1,t_2)$ belongs to $L^2(\mathbb{R})$ and the inequality

$$\|f_\alpha * h_\alpha\|_{L^2(\mathbb{R})}\leq\|f_\alpha\|_{L^2(\mathbb{R})}\|h_\alpha\|_{L^1(\mathbb{R})},$$

implies that

$$\|\mathcal{U}_\alpha^s(t_1,t_2)\|_{L^2(\mathbb{R}^2)}\leq\mathcal{C}(\alpha)\|\psi\|_{H_\alpha^{s+m}}.$$

Therefore in viewing Proposition 5.4, we have

$$\begin{aligned}
&\|e^{-ix_1x_2\cot\alpha}\mathcal{A}_{\nu,\alpha}\psi(x_1,x_2)\|_{H_\alpha^s}\\
=&\|(1+(t_1^2+t_2^2)\csc^2\alpha)^{\frac{s}{2}}\mathcal{R}^\alpha e^{-ix_1x_2\cot\alpha}\mathcal{A}_{\nu,\alpha}\psi(x_1,x_2)\|_{H_\alpha^s}\\
=&\|\mathcal{U}_\alpha^s(t_1,t_2)\|_{L^2(\mathbb{R}^2)}\leq\mathcal{C}(\alpha)\|\psi\|_{H_\alpha^{s+m}}.
\end{aligned}$$

Which completes the proof of this theorem. □

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