Çankaya University Journal of Science and Engineering
Volume 9 (2012), No. 2, 167–182

m-Generators of Fuzzy Dynamical Systems

Mohammad Ebrahimi[1,*] and Uosef Mohamadi[1]

[1] *Department of Mathematics, Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran*
[*] *Corresponding author: mohamad_ebrahimi@mail.uk.ac.ir*

Özet. Bu makalede, sonlu atomlu bir alt-σ-cebirine göre bulanık ölçüm koruyan dönüşümün entropisinin afin olduğunu ispatlıyor, daha sonra bir sonlu alt-σ-cebirinin entropisini hesaplama yöntemini sayılabilir çoklukta atomlu alt-σ-cebirine uygulanacak şekille genelleştiriyor, ve bulanık olasılık dinamik sistemlerin ergodik özelliklerini araştırıyoruz. Son olarak, bu kavram kullanılarak Kolmogorov-Sinai önermesinin [6, 9, 10] bir çeşidi veriliyor.[†]

Anahtar Kelimeler. Bulanık olasılık uzayı, entropi, bulanık dinamik sistemler, m-denklik, m-saflaştırma, bulanık m-üreteç.

Abstract. In this paper we prove that the entropy of a fuzzy measure preserving transformation with respect to a sub-σ-algebra having finite atoms is affine and then we extend the method of computing the entropy of a finite sub-σ-algebra to a sub-σ-algebra having countable atoms, and we investigate the ergodic properties of fuzzy probability dynamical systems. At the end by using this notion, a version of Kolmogorov-Sinai proposition [6, 9, 10] is given.

Keywords. Fuzzy probability space, entropy, fuzzy dynamical systems, m-equivalence, m-refinement, fuzzy m-generator.

1. Introduction and Preliminaries

The main idea of fuzzy entropy is the substitution of partitions by fuzzy partitions. In some previous papers [1, 2, 3] entropy of a fuzzy dynamical system has been defined. Also the notions of m-refinements and m-equivalence have been defined in [8]. In this paper we give a definition for the entropy of a sub-σ-algebra with countable atoms.

Received April 7, 2012; accepted February 20, 2013.

[†]Türkçe özet ve anahtar kelimeler, orijinal İngilizce metindeki ilgili kısmın doğrudan tercümesi olup *Çankaya University Journal of Science and Engineering* editörlüğü tarafından yazılmıştır. | Turkish abstract and the keywords are written by the editorial staff of *Çankaya University Journal of Science and Engineering* which are the direct translations of the related original English text.

ISSN 1309 - 6788

2. Fuzzy Dynamical Systems

We recall that a fuzzy set in a nonempty set X is an element of the family I^X of all functions from X to closed unit interval $I = [0, 1]$. A sequence $\{\lambda_i\}$ of fuzzy sets in X increase to $\lambda \in I^X$ (written as $\lambda_i \uparrow \lambda$) if $\{\lambda_i(x)\}_{i=1}^{\infty}$ is monotonic increasing and converges to $\lambda(x)$ for each x in X. A fuzzy σ-algebra M on a non-empty set X is a subset of I^X which satisfies the following conditions:

(i) $1 \in M$,
(ii) $\lambda \in M \Rightarrow 1 - \lambda \in M$,
(iii) if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in M then $\bigvee_{i=1}^{\infty} \lambda_i = \sup_i \lambda_i \in M$.

If N_1 and N_2 are fuzzy σ-algebras on X then $N_1 \vee N_2$ is the smallest fuzzy σ-algebra that contains $N_1 \cup N_2$, denoted by $[N_1 \cup N_2]$. A fuzzy probability measure m over M is a function $m : M \to I$ which fulfills the conditions:

(i) $m(1) = 1$,
(ii) $m(1 - \lambda) = 1 - m(\lambda)$,
(iii) $m(\lambda \vee \mu) + m(\lambda \wedge \mu) = m(\lambda) + m(\mu)$ for each $\lambda,\ \mu \in M$,
(iv) for each sequence $\{\lambda_i\}_{i=1}^{\infty}$ in M such that $\lambda_i \uparrow \lambda$, $m(\lambda) = \sup_i m(\lambda_i)$.

The triple (X, M, m) is called a fuzzy probability measure space and the elements of M are called fuzzy measurable sets [8].

Definition 2.1. Let (X, M, m) be a fuzzy probability measure space, the elements μ, λ of M are called m-disjoint if $m(\lambda \wedge \mu) = 0$.

A relation '$= (\mathrm{mod}\, m)$' on M is defined as follows;

$$\lambda = \mu \ (\mathrm{mod}\ m) \quad \text{iff} \quad m(\lambda) = m(\mu) = m(\lambda \wedge \mu), \qquad \lambda, \mu \in M.$$

Relation '$= (\mathrm{mod}\, m)$' is an equivalence relation. $\tilde{M}$ denotes the set of all equivalence classes induced by this relation, and $\tilde{\mu}$ is the equivalence class determined by μ. For $\lambda, \mu \in M$, $\lambda \wedge \mu = 0 \ (\mathrm{mod}\, m)$ iff λ, μ are m-disjoint. We shall identify $\tilde{\mu}$ with μ [8].

Definition 2.2. Let (X, M, m) be a fuzzy probability measure space, and N be a fuzzy sub-σ-algebra of M. Then an element $\tilde{\mu} \in \tilde{N}$ is an atom of N if

(i) $m(\mu) > 0$,
(ii) for each $\tilde{\lambda} \in \tilde{N}$ such that $m(\lambda \wedge \mu) = m(\lambda) \neq m(\mu)$ then $m(\lambda) = 0$, [8].

Proposition 2.3. *Let (X, M, m) be a fuzzy probability measure space, and N be a fuzzy sub-σ-algebra of M. If $\tilde{\mu_1}$, $\tilde{\mu_2}$ are disjoint atoms of N then they are m-disjoint.*

Proof. See [8]. □

The set of all atoms of N is denoted by $\overline{N}$. We define $F(M)$ as below

$$F(M) = \{N : \ N \text{ is a sub-}\sigma\text{-algebra of } M \text{ with finite atoms}\}.$$

Definition 2.4. Suppose (X, M, m) and (Y, N, n) are fuzzy probability measure spaces. A transformation $\varphi : \ (X, M, m) \to (Y, N, n)$ is said to be a fuzzy measure preserving if

(i) $\varphi^{-1}(\mu) \in M$ for every $\mu \in N$,
(ii) $m(\varphi^{-1}(\mu)) = n(\mu)$ for all $\mu \in \overline{N}$.

Definition 2.5. A fuzzy dynamical system is denoted by (X, M, m, φ) where (X, M, m) is a fuzzy probability measure space and φ is a fuzzy measure preserving transformation.

Definition 2.6. The entropy of $N \in F(M)$ is given by

$$H(N, m) = -\sum_{\mu \in \overline{N}} m(\mu) \log m(\mu),$$

and the mean entropy of φ on N of the fuzzy dynamical system (X, M, m, φ) is defined by

$$h(N, M, \varphi) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=1}^{n-1} \varphi^{-i}(N), m).$$

Note that $\varphi^{-i}(N) = \{\varphi^{-i}(\mu) : \ \mu \in N\}$ is an element of $F(M)$ and $\bigvee_{i=1}^{n-1} \varphi^{-i}(N)$ is the smallest fuzzy σ-algebra containing $\bigcup_{i=0}^{n-1} \varphi^{-i}(N)$, and the above limit exists [8].

Proposition 2.7. *The mean entropy of φ on N of the fuzzy dynamical system (X, M, m, φ) is affine, i.e,*

$$h(N, \lambda m_1 + (1 - \lambda) m_2, \varphi) = \lambda h(N, m_1, \varphi) + (1 - \lambda) h(N, m_2, \varphi),$$

for each pair m_1 and m_2 of fuzzy probability measures, $N \in F(M)$ and $\lambda \in [0, 1]$.

Proof. If m_1 and m_2 are two fuzzy probability measures and $\lambda \in [0, 1]$ then

$$H(N, \lambda m_1 + (1 - \lambda) m_2) \geq \lambda H(N, m_1) + (1 - \lambda) H(N, m_2). \tag{1}$$

The 'concavity' inequality (1) is a direct consequence of the definition of $H(N, m)$ and the 'concavity' of the function $x \to -x\log x$. Conversely, one has inequalities

$$-\log(\lambda m_1(\mu_i) + (1-\lambda)m_2(\mu_i)) \leq -\log\lambda - \log(m_1(\mu_i)),$$

and

$$-\log(\lambda m_1(\mu_i) + (1-\lambda)m_2(\mu_i)) \leq -\log(1-\lambda) - \log(m_2(\mu_i)),$$

since $x \to -\log x$ is decreasing. Therefore one obtains the 'convexity' bound

$$H(N, \lambda m_1 + (1-\lambda)m_2) \leq \lambda H(N, m_1) + (1-\lambda)H(N, m_2) - \lambda\log\lambda - (1-\lambda)\log(1-\lambda). \quad (2)$$

Now replacing N by $\bigvee_{i=0}^{n-1} \varphi^{-i}(N)$ in (1), dividing by n and taking the $\lim_{n\to\infty}$ gives

$$h(N, \lambda m_1 + (1-\lambda)m_2, \varphi) \geq \lambda h(N, m_1, \varphi) + (1-\lambda)h(N, m_2, \varphi).$$

Similarly from (2), since

$$\frac{-(\lambda\log\lambda + (1-\lambda)\log(1-\lambda))}{n} \to 0 \text{ as } n \to \infty,$$

one deduces the converse inequality

$$h(N, \lambda m_1 + (1-\lambda)m_2, \varphi) \leq \lambda h(N, m_1, \varphi) + (1-\lambda)h(N, m_2, \varphi).$$

Hence one concludes the map $m \to h(N, m, \varphi)$ is affine. This is a somewhat surprising and is of great significance in the application of fuzzy mean entropy. □

3. Ergodic Measures and Weak-Mixing

Definition 3.1. Given a fuzzy probability space (X, M, m), a fuzzy measure preserving transformation $\varphi : X \to X$ is called ergodic if for every atom $\gamma \in \overline{M}$ with $\varphi^{-1}(\gamma) = \gamma$ we have that either $m(\gamma) = 0$ or $m(\gamma) = 1$. Alternatively we say that m is φ-ergodic.

Proposition 3.2. *Let Σ denote the set of fuzzy invariant probability measures on X. $m \in \Sigma$ is ergodic if whenever there exists $m_1,\ m_2 \in \Sigma$ and $0 < \lambda < 1$ with $m = \lambda m_1 + (1-\lambda)m_2$ then $m_1 = m_2$.*

Proof. If m is not ergodic then we can find $\gamma \in \overline{M}$ with $\varphi^{-1}(\gamma) = \gamma$ and $0 < m(\gamma) < 1$ but for every atom $\mu \in \overline{M}$ we can write

$$\mu = (\mu \wedge \gamma) \vee (\mu \wedge (1-\gamma)).$$

Therefore

$$\begin{aligned} m(\mu) &= m((\mu \wedge \gamma) \vee (\mu \wedge (1-\gamma))) \\ &= m(\gamma)\left(\frac{m(\mu \wedge \gamma)}{m(\gamma)}\right) + m(1-\gamma)\left(\frac{m(\mu \wedge (1-\gamma))}{m(1-\gamma)}\right) \\ &= \lambda m_1(\mu) + (1-\lambda) m_2, \end{aligned}$$

where $\lambda = m(\gamma)$ and $m_1(\mu) = m(\mu \wedge \gamma)/m(\gamma)$, $m_2(\mu) = m(\mu \wedge (1-\gamma))/m(1-\gamma)$ this shows that $m = \lambda m_1 + (1-\lambda)m_2(\mu)$. □

Definition 3.3. Let (X, M, m, φ) be a fuzzy dynamical system, we say that φ is weak-mixing if for any $\mu,\ \lambda \in \overline{M}$ we have that,

$$\frac{1}{k}\sum_{n=0}^{k-1} |m(\varphi^{-n}(\mu) \wedge \lambda) - m(\mu)m(\lambda)| \to 0 \quad \text{as} \quad k \to \infty.$$

Proposition 3.4. *If a transformation $\varphi : X \to X$ on a fuzzy probability measure space (X, M, m) is weak-mixing then it is necessarily ergodic.*

Proof. If φ is weak-mixing then by definition we have that for any $\mu,\ \lambda \in \overline{M}$,

$$\frac{1}{k}\sum_{n=0}^{k-1} |m(\varphi^{-n}(\mu) \wedge \lambda) - m(\mu)m(\lambda)| \to 0 \qquad as \quad k \to \infty.$$

By the triangle inequality we have that,

$$\left|\frac{1}{k}\sum_{n=0}^{k-1} m(\varphi^{-n}(\mu) \wedge \lambda) - m(\mu)m(\lambda)\right| \le \frac{1}{k}\sum_{n=0}^{k-1} |m(\varphi^{-n}(\mu) \wedge \lambda) - m(\mu)m(\lambda)| \to 0.$$

If we assume (for a contradiction) that φ was not ergodic then there would exist a φ-invariant atom $\gamma \in \overline{M}$ with $\varphi^{-1}(\gamma) = \gamma$ with $0 < m(\gamma) < 1$. If we take $\mu = \gamma$ and $\lambda = 1-\gamma$ then since $m(\varphi^{-n}(\gamma) \wedge (1-\gamma)) = m(\gamma \wedge (1-\gamma))$, for all $n \geq 0$, we deduce that $m(\gamma)\, m(1-\gamma) = 0$ giving the required contradiction. Thus φ is ergodic. □

4. Entropy of a Sub-σ-Algebra with Countable Atoms

In this section we introduce the notion of entropy of a sub-σ-algebra with countable atoms. We introduce $F^*(M)$ as below,

$$F^*(M) = \{N :\ N \text{ is a sub-}\sigma\text{-algebra of } M \text{ with countable atoms}\}.$$

Assume that M is a σ-algebra and $N_1,\ N_2 \in F^*(M)$, and $\{\lambda_i :\ i \in \mathbb{N}\}$ and $\{\mu_j :\ j \in \mathbb{N}\}$ denote the atoms of N_1 and N_2 respectively, then the atoms of $N_1 \vee N_2$

are $\lambda_i \wedge \mu_j$ which $m(\lambda_i \wedge \mu_j) > 0$ for each $i,\ j \in \mathbb{N}$. If $\gamma \in \overline{M}$ we set

$$N_1 \vee \gamma = \{\lambda_i \wedge \gamma : \ m(\lambda_i \wedge \gamma) > 0,\ i \in \mathbb{N}\}.$$

Proposition 4.1. *Let $\{\lambda_i : \ i \in \mathbb{N}\}$ be an m-disjoint collection of fuzzy measurable sets of fuzzy probability measure space (X, M, m), then,*

$$m\left(\bigvee_{i=1}^{\infty}(\lambda_i)\right) = \sum_{i=1}^{\infty} m(\lambda_i).$$

Proof. See [8]. □

Definition 4.2. Let (X, M, m) be a fuzzy probability measure space and $N_1,\ N_2 \in F^*(M)$. We say that N_2 is an m-refinement of N_1, denoted by $N_1 \leq_m N_2$, if for each $\mu \in \overline{N_2}$ there exists $\lambda \in \overline{N_1}$ such that $m(\lambda \wedge \mu) = m(\mu)$.

Proposition 4.3. *Let (X, M, m) be a fuzzy probability measure space and $N_1,\ N_2$, $N_3 \in F^*(M)$. If $N_1 \leq_m N_2$ then,*

$$N_1 \vee N_3 \leq_m N_2 \vee N_3.$$

Proof. See [8]. □

Definition 4.4. Let (X, M, m) be a fuzzy probability space, and N be a sub-σ-algebra of M for which $N \in F^*(M)$. The entropy of N is defined as

$$H(N) = -\log \sup_{i \in \mathbb{N}} m(\mu_i),$$

where $\{\mu_i : \ i \in \mathbb{N}\}$ are atoms of N.

Definition 4.5. Let (X, M, m) be a fuzzy probability measure space and $N \in F^*(M)$. The conditional entropy of N given $\gamma \in \overline{M}$ is defined by

$$H(N|\gamma) = -\log \sup_{i \in \mathbb{N}} m(\mu_i|\gamma),$$

where,

$$m(\mu_i|\gamma) = \frac{m(\mu_i \wedge \gamma)}{m(\gamma)} \qquad (m(\gamma) \neq 0).$$

Proposition 4.6. *Let (X, M, m) be a fuzzy probability measure space, and $N_1, N_2 \in F^*(M)$ for which $\overline{N_1} = \{\lambda_i : \ i \in \mathbb{N}\}$ and $\overline{N_2} = \{\mu_j : \ j \in \mathbb{N}\}$. Then,*

(i) $N_1 \leq_m N_2 \Rightarrow H(N_1) \leq H(N_2)$,
(ii) $N_1 \leq_m N_2 \Rightarrow H(N_1|\gamma) \leq H(N_2|\gamma)$.

Proof. (i) Suppose $N_1 \leq_m N_2$, and then for each $\mu_j \in \overline{N_2}$ there exists $\lambda_{i_j} \in \overline{N_1}$ such that, $m(\mu_j \wedge \lambda_{i_j}) = m(\mu_j)$ but $\lambda_{i_j} \wedge \mu_j \leq \lambda_{i_j}$. Then,

$$m(\lambda_{i_j} \wedge \mu_j) \leq m(\lambda_{i_j}) \Rightarrow m(\mu_j) \leq m(\lambda_{i_j}) \Rightarrow m(\mu_j) \leq \sup_{\lambda_{i_j} \in \overline{N_1}} m(\lambda_{i_j}).$$

Since μ_j is arbitrary we have: $\sup_{j\in\mathbb{N}} m(\mu_j) \leq \sup_{i\in\mathbb{N}} m(\lambda_i)$ and then we have $H(N_1) \leq H(N_2)$ since $f(x) = -\log x$ is a decreasing function.

(ii) Supppose $N_1 \leq_m N_2$, by Proposition 4.3 we have, $N_1 \vee \gamma \leq_m N_2 \vee \gamma$, and by (i) we conclude that

$$\begin{aligned} H(N_1 \vee \gamma) \leq H(N_2 \vee \gamma) &\Rightarrow -\log \sup_{i\in\mathbb{N}} m(\lambda_i \wedge \gamma) \leq -\log \sup_{j\in\mathbb{N}} m(\mu_j \wedge \gamma) \\ &\Rightarrow \sup_{j\in\mathbb{N}} m(\mu_j \wedge \gamma) \leq \sup_{i\in\mathbb{N}} m(\lambda_i \wedge \gamma) \\ &\Rightarrow \sup_{j\in\mathbb{N}} \frac{m(\mu_j \wedge \gamma)}{m(\gamma)} \leq \sup_{i\in\mathbb{N}} \frac{m(\lambda_i \wedge \gamma)}{m(\gamma)} \\ &\Rightarrow -\log \sup_{i\in\mathbb{N}} \frac{m(\lambda_i \wedge \gamma)}{m(\gamma)} \leq -\log \sup_{j\in\mathbb{N}} \frac{m(\mu_j \wedge \gamma)}{m(\gamma)} \\ &\Rightarrow H(N_1|\gamma) \leq H(N_2|\gamma). \end{aligned}$$

□

Definition 4.7. Let (X, M, m) be a fuzzy probability measure space and $N_1,\ N_2 \in F^*(M)$. We say that N_1 and N_2 are m-equivalent, denoted by $N_1 \approx_m N_2$, if

(i) for each $\mu \in \overline{N_2}$, $m(\mu \wedge (\bigvee\{\lambda :\ \lambda \in \overline{N_1}\})) = m(\mu)$,
(ii) for each $\lambda \in \overline{N_1}$, $m(\lambda \wedge (\bigvee\{\mu :\ \mu \in \overline{N_2}\})) = m(\lambda)$.

Proposition 4.8. *Let (X, M, m) be a fuzzy probability measure space, and $N_1, N_2 \in F^*(M)$. Then,*

$$N_1 \approx_m N_2 \Rightarrow N_1 \approx_m N_1 \vee N_2.$$

Proof. Assume that, $\overline{N_1} = \{\lambda_i :\ i \in \mathbb{N}\}$, $\overline{N_2} = \{\mu_j :\ j \in \mathbb{N}\}$. We know that

$$\overline{N_1 \vee N_2} = \{\lambda_i \wedge \mu_j :\ \lambda_i \in \overline{N_1},\ \mu_j \in \overline{N_2}, m(\lambda_i \wedge \mu_j) > 0\}.$$

If $\alpha = \{(i,j) :\ V_{ij} = \lambda_i \wedge \mu_j \in \overline{N_1 \vee N_2}\}$ then $\alpha = \bigcup_{i\in\mathbb{N}} \{(i,j) :\ j \in \beta_i\}$ where $\beta_i = \{j :\ m(V_{ij}) > 0\}$ and $i \in \mathbb{N}$. Note that if $j \notin \beta_i$ then $m(V_{ij}) = 0$ we have

$$\bigvee_{i,j\in\mathbb{N}} V_{ij} = \bigvee_{i\in\mathbb{N}} (\bigvee_{j\in\beta_i} V_{ij}) = \bigvee_{i\in\mathbb{N}} (\lambda_i \wedge (\bigvee_{j\in\beta_i} \mu_j)).$$

Since the collections of $\{\lambda_i :\ i \in \mathbb{N}\}$ and $\{\mu_j :\ j \in \mathbb{N}\}$ are m-disjoint, we have,

$$\begin{aligned}
m(\lambda_k \wedge (\bigvee_{i,j\in\mathbb{N}} V_{ij})) &= m(\lambda_k \wedge (\bigvee_{i\in\mathbb{N}} \lambda_i \wedge (\bigvee_{j\in\beta_i} \mu_j))) \\
&= m(\lambda_k \wedge (\bigvee_{j\in\beta_i} \mu_j)) \\
&= m(\lambda_k \wedge (\bigvee_{j\in\beta_k} \mu_j)) \\
&= m(\bigvee_{j\in\beta_k} (\lambda_k \wedge \mu_j)) \\
&= \sum_{j\in\beta_k} m(\vee V_{kj}) \\
&= \sum_{j\in\mathbb{N}} m(\vee V_{kj}) \\
&= m(\lambda_k \wedge (\bigvee_{j\in\mathbb{N}} \mu_j)) \\
&= m(\lambda_k).
\end{aligned}$$

□

Proposition 4.9. *Let (X, M, m) be a fuzzy probability measure space, and $N_1, N_2 \in F^*(M)$. If $N_1 \approx_m N_2$ then,*

$$H(N_1) \leq H(N_1 \vee N_2).$$

Proof. Suppose $N_1 \approx_m N_2$, by Proposition 4.8 we have $N_1 \approx_m N_1 \vee N_2$. Now suppose that $\theta \in \overline{N_1 \vee N_2}$ then $\theta = \lambda_i \wedge \mu_j$ where $\lambda_i \in \overline{N_1}$ and $\mu_j \in \overline{N_2}$. So for $\lambda_i \in \overline{N_1}$, $m(\theta) = M(\theta \wedge \lambda_i)$ and therefore we have $N_1 \leq_m N_1 \vee N_2$. Now use Proposition 4.6, (i). □

Definition 4.10. Let (X, M, m) be a fuzzy probability measure space and $N \in F^*(M)$. The diameter of N is defined as follows

$$\operatorname{diam} N = \sup_{\lambda_i \in \overline{N}} m(\lambda_i).$$

Definition 4.11. Let (X, M, m) be a fuzzy probability measure space and $N,\ C \in F^*(M)$, where $\overline{N} = \{\lambda_i :\ i \in \mathbb{N}\}$, $\quad \overline{C} = \{\gamma_k :\ k \in \mathbb{N}\}$. The conditional entropy of

N given C is defined as

$$\begin{aligned} H(N|C) &= -\log \sup_{i\in\mathbb{N}} \frac{\operatorname{diam}(\lambda_i \vee C)}{\operatorname{diam} C} \\ &= -\log \sup_{j\in\mathbb{N}} \frac{\operatorname{diam}(N \vee \mu_j)}{\operatorname{diam} C}. \end{aligned}$$

Proposition 4.12. *Let (X, M, m) be a fuzzy probability measure space, and N, C, $D \in F^*(M)$. Then,*

(i) $C \leq_m D \Rightarrow H(N|C) \leq H(N \vee D)$,
(ii) $H(N|C) \leq H(N \vee C)$,
(iii) $N \leq_m C \Rightarrow H(N|D) \leq H(C|D)$.

Proof. Suppose that $\overline{N} = \{\lambda_i : \ i \in \mathbb{N}\}$, $\overline{C} = \{\mu_j : \ j \in \mathbb{N}\}$ and $\overline{D} = \{\gamma_k : \ k \in \mathbb{N}\}$.

(i) Suppose that $C \leq_m D$, then we have,

$$\begin{aligned} H(C \vee N) \leq H(D \vee N) &\Rightarrow -\log \sup_{i\in\mathbb{N}} m(\lambda_i \wedge \mu_j) \leq -\log \sup_{i,k\in\mathbb{N}} m(\lambda_i \wedge \gamma_k) \\ &\Rightarrow \sup_{i,j\in\mathbb{N}} m(\lambda_i \wedge \mu_j) \geq \sup_{i,k\in\mathbb{N}} m(\lambda_i \wedge \gamma_k) \\ &\Rightarrow \sup_{i,j\in\mathbb{N}} m(\lambda_i \wedge \mu_j) \geq \sup_{i,k\in\mathbb{N}} m(\lambda_i \wedge \gamma_k) \operatorname{diam} C \\ &\Rightarrow H(N|C) \leq H(N \vee D). \end{aligned}$$

Note that $0 < \operatorname{diam} C \leq 1$.

(ii) Obvious.

(iii) Suppose $N \leq_m C$, then we have,

$$\begin{aligned} H(N \vee D) \leq H(C \vee D) &\Rightarrow -\log \sup_{i,k\in\mathbb{N}} m(\lambda_i \wedge \gamma_k) \leq -\log \sup_{k,j\in\mathbb{N}} m(\gamma_k \wedge \mu_j) \\ &\Rightarrow \sup_{k,j\in\mathbb{N}} m(\gamma_k \wedge \mu_j) \leq \sup_{i,k\in\mathbb{N}} m(\lambda_i \wedge \gamma_k) \\ &\Rightarrow \sup_{k,j\in\mathbb{N}} \frac{m(\gamma_k \wedge \mu_j)}{\operatorname{diam} D} \leq \sup_{i,k\in\mathbb{N}} \frac{m(\lambda_i \wedge \gamma_k)}{\operatorname{diam} D} \\ &\Rightarrow -\log \sup_{i,k\in\mathbb{N}} \frac{m(\lambda_i \wedge \gamma_k)}{\operatorname{diam} D} \leq -\log \sup_{k,j\in\mathbb{N}} \frac{m(\gamma_k \wedge \mu_j)}{\operatorname{diam} D} \\ &\Rightarrow H(N|D) \leq H(C|D). \end{aligned}$$

□

Proposition 4.13. *Suppose* (X, M, m) *is a fuzzy probability measure space, and* $N_1,\ N_2,\ N_3 \in F^*(M)$*. Then,*

$$H(N_1 \vee N_2 | N_3) = H(N_1 | N_2) + H(N_2 | N_1 \vee N_3).$$

Proof. Suppose that $\overline{N_1} = \{\lambda_i :\ i \in \mathbb{N}\},\ \overline{N_2} = \{\mu_j :\ j \in \mathbb{N}\}$ and $\overline{N_3} = \{\gamma_k :\ k \in \mathbb{N}\}$. We know that,

$$H(N_1 \vee N_2 | N_3) = -\log \sup_{i,j,k \in \mathbb{N}} \frac{m(\lambda_i \wedge \mu_j \wedge \gamma_k)}{\operatorname{diam} N_3}.$$

But we can write,

$$\frac{m(\lambda_i \wedge \mu_j \wedge \gamma_k)}{\sup_{k \in \mathbb{N}} m(\gamma_k)} = \frac{m(\lambda_i \wedge \mu_j \wedge \gamma_k)}{\sup_{i,k \in \mathbb{N}} m(\lambda_i \wedge \gamma_k)} \frac{\sup_{i,k \in \mathbb{N}} m(\lambda_i \wedge \gamma_k)}{\sup_{k \in \mathbb{N}} m(\gamma_k)},$$

and therefore the proof is obvious. □

5. Entropy of a Measure Preserving Transformation

Definition 5.1. Suppose $\varphi : X \to X$ is a fuzzy measure preserving transformation of the fuzzy probability measure space (X, M, m). If $N \in F^*(M)$, we define the entropy of φ with respect to N as

$$h(\varphi, N) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \varphi^{-i}(N)).$$

We say (X, M, m, φ) is a fuzzy dynamical system. It is of course necessary to establish that the limit above exists, but this is a consequence of subadditivity [1].

Proposition 5.2. *Suppose* $\varphi : (X, M, m) \to (Y, N, n)$ *is a fuzzy measure preserving transformation. Then for each* $L \in F^*(N)$ *we have*

$$H(L) = H(\varphi^{-1}(L)).$$

Proof. Since φ is measure preserving, for all $\mu \in \overline{L}$, we have

$$\begin{aligned} m(\varphi^{-1}(\mu)) = n(\mu) \Rightarrow H(\varphi^{-1}(L)) &= -\log \sup_{\mu \in \overline{L}} m(\varphi^{-1}(\mu)) \\ &= -\log \sup_{\mu \in \overline{L}} n(\mu) \\ &= H(L). \end{aligned}$$

□

Proposition 5.3. *Let (X, M, m, φ) be a fuzzy dynamical system and $N,\ C \in F^*(M)$. Then,*

(i) $N \leq_m C \Rightarrow h(\varphi, N) \leq h(\varphi, C)$,

(ii) $h(\varphi, \varphi^{-1}(N)) = h(\varphi, N)$,

(iii) $h(\varphi, \bigvee_{i=0}^{r-1} \varphi^{-i}(N)) = h(\varphi, N)$ *for every* $r \geq 1$,

(iv) *if* $N_1,\ N_2 \in F^*(M)$ *such that* $N_1 \approx_m N_2$ *then,*

$$\varphi^{-1}(N_1) \approx_m \varphi^{-1}(N_2).$$

Proof. (i) Follows from Proposition 4.3 and Proposition 4.6, (i).

(ii) Obvious.

(iii)

$$\begin{aligned} h(\varphi, \bigvee_{i=1}^{\infty} \varphi^{-i}(N)) &= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{j=0}^{n-1} \varphi^{-j}(\bigvee_{i=0}^{r-1} \varphi^{-i}(N))) \\ &= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{r+n-2} \varphi^{-i}(N)) \\ &= \lim_{n\to\infty} (\frac{r+n-2}{n})(\frac{1}{r+n-2}) H(\bigvee_{i=0}^{r+n-2} \varphi^{-i}(N)) \\ &= h(\varphi, \varphi(N)). \end{aligned}$$

(iv) Let $\varphi^{-1}(\mu) \in \overline{\varphi^{-1}(N_2)}$ such that $\mu \in \overline{N_2}$. Then,

$$\begin{aligned} m(\varphi^{-1}(\mu) \wedge (\bigvee\{\varphi^{-1}(\lambda) :\ \lambda \in \overline{N_1}\})) &= m(\varphi^{-1}(\mu \wedge (\bigvee\{\lambda :\ \lambda \in \overline{N_1}\}) \\ &= n(\mu \wedge (\bigvee\{\lambda :\ \lambda \in \overline{N_1}\})) \\ &= n(\mu) \\ &= m(\varphi^{-1}(\mu)). \end{aligned}$$

The proof of

$$m(\varphi^{-1}(\lambda) \wedge (\bigvee\{\varphi^{-1}(\mu) :\ \mu \in \overline{N_2}\})) = m(\varphi^{-1}(\lambda)),$$

where $\varphi^{-1}(\lambda) \in \overline{\varphi^{-1}(N_1)}$ is similar. □

6. Entropy and m-Isomorphic Dynamical Systems

Definition 6.1. Let (X, M, m, φ) be a fuzzy dynamical system and $L \in F^*(M)$. Suppose $[L]$ denotes the m-equivalence class induced by L. Then the entropy

$h(\varphi, [L])$ of φ on L is defined as

$$h(\varphi, [L]) = \sup_{N \in [L]} h(\varphi, N).$$

Definition 6.2. A fuzzy dynamical system $\phi_1 = (X_1, M_1, m_1, \varphi_1)$ is a factor of fuzzy dynamical system $\phi_2 = (X_2, M_2, m_2, \varphi_2)$ if there exists an onto fuzzy measure preserving transformation (called homomorphism) $\psi : \phi_2 \to \phi_1$ such that,

$$\psi \circ \varphi_2 = \varphi_1 \circ \psi,$$

and for each $\mu \in \overline{M_1}$,

$$m_1(\mu) = m_2(\psi^{-1}(\mu)).$$

Proposition 6.3. *Let $\phi_1 = (X_1, M_1, m_1, \varphi_1)$ be a factor of fuzzy dynamical system $\phi_2 = (X_2, M_2, m_2, \varphi_2)$, then for each $L \in F^*(M_1)$,*

$$h(\phi_1, [L]) \leq h(\phi_2, [\psi^{-1}(L)]),$$

where $\psi : \phi_2 \to \phi_1$ is the corresponding homomorphism.

Proof. Suppose that $N \in [L]$. Then by Proposition 5.4, $H(N) = H(\psi^{-1}(N))$. Now,

$$\begin{aligned} h(\phi_1, N) &= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \phi_1^{-i}(N)) \\ &= \lim_{n\to\infty} \frac{1}{n} H(\psi^{-1}(\bigvee_{i=0}^{n-1} \phi_1^{-i}(N))) \\ &= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \psi^{-1}\phi_1^{-i}(N)) \\ &= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \phi_2^{-i}\psi^{-1}(N)) \\ &= h(\phi_2, \psi^{-1}(N)). \end{aligned}$$

As N ranges over an m-equivalence class $[L]$ in $F^*(M_1)$, $\psi^{-1}(N)$ ranges over a subset of the m-equivalence class $[\psi^{-1}(L)]$ in $F^*(M_2)$. □

Definition 6.4. Two dynamical systems $\phi_1 = (X_1, M_1, m_1, \varphi_1)$ and $\phi_2 = (X_2, M_2, m_2, \varphi_2)$ are said to be m-isomorphic if there exists an invertible fuzzy measure preserving transformation $\psi : \phi_1 \to \phi_2$ (i.e both ψ and ψ^{-1} are fuzzy measure preserving transformations) such that,

$$\psi \circ \varphi_1 = \varphi_2 \circ \psi.$$

The mapping ψ is called m-isomorphism.

Proposition 6.5. *Suppose $\phi_1 = (X_1, M_1, m_1, \varphi_1)$ and $\phi_2 = (X_2, M_2, m_2, \varphi_2)$ are m-isomorphic dynamical systems and φ_1 is an ergodic fuzzy transformation. Then φ_2 is also ergodic.*

Proof. Let $\mu \in \overline{M_2}$; ${\varphi_2}^{-1}(\mu) = \mu$. By definition there exists an invertible fuzzy measure preserving transformation ψ of ϕ_1 onto ϕ_2 such that,

$$\psi \circ \varphi_1 = \varphi_2 \circ \psi.$$

But $\psi^{-1}(\mu) = \gamma \in \overline{M_1}$, and,

$$\begin{aligned} {\varphi_2}^{-1}(\mu) &= {\varphi_2}^{-1}(\psi(\gamma)) \\ &= \psi \circ {\varphi_1}^{-1}(\mu) \\ &= \psi(\gamma). \end{aligned}$$

So we have

$$\begin{aligned} {\varphi_1}^{-1}(\gamma) = \gamma &\Rightarrow m_1(\gamma) = 0 \text{ or } 1 \\ &\Rightarrow m_1(\psi^{-1}(\mu)) = 0 \text{ or } 1 \\ &\Rightarrow m_2(\mu) = 0 \text{ or } 1. \end{aligned}$$

. □

Proposition 6.6. *Let $\phi_1 = (X_1, M_1, m_1, \varphi_1)$ and $\phi_2 = (X_2, M_2, m_2, \varphi_2)$ be m-isomorphic dynamical systems and φ_1 be weak mixing. Then φ_2 is also a weak mixing.*

Proof. Since φ_1 is weak mixing then we have that for any $\mu, \lambda \in \overline{M_1}$,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k-1} |m_1({\varphi_1}^{-n}(\mu) \wedge \lambda) - m_1(\mu) m_2(\lambda)| = 0.$$

We prove that for any $\eta, \nu \in \overline{M_2}$ we have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k-1} |m_1({\varphi_1}^{-n}(\eta) \wedge \nu) - m_1(\eta) m_2(\nu)| = 0.$$

Since ϕ_1 and ϕ_2 are m-isomorphic, there is an invertible fuzzy measure preserving transformation ψ such that $\psi \circ \varphi_1 = \varphi_2 \circ \psi$ we have

$$\psi^{-1} \circ \varphi_2^{-n} = \varphi_1^{-n} \circ \psi^{-1}.$$

Since ψ is surjective and measure preserving, $\psi^{-1}(\eta) \in \overline{M_1}$, $\psi^{-1}(\nu) \in \overline{M_1}$. Suppose that $\psi^{-1}(\eta) = \mu$, $\psi^{-1}(\nu) = \lambda$, then,

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k-1} |m_1({\varphi_1}^{-n}(\mu) \wedge \lambda) - m_1(\mu)m_1(\lambda)|& \\
&= \lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k-1} |m_1(\psi^{-1}((\varphi_2^{-n}(\eta) \wedge \nu) - m_1(\psi^{-1}(\eta))m_1(\psi^{-1}(\nu))| \\
&= \lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k-1} |m_2({\varphi_2}^{-n}(\eta) \wedge \nu) - m_2(\eta)m_2(\nu)| \\
&= 0.
\end{aligned}$$

□

Proposition 6.7. *Let ϕ_1 and ϕ_2 be m-isomorphic dynamical systems. Then for each $L \in F^*(M)$,*

$$h(\varphi_1, [L]) = h(\varphi_2, [\psi^{-1}(L)]),$$

where $\psi : \phi_1 \to \phi_2$ is the corresponding m-isomorphism. In the other words $h(\varphi, [L])$ is m-isomorphism invariant.

Proof. Follows from Proposition 6.4. □

7. Entropy and m-Generators of Fuzzy Dynamical Systems

Definition 7.1. The entropy of the fuzzy dynamical system (X, M, m, φ) is the number $h(\varphi)$ defined by:

$$h(\varphi) = \sup_{\xi} h(\varphi, \xi),$$

where the supremum is taken over all sub-σ-algebras of M where $\xi \in F^*(M)$.

Definition 7.2. $\xi \in F^*(M)$ is said to be a fuzzy m-generator of the fuzzy dynamical system (X, M, m, φ) if there exists an integer $r > 0$ such that,

$$\eta \leq_m \bigvee_{i=0}^{r} \varphi^{-i}\xi,$$

for each $\eta \in F^*(M)$.

Proposition 7.3. *If ξ is a m-generator of the fuzzy dynamical system (X, M, m, φ) then,*

$$h(\varphi, \eta) \leq h(\varphi, \xi),$$

for each $\eta \in F^*(M)$.

Proof. Let $\eta \in F^*(M)$ be any arbitrary sub-σ-algebra of M. Since ξ, is an m-generator, $\eta \leq_m \bigvee_{i=0}^{r} \varphi^{-i}\xi$ from Proposition 5.3, (iii),

$$h(\varphi,\eta) \leq h(\varphi, \bigvee_{i=0}^{r} \varphi^{-i}\xi) = h(\varphi,\xi).$$

□

Now we can deduce the following version of Kolmogorov-Sinai proposition.

Proposition 7.4. *If* ξ *is an* m*-generator of fuzzy dynamical system* (X, M, m, φ) *then,*

$$h(\varphi) = h(\varphi,\xi).$$

Proof. Obvious. □

8. Concluding Remarks and Open Problems

In this paper we investigate the ergodic properties of fuzzy dynamical systems using the concept of atoms in a fuzzy σ-algebra. In this respect we introduce the m-generators of fuzzy dynamical systems. We have to consider a slight modification of some previously defined notions. A fuzzy version of Kolmogorov-Sinai proposition concerning the entropy of fuzzy dynamical system is given. This proposition enables us to compute the entropy for a class of fuzzy systems.

An interesting open problem is to establish a proposition on existence of m-generators having finite entropy.

References

[1] D. Dumitrescu, C. Hăloiu and A. Dumitrescu, Generators of fuzzy dynamical systems, *Fuzzy Sets and Systems* **113** (2000), 447–452.

[2] D. Dumitrescu, Entropy of a fuzzy dynamical system, *Fuzzy Sets and Systems* **70** (1995), 45–57.

[3] D. Dumitrescu, Entropy of a fuzzy process, *Fuzzy Sets and Systems* **55** (1993), 169–177.

[4] M. Ebrahimi, Generators of probability dynamical systems, *Differential Geometry-Dynamical Systems* **8** (2006), 90–97.

[5] M. Ebrahimi and N. Mohamadi, The entropy function on an algebraic structure with infinite partition and m-preserving transformation generators, *Applied Sciences* **12** (2010), 48–63.

[6] A. N. Kolmogorov, Entropy per unit time as a metric invariant of automorphism, *Doklady of Russian Academy of Sciences* **124** (1959), 754–755.

[7] P. Serivastava, M. Khare and Y. K. Srivastava, A fuzzy measure algebra on a metric space, *Fuzzy Sets and Systems* **79** (1996), 395–400.

[8] P. Serivastava, M. Khare and Y. K. Srivastava, m-Equivalence, entropy and F-dynamical systems, *Fuzzy Sets and Systems* **121** (2001), 275–283.

[9] Ya. Sinai, On the notion of entropy of a dynamical system, *Doklady ofRussian Academy of Sciences* **124** (1959), 768–771.

[10] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, 1982.