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Research Article

Radii of convexity associated with various subclasses of analytic functions for functions related to Kaplan classes

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Abstract

A normalized analytic function f defined on the open unit disc $\mathbb{D}$ is called Ma-Minda convex if $1+zf''(z)/f'(z)$ is subordinate to the function φ. For $0 \leqslant \alpha \leqslant \beta$, the Kaplan class $\mathcal{K}(\alpha,\beta)$ of type α and β consists of normalized analytic functions of the form $p^{\alpha}g$ defined on $\mathbb{D}$ where p with $p(0)=1$ is an analytic function taking values in the right half-plane and g is an analytic function with $g(0)=1$ satisfying $\operatorname{Re}(zg'(z)/g(z)) > (\alpha-\beta)/2$. For functions f with $f' \in \mathcal{K}(\alpha,\beta)$, we obtain the radius of Ma-Minda convexity for various choices of φ. The radius of lemniscate convexity, lune convexity, nephroid convexity, exponential convexity and several other radius estimates are examined. The results obtained are sharp.

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1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f defined on the open unit disc $\mathbb{D}$ and normalized by the conditions $f(0)=0$, $f'(0)=1$. Let $\mathcal{A}_0$ be the class of functions g such that $zg \in \mathcal{A}$, this class consists of analytic functions with $g(0)=1$. The set of univalent (one-to-one) functions in the class $\mathcal{A}$ is symbolized as $\mathcal{S}$. For $0 \leqslant \alpha < 1$, let $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ denote the subclasses of the class $\mathcal{A}$ consisting of starlike and convex functions respectively such that $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ and $\operatorname{Re}(1+(zf''(z)/f(z))) > \alpha$. For $\alpha = 0$, these classes reduces to the class of Starlike and convex functions denoted by $\mathcal{ST}$ and $\mathcal{CV}$, respectively. Many classes in geometric function theory can be unified using the so called Kaplan classes. Define

$$\mathcal{H} := \{f \in \mathcal{A}_0 : \text{for some} \quad \delta \in \mathbb{R}, \operatorname{Re} e^{i\delta} f(z) > 0, z \in \mathbb{D}\}.$$

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For $\alpha > 0$, we define $\mathcal{H}^\alpha$ to be the class of functions $f^\alpha \in \mathcal{A}_0$ such that $f \in \mathcal{H}$. Let

$$\mathcal{K}(0,\gamma) = \left\{ g \in \mathcal{A}_0 : \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) > -\frac{\gamma}{2}, \gamma \geqslant 0 \right\}.$$

Let the Kaplan class $\mathcal{K}(\alpha,\beta)$ of type α and β be defined by

$$\mathcal{K}(\alpha,\beta) = \mathcal{H}^\alpha \cdot \mathcal{K}(0,\beta-\alpha) \quad \text{if} \quad 0 \leqslant \alpha \leqslant \beta.$$

For $0 \leqslant \beta \leqslant \alpha$, the Kaplan class is defined by

$$\mathcal{K}(\alpha,\beta) = \left\{ g : g(z) = \frac{1}{f(z)}, f \in \mathcal{K}(\beta,\alpha) \right\}.$$

However, we restrict our results to the case where $0 \leqslant \alpha \leqslant \beta$.

Kaplan classes were first introduced by Sheil-Small [25] and were later discussed by Ruscheweyh [21]. In 1952, Kaplan [9] introduced the class $\mathcal{CC}$ of close to convex functions consisting of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad (z \in \mathbb{D})$$

for some (not necessarily normalized) starlike function g or equivalently,

$$\operatorname{Re}\left(\frac{f'(z)}{h'(z)}\right) > 0 \quad (z \in \mathbb{D})$$

for some (not necessarily normalized) convex function h. The sets $\mathcal{K}(\alpha,\beta)$ are called Kaplan classes because of their close relation with close to convex functions: $f \in \mathcal{CC} \Leftrightarrow f' \in \mathcal{K}(1,3)$. In addition to above, we have some more interesting relations between Kaplan classes and various subclasses of $\mathcal{S}$. The class of strongly close to convex functions of order α, denoted by $\mathcal{SCC}(\alpha)$, contains functions $f \in \mathcal{A}$ such that $|\arg e^{i\phi} zf'(z)/g(z)| \leqslant \alpha\pi/2$ for some $g \in \mathcal{ST}$ and $\phi \in \mathbb{R}$. It is easy to see that

$$f \in \mathcal{SCC}(\alpha) \Leftrightarrow f' \in \mathcal{K}(\alpha, \alpha+2).$$

Another generalisation of class of close to convex functions, denoted by $\mathcal{CC}(\sigma,\lambda)$, $\mathcal{SCC}(1) = \mathcal{CC}$ and contains $f \in \mathcal{A}$ such that $\operatorname{Re}(e^{i\phi} zf'(z)/g(z)) > \lambda$ for some $\phi \in \mathbb{R}$ and $g \in \mathcal{ST}(\sigma)$, $0 \leqslant \sigma \leqslant 1$, $z \in \mathbb{D}$. For $\lambda = 0$, we have $f \in \mathcal{CC}(\sigma, 0) \Leftrightarrow f' \in \mathcal{K}(1, 3-2\sigma)$. Another class which can be related with Kaplan classes is the class of functions with bounded boundary rotation, V_k. Paatero [17, 18] introduced the class V_k consisting of $f \in \mathcal{A}$ satisfying

$$\int_0^{2\pi} \left| \operatorname{Re}\left\{ 1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \right\} \right| d\theta \leqslant k\pi, \quad k \geqslant 2.$$

From [20, Equation(3.4)], we have

$$f \in V_k \Rightarrow f' \in \mathcal{K}(k/2-1, k/2+1), \qquad k > 2$$

and $V_2 = \mathcal{CV}(0)$.

For two subclasses $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$, the $\mathcal{G}$-radius for the class $\mathcal{F}$, denoted by $\mathcal{R}_{\mathcal{G}}(\mathcal{F})$, is the maximum value $R \in (0,1]$ such that $r^{-1} f(rz) \in \mathcal{G}$ holds for all $f \in \mathcal{F}$ and for $0 < r < R$. Various radius problems were recently investigated in [3, 5, 8, 11–13, 22]. The class $\mathcal{K}(\alpha,\beta)$ was investigated by Ravichandran *et al.* [20], where they determined the radius of convexity and radius of uniform convexity for functions $f \in \mathcal{A}$ satisfying $f' \in \mathcal{K}(\alpha,\beta)$. Motivated by their work, our present study seeks to determine the radius of Ma-Minda convexity of functions f with $f' \in \mathcal{K}(\alpha,\beta)$. Ma-Minda classes are defined by using subordination. For two analytic functions f and g, we say that f is subordinate to g, written as $f \prec g$, if there exists a self-map w on $\mathbb{D}$ such that $w(0) = 0$ and $f(z) = g(w(z))$.

If the function g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For an analytic function φ with $\varphi(0) = 1$, the classes

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z), z \in \mathbb{D} \right\} \tag{1.1}$$

and

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z), z \in \mathbb{D} \right\}, \tag{1.2}$$

are called Ma and Minda starlike/convex functions respectively as they [14] obtained growth and distortion theorems where φ is a function with positive real part whose range is symmetric about the real axis and starlike with respect to the origin. These classes unifies various classes of starlike functions and convex functions. For example, when φ is a mapping of $\mathbb{D}$ onto the right-half plane, the classes $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ reduce to the usual class $\mathcal{ST}$ of starlike functions and the class $\mathcal{CV}$ of convex functions respectively. When φ is the mapping of $\mathbb{D}$ onto the half-plane $\{w : \operatorname{Re} w > \alpha, 0 \leqslant \alpha < 1\}$, then the classes $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ reduce to the usual class $\mathcal{ST}(\alpha)$ of starlike functions of order α and the class $\mathcal{CV}(\alpha)$ of convex functions of order α respectively. We determine Ma-Minda convexity when φ is given by $\varphi(z) = 1 + z - (z^3/3)$, $1 + \sinh^{-1}(z)$, $\varphi(z) = 2/(1 + e^{-z})$, $\varphi(z) = 1 + \sin z$, $\varphi(z) = \sqrt{1 + z}$, $\varphi(z) = e^z$, and other functions having nice geometry. Our results generalizes the results of Sebastian and Ravichandran [22] who studied analytic functions f defined on $\mathbb{D}$ for which f/g and $(1 + z)g/z$ are both functions with positive real part for some analytic function g. For $\alpha = 2$ and $\beta = 3$, our findings yield radius constants for the functions belonging to class $\mathcal{F}_1$. This class consists of functions $f \in \mathcal{A}$ satisfying $f/g \in \mathcal{P}$ for some $g \in \mathcal{A}$ with $(1+z)g(z)/z \in \mathcal{P}$. Similarly, for $\alpha = 1$ and $\beta = 2$, our results deduce to the radius constants for the functions in class $\mathcal{F}_3$ which includes functions $f \in \mathcal{A}$ satisfying the inequality $\operatorname{Re}(((1 + z)/z)f(z)) > 0$ for $z \in \mathbb{D}$. Moreover, for $\alpha = 1$ and $\beta = 3$, the radius constants we have obtained reduces to the results for functions in the class $\mathcal{F}_4$ which consists of functions $f \in \mathcal{A}$ that satisfies the inequality $\operatorname{Re}(((1 + z)^2/z)f(z)) > 0$ for $z \in \mathbb{D}$.

2. The image of $|z| \leqslant r$ under $1 + zf''(z)/f'(z)$ for $f' \in \mathcal{K}(\alpha, \beta)$

The radius of convexity of a function $f \in \mathcal{A}$ is determined by finding the largest $r < 1$ that satisfy the inequality $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ for all z with $|z| \leqslant r$. Similarly, other radius of convexity are determined by finding largest $r < 1$ such that $1 + zf''(z)/f'(z)$ lies in certain region for all z with $|z| < r$. Thus, we need to know the region to which $1 + zf''(z)/f'(z)$ maps the disc $|z| \leqslant r$. We now describe the region in the case of $f' \in \mathcal{K}(\alpha, \beta)$ for $0 \leqslant \alpha \leqslant \beta$.

We first note that for $0 \leqslant \alpha < 1$, the class $\mathcal{P}(\alpha)$ consists of functions p with $p(0) = 1$ such that $\operatorname{Re}(p(z)) > \alpha$. The class $\mathcal{P} := \mathcal{P}(0)$ is the class of Carathéodory functions or the class of functions with positive real part. If $q \in \mathcal{P}(\alpha)$, then we have (see [20])

$$\left| q(z) - \frac{1 + (1 - \alpha)r^2}{1 - r^2} \right| \leqslant \frac{2(1 - \alpha)r}{1 - r^2}, \quad |z| \leqslant r < 1. \tag{2.1}$$

Lemma 2.1. *[20] For $0 \leqslant \alpha \leqslant \beta$, we have $f' \in \mathcal{K}(\alpha, \beta)$ if and only if there exists $g \in \mathcal{ST}((2 - \beta + \alpha)/2)$ and $\phi \in \mathbb{R}$ such that*

$$\left| \arg e^{i\phi} \frac{zf'(z)}{g(z)} \right| \leqslant \frac{\alpha\pi}{2}, \qquad z \in \mathbb{D}. \tag{2.2}$$

Define the function p by $p(z) = e^{i\phi} zf'(z)/g(z)$ where $f' \in \mathcal{K}(\alpha, \beta)$ and $g \in \mathcal{ST}((2 - \beta + \alpha)/2)$. We note that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}. \tag{2.3}$$

Since $g \in \mathcal{ST}((2-\beta+\alpha)/2)$, we have $zg'/g \in \mathcal{P}((2-\beta+\alpha)/2)$, and by (2.1)

$$\left|\frac{zg'(z)}{g(z)} - \frac{1+(\beta-\alpha-1)r^2}{1-r^2}\right| \leqslant \frac{(\beta-\alpha)r}{1-r^2}, \quad |z| = r < 1. \tag{2.4}$$

Note that for any $q \in \mathcal{P}$, we have (see [15])

$$\left|\frac{zq'(z)}{q(z)}\right| \leqslant \frac{2r}{1-r^2}, \quad |z| \leqslant r < 1. \tag{2.5}$$

Using (2.2), we have $p^{1/\alpha} \in \mathcal{P}$. Hence, by (2.5) we have

$$\left|\frac{zp'(z)}{p(z)}\right| \leqslant \frac{2\alpha r}{1-r^2}, \quad |z| \leqslant r < 1. \tag{2.6}$$

From (2.3), (2.4) and (2.6), we get

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{1+(\beta-\alpha-1)r^2}{1-r^2}\right| \leqslant \frac{(\alpha+\beta)r}{1-r^2}, \quad |z| \leqslant r < 1. \tag{2.7}$$

From (2.7), it is evident that $w = 1+(zf''(z)/f'(z))$ resides within the disc $|w-a(r)| \leqslant r_a$, where $a(r)$ and $r_a(r)$ denote the center and the radius respectively:

$$a(r) := \frac{1+(\beta-\alpha-1)r^2}{1-r^2} \quad \text{and} \quad r_a(r) := \frac{(\alpha+\beta)r}{1-r^2}. \tag{2.8}$$

We note that the center $a(r)$ is an increasing function of r and $a(r) \geqslant 1$ as $\beta \geqslant \alpha \geqslant 0$.

3. Radius of Convexity

In this section we obtain various radius constants for the functions whose derivatives belong to the class $\mathcal{K}(\alpha,\beta)$. Wani and Swaminathan [28] studied the class $\mathcal{CV}_{Ne} = \mathcal{CV}(\varphi_{Ne})$ which consists of convex functions associated with a nephroid domain, where $\varphi_{Ne}(z) = 1+z-(z^3/3)$ that maps the unit circle onto a 2-cusped curve, $((u-1)^2+v^2-(4/9))^3 - (4v^2/3) = 0$.

Theorem 3.1. *Let $0 \leqslant \alpha \leqslant \beta$. The sharp $\mathcal{CV}_{Ne}$ radius for functions whose derivative belongs to $\mathcal{K}(\alpha,\beta)$ is given by*

$$\mathcal{R}_{\mathcal{CV}_{Ne}} := \frac{4}{3(\alpha+\beta)+\sqrt{9(\alpha+\beta)^2+8(3(\beta-\alpha)+2)}}.$$

Proof. We first note that $R = \mathcal{R}_{\mathcal{CV}_{Ne}}$ is the smallest positive root of the equation $(3(\beta-\alpha)+2)r^2+3(\alpha+\beta)r-2 = 0$ in the interval $(0,1)$. It then follows that $(3(\beta-\alpha)+2)r^2+3(\alpha+\beta)r-2 \leqslant 0$ for $0 \leqslant r \leqslant R$ and by rewriting it, we have

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \frac{5}{3} - \frac{1+(\beta-\alpha-1)r^2}{1-r^2}, \quad r \leqslant R.$$

Consequently, for $0 \leqslant r \leqslant R = \mathcal{R}_{\mathcal{CV}_{Ne}}$, the disc in (2.7) becomes

$$\left|1 + \frac{zf''(z)}{f'(z)} - a(r)\right| \leqslant \frac{5}{3} - a(r) \tag{3.1}$$

where $a(r)$ is given by (2.8). By [27, Lemma 2.1], we have $\{w : |w-a| < 5/3-a\} \subseteq \varphi_{Ne}(\mathbb{D}) = \Omega_{Ne}$, for $1 < a < 5/3$. The condition $a < 5/3$ ensures that the center lies inside the nephroid. Since the center $a(r)$ given by (2.8) is always greater than or equal to 1, the disc in (3.1) lies inside the region Ω_{Ne} for $r \leqslant R$ proving that the $\mathcal{CV}_{Ne}$ radius for functions whose derivative belongs to $\mathcal{K}(\alpha,\beta)$ is at least R.

To show that sharpness, consider the function f_0 defined by

$$f_0'(z) = \frac{(1+z)^\alpha}{(1-z)^\beta}. \tag{3.2}$$

This function f_0 has derivative $f_0' \in \mathcal{K}(\alpha,\beta)$. Indeed, with $g_0(z) = z/(1-z)^{\beta-\alpha}$, we have $\operatorname{Re}(zg_0'(z)/g_0(z)) = (1+(\beta-\alpha-1)z)/(1-z)$ and hence $g_0 \in \mathcal{ST}((2-\beta+\alpha)/2)$. Also we have $|\arg(zf_0'(z)/g_0(z))| = |\arg(((1+z)/(1-z))^{\alpha})| < (\alpha\pi)/2$. Since f_0 satisfies (2.2), it follows that $f_0' \in \mathcal{K}(\alpha,\beta)$. From (3.2), we find

$$1+\frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)z^2+(\alpha+\beta)z}{1-z^2}. \tag{3.3}$$

For $z = R$, the equation (3.3) gives

$$1+\frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R^2+(\alpha+\beta)R}{1-R^2} = \frac{5}{3} = \varphi_{Ne}(1).$$

Thus, the radius R is sharp for the function f_0 defined by (3.2) (See Fig.1) (This figure as well as other later figures show the image of the extremal function f_0, and the disc given by (2.7) as well as the boundary of $\varphi(\mathbb{D})$). □

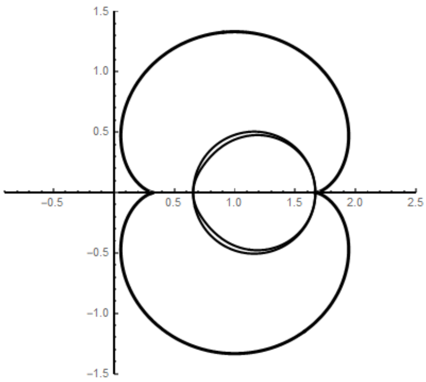


Figure 1. Sharpness of $\mathcal{R}_{\mathcal{CV}_{Ne}}$ radius when $\alpha = 0.1$ and $\beta = 1$

Corollary 3.2. *Radius of convexity associated with the class $\mathcal{CV}_{Ne}$ for some special cases:*

(1) *The $\mathcal{CV}_{Ne}$ radius for the class $\mathcal{CC}(\sigma,0)$ is*

$$\mathcal{R}_{\mathcal{CV}_{Ne}} = \frac{4}{3(2-\sigma)+\sqrt{9\sigma^2-38\sigma+54}}.$$

(2) *The $\mathcal{CV}_{Ne}$ radius for the class V_k is*

$$\mathcal{R}_{\mathcal{CV}_{Ne}} = \frac{4}{3k+\sqrt{9k^2+64}}.$$

(3) *The $\mathcal{CV}_{Ne}$ radius for the class $\mathcal{SCC}(\alpha)$ is*

$$\mathcal{R}_{\mathcal{CV}_{Ne}} = \frac{2}{3(\alpha+1)+\sqrt{9(\alpha+1)^2+16}}.$$

Kumar and Arora [2] introduced the class which consists of convex functions associated with the petal shaped domain, given by $\mathcal{CV}_h = \mathcal{CV}(\varphi_h)$, where $\varphi_h(z) = 1+\sinh^{-1}(z)$.

Theorem 3.3. *Let $0 \leqslant \alpha \leqslant \beta$. The $\mathcal{CV}_h$ radius for functions whose derivative belongs to $\mathcal{K}(\alpha,\beta)$ is given by*

$$\mathcal{R}_{\mathcal{CV}_h} = \frac{2\sinh^{-1}(1)}{(\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4(\beta-\alpha+\sinh^{-1}(1))(\sinh^{-1}(1))}}.$$

The radius obtained is sharp.

Proof. We first note that $R = \mathcal{R}_{\mathcal{CV}_h}$ is the smallest positive root of the equation $(\beta - \alpha + \sinh^{-1}(1))r^2 + (\alpha+\beta)r - \sinh^{-1}(1) = 0$ in the interval $(0,1)$. Then it follows that $(\beta - \alpha + \sinh^{-1}(1))r^2 + (\alpha+\beta)r - \sinh^{-1}(1) \leqslant 0$ for $0 \leqslant r \leqslant R$ and hence

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant 1 + \sinh^{-1}(1) - \frac{1+(\beta-\alpha-1)r^2}{1-r^2}, \quad r \leqslant R.$$

Consequently, for $0 \leqslant r \leqslant R = \mathcal{R}_{\mathcal{CV}_h}$, the disc in (2.7) becomes

$$\left|1 + \frac{zf''(z)}{f'(z)} - a(r)\right| \leqslant 1 + \sinh^{-1}(1) - a(r) \tag{3.4}$$

where $a(r)$ is given by (2.8). For $1 < a \leqslant 1 + \sinh^{-1}(1)$, by [2, Lemma 2.1] we have $\{w : |w - a| < 1 + \sinh^{-1}(1) - a\} \subseteq \varphi_h(\mathbb{D}) = \Omega_h$. The condition $a < 1 + \sinh^{-1}(1)$ tells us that the center lies inside the region Ω_h. Since the center $a(r)$ is always greater than 1, the disc in (3.4) lies inside the region Ω_h for $r \leqslant R$ proving that $\mathcal{CV}_h$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ is at least R.

To prove the sharpness of the radius $R = \mathcal{R}_{\mathcal{CV}_h}$, we consider the function f_0 defined by (3.2). For $z = R$ in (3.3), we have

$$1 + \frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R^2+(\alpha+\beta)R}{1-R^2} = 1 + \sinh^{-1}(1) = \varphi_h(1).$$

Therefore, the radius R is sharp for the function f_0 defined by (3.2) (See Fig.2). □

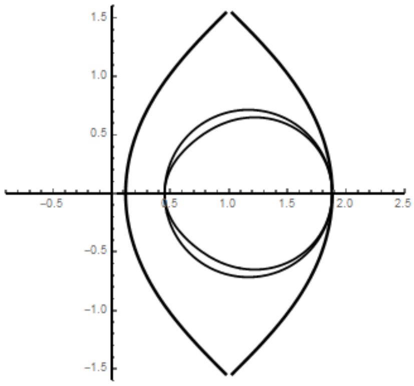


Figure 2. Sharpness of $\mathcal{R}_{\mathcal{CV}_h}$ radius when $\alpha = 0.3$ and $\beta = 1.5$

Corollary 3.4. *Radius of convexity associated with the class $\mathcal{CV}_h$ for some special cases:*

(1) *The $\mathcal{CV}_h$ radius for the class $\mathcal{CC}(\sigma, 0)$ is*

$$\mathcal{R}_{\mathcal{CV}_h} = \frac{\sinh^{-1}(1)}{(2-\sigma) + \sqrt{(2-\sigma)^2 + (2(1-\sigma)+\sinh^{-1}(1))(\sinh^{-1}(1))}}.$$

(2) *The $\mathcal{CV}_h$ radius for the class V_k is*

$$\mathcal{R}_{\mathcal{CV}_h} = \frac{2\sinh^{-1}(1)}{k + \sqrt{k^2 + 4(\sinh^{-1}(1))(2+\sinh^{-1}(1))}}.$$

(3) *The $\mathcal{CV}_h$ radius for the class $\mathcal{SCC}(\alpha)$ is*

$$\mathcal{R}_{\mathcal{CV}_h} = \frac{\sinh^{-1}(1)}{(\alpha+1) + \sqrt{(\alpha+1)^2 + (2+\sinh^{-1}(1))(\sinh^{-1}(1))}}.$$

Goel and Kumar [6] introduced the class which consists of convex functions associated with the modified sigmoid, given by $\mathcal{CV}_{SG} = \mathcal{CV}(\varphi_{SG})$, where $\varphi_{SG}(z) = 2/(1+e^{-z})$.

Theorem 3.5. *Let $0 \leqslant \alpha \leqslant \beta$. The sharp $\mathcal{CV}_{SG}$ radius for functions f such that $f' \in \mathcal{K}(\alpha, \beta)$ is given by*

$$\mathcal{R}_{\mathcal{CV}_{SG}} = \frac{2(e-1)}{(\alpha+\beta)(e+1) + \sqrt{(e+1)^2(\alpha+\beta)^2 + 4(e-1)((e+1)(\beta-\alpha)+e-1)}}.$$

Proof. We first note that $R = \mathcal{R}_{\mathcal{CV}_{SG}}$ is the smallest positive root of the equation $((e+1)(\beta-\alpha)+e-1)r^2 + (e+1)(\alpha+\beta)r - e + 1 = 0$ in $(0,1)$. Then it follows that $((e+1)(\beta-\alpha)+e-1)r^2 + (e+1)(\alpha+\beta)r - e + 1 \leqslant 0$ for $0 \leqslant r \leqslant R$ and so

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \frac{e-1}{e+1} + 1 - \frac{1+(\beta-\alpha-1)r^2}{1-r^2}, \quad r \leqslant R.$$

Consequently, for $0 \leqslant r \leqslant R = \mathcal{R}_{\mathcal{CV}_{SG}}$, the disc in (2.7) becomes

$$\left|1 + \frac{zf''(z)}{f'(z)} - a(r)\right| \leqslant \frac{e-1}{e+1} + 1 - a(r) \tag{3.5}$$

where $a(r)$ is given by (2.8). By [6, Lemma 2.2], we have $\{w : |w-a| < ((e-1)/(e+1)) + 1 - a\} \subseteq \varphi_{SG}(\mathbb{D}) = \Omega_{SG}$, for $2/(1+e) < a < 2e/(1+e)$. The condition $a < 2e/(1+e)$ assures that the center lies inside the region Ω_{SG}. As the center $a(r)$ is always greater than 1, the disc in (3.5) lies inside the region Ω_{SG} for $r \leqslant R$ proving that $\mathcal{CV}_{SG}$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha, \beta)$ is at least R.

To prove the sharpness of the radius $R = \mathcal{R}_{\mathcal{CV}_{SG}}$, consider the function f_0 defined by (3.2). For $z = R$ in (3.3), we have

$$1 + \frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R^2+(\alpha+\beta)R}{1-R^2} = \frac{2e}{e+1} = \varphi_{SG}(1),$$

which proves that the radius R is sharp for the function f_0 defined by (3.2) (See Fig.3). □

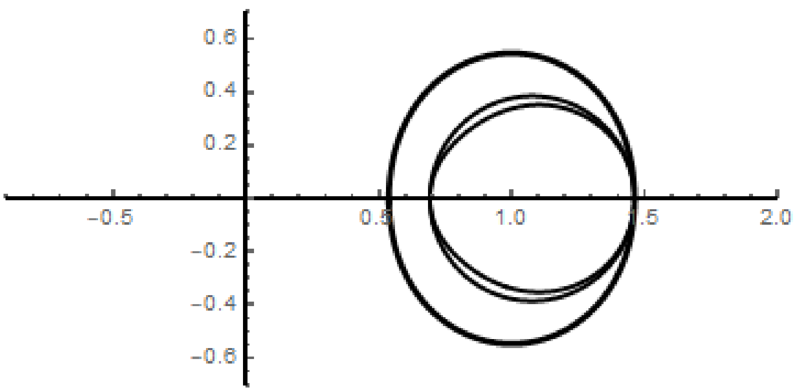


Figure 3. Sharpness of $\mathcal{R}_{\mathcal{CV}_{SG}}$ radius when $\alpha = 0.2$ and $\beta = 0.9$

Corollary 3.6. *Radius of convexity associated with the class $\mathcal{CV}_{SG}$ for some special cases:*

(1) *The $\mathcal{CV}_{SG}$ radius for the class $\mathcal{CC}(\sigma, 0)$ is*

$$\mathcal{R}_{\mathcal{CV}_{SG}} = \frac{(e-1)}{(2-\sigma)(e+1) + \sqrt{(e+1)^2(2-\sigma)^2 + ((3e+1)-2\sigma(e+1))(e-1)}}.$$

(2) *The $\mathcal{CV}_{SG}$ radius for the class V_k*

$$\mathcal{R}_{\mathcal{CV}_{SG}} = \frac{2(e-1)}{k(e+1) + \sqrt{(e+1)^2k^2 + 4(3e+1)(e-1)}}.$$

(3) *The $\mathcal{CV}_{SG}$ radius for the class $\mathcal{SCC}(\alpha)$ is*

$$\mathcal{R}_{\mathcal{CV}_{SG}} = \frac{(e-1)}{(\alpha+1)(e+1) + \sqrt{(e+1)^2(\alpha+1)^2 + (3e+1)(e-1)}}.$$

Cho et al. [3] introduced the class that consists of convex functions associated with the sine function denoted by $\mathcal{CV}_{\sin} = \mathcal{CV}(\varphi_{\sin})$, where $\varphi_{\sin}(z) = 1 + \sin z$.

Theorem 3.7. *Let $0 \leqslant \alpha \leqslant \beta$. The sharp $\mathcal{CV}_{\sin}$ radius for functions f such that $f' \in \mathcal{K}(\alpha, \beta)$ is given by*

$$\mathcal{R}_{\mathcal{CV}_{\sin}} = \frac{2(\sin 1)}{(\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 4(\beta - \alpha + \sin 1)(\sin 1)}}.$$

Proof. We first note that $R = \mathcal{R}_{\mathcal{CV}_{\sin}}$ is the smallest positive root of the equation $(\beta - \alpha + (\sin 1))r^2 + (\alpha+\beta)r - (\sin 1) = 0$ in the interval $(0, 1)$. Then it follows that $(\beta - \alpha + (\sin 1))r^2 + (\alpha+\beta)r - (\sin 1) \leqslant 0$ for $0 \leqslant r \leqslant R$ and so

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant (\sin 1) + 1 - \frac{1 + (\beta - \alpha - 1)r^2}{1-r^2}, \quad r \leqslant R.$$

Consequently, for $0 \leqslant r \leqslant R = \mathcal{R}_{\mathcal{CV}_{\sin}}$, the disc in (2.7) becomes

$$\left|1 + \frac{zf''(z)}{f'(z)} - a(r)\right| \leqslant (\sin 1) + 1 - a(r) \tag{3.6}$$

where $a(r)$ is given by (2.8). By [3, Lemma 3.3], we have $\{w : |w - a| < (\sin 1) + 1 - a\} \subseteq \varphi_{\sin}(\mathbb{D}) = \Omega_{\sin}$, for $1 - \sin 1 < a < 1 + \sin 1$. The condition $a < 1 + \sin 1$ assures that the center lies inside the region $\Omega_{\sin}$. As the center $a(r)$ is always greater than 1, the disc in (3.6) lies inside the region $\Omega_{\sin}$ for $r \leqslant R$ proving that $\mathcal{CV}_{\sin}$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha, \beta)$ is at least R.

In order to prove the sharpness of the radius $R = \mathcal{R}_{\mathcal{CV}_{\sin}}$, we consider the function f_0 defined by (3.2). For $z = R$ in (3.3),we have

$$1 + \frac{zf_0''(z)}{f_0'(z)} = \frac{1 + (\beta - \alpha - 1)R^2 + (\alpha+\beta)R}{1 - R^2} = 1 + (\sin 1) = \varphi_{\sin}(1),$$

which proves that the radius R is sharp for the function f_0 defined by (3.2) (See Fig.4). □

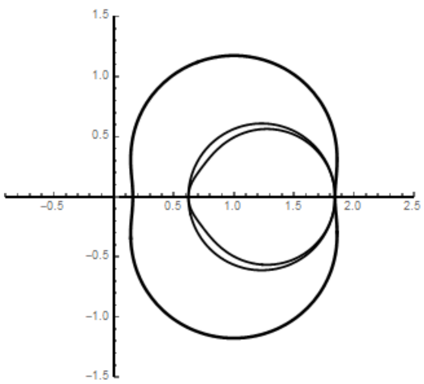


Figure 4. Sharpness of $\mathcal{R}_{\mathcal{CV}_{\sin}}$ radius when $\alpha = 0.1$ and $\beta = 0.9$

Corollary 3.8. *Radius of convexity associated with the class $\mathcal{CV}_{\sin}$ for some special cases:*

(1) *The $\mathcal{CV}_{\sin}$ radius for the class $\mathcal{CC}(\sigma, 0)$ is*

$$\mathcal{R}_{\mathcal{CV}_{\sin}} = \frac{\sin 1}{(2-\sigma) + \sqrt{(2-\sigma)^2 + (2 - 2\sigma + \sin 1)(\sin 1)}}.$$

(2) *The $\mathcal{CV}_{\sin}$ radius for the class V_k is*

$$\mathcal{R}_{\mathcal{CV}_{\sin}} = \frac{2 \sin 1}{k + \sqrt{k^2 + 4(2 + \sin 1)(\sin 1)}}.$$

(3) *The* $\mathcal{CV}_{\sin}$ *radius for the class* $\mathcal{SCC}(\alpha)$ *is*

$$\mathcal{R}_{\mathcal{CV}_{\sin}} = \frac{\sin 1}{(\alpha+1)+\sqrt{(\alpha+1)^2+(2+\sin 1)(\sin 1)}}.$$

Sokól and Stankiewicz [26] introduced the class of convex functions associated with a lemniscate domain, given by $\mathcal{CV}_L = \mathcal{CV}(\varphi_L)$, where $\varphi_L(z) = \sqrt{1+z}$ consisting of functions $f \in \mathcal{A}$ such that for each $z \in \mathbb{D}$, $w = 1 + (zf''(z)/f'(z))$ lies in the region bounded by right half of the lemniscate of Bernoulli given by $|w^2-1| < 1$.

Theorem 3.9. *The radius of lemniscate convexity for the functions whose derivative belongs to* $\mathcal{K}(\alpha,\beta)$, $0 \leqslant \alpha \leqslant \beta$ *is given by*

$$\mathcal{R}_{\mathcal{CV}_L} = \frac{2(\sqrt{2}-1)}{(\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4(\sqrt{2}-1)(\sqrt{2}+(\beta-\alpha-1))}}.$$

Proof. Firstly, we note that $R = \mathcal{R}_{\mathcal{CV}_L}$ is the smallest positive root of the equation $(\beta-\alpha+\sqrt{2}-1)r^2+(\alpha+\beta)r-(\sqrt{2}-1) = 0$ in the interval $(0,1)$. Then it follows that $(\beta-\alpha+\sqrt{2}-1)r^2+(\alpha+\beta)r-(\sqrt{2}-1) \leqslant 0$ for $0 \leqslant r \leqslant R$ and so

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \sqrt{2} - \frac{1+(\beta-\alpha-1)r^2}{1-r^2}, \quad r \leqslant R.$$

Consequently, for $0 \leqslant r \leqslant R = \mathcal{R}_{\mathcal{CV}_L}$, the disc in (2.7) becomes

$$\left|1+\frac{zf''(z)}{f'(z)}-a(r)\right| \leqslant \sqrt{2}-a(r) \tag{3.7}$$

where $a(r)$ is given by (2.8). By [1, Lemma 2.2], we have $\{w : |w-a| < \sqrt{2}-a\} \subseteq \varphi_L(\mathbb{D}) = \Omega_L$ for $(2\sqrt{2})/3 < a < \sqrt{2}$. The condition $a < \sqrt{2}$ ensures that the center lies inside the region Ω_L. As the center $a(r)$ is always greater than 1, the disc in (3.7) lies inside the region Ω_L for $r \leqslant R$ proving that $\mathcal{CV}_L$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ is at least R_3.

To show the sharpness of the radius $R = \mathcal{R}_{\mathcal{CV}_L}$, we consider the function f_0 defined by (3.2). For $z = R$ in (3.3),we have

$$1+\frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R^2+(\alpha+\beta)R}{1-R^2} = \sqrt{2} = \varphi_L(1),$$

which proves that the radius R is sharp for the function f_0 defined by (3.2) (See Fig.5). □

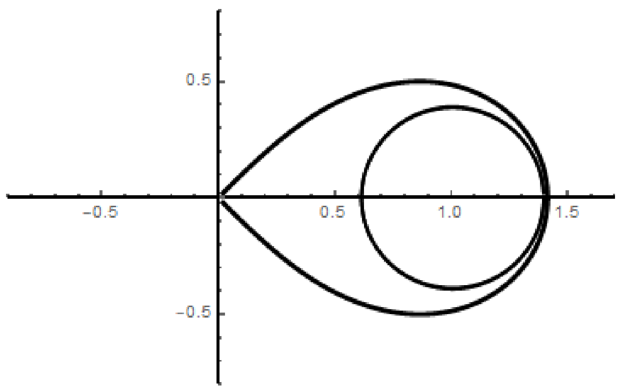


Figure 5. Sharpness of $\mathcal{R}_{\mathcal{CV}_L}$ radius when $\alpha = 2$ and $\beta = 3$

Corollary 3.10. *Radius of convexity associated with the class* $\mathcal{CV}_L$ *for some special cases:*

(1) *The* $\mathcal{CV}_L$ *radius for the class* $\mathcal{CC}(\sigma, 0)$ *is*

$$\mathcal{R}_{\mathcal{CV}_L} = \frac{(\sqrt{2}-1)}{(2-\sigma)+\sqrt{(2-\sigma)^2-2(\sqrt{2}-1)\sigma+1}}.$$

(2) *The* $\mathcal{CV}_L$ *radius for the class* V_k *is*

$$\mathcal{R}_{\mathcal{CV}_L} = \frac{2(\sqrt{2}-1)}{k+\sqrt{k^2+4}}.$$

(3) *The* $\mathcal{CV}_L$ *radius for the class* $\mathcal{SCC}(\alpha)$ *is*

$$\mathcal{R}_{\mathcal{CV}_L} = \frac{(\sqrt{2}-1)}{(\alpha+1)+\sqrt{(\alpha+1)^2+1}}.$$

The class $\mathcal{CV}_R = \mathcal{CV}(\varphi_R)$ consisting of convex functions associated with the rational function $\varphi_R(z) = 1 + ((z^2+kz)/k^2 - kz)$ for $k = \sqrt{2}+1$, was introduced by Kumar and Ravichandran [10].

Theorem 3.11. *Let* $0 \leqslant \alpha \leqslant \beta$. *The sharp* $\mathcal{CV}_R$ *radius for functions* f *such that* $f' \in \mathcal{K}(\alpha, \beta)$ *is given by*

$$\mathcal{R}_{\mathcal{CV}_R} = \begin{cases} R_2, & R_2 \leqslant R_1 \\ R_3, & R_2 \geqslant R_1 \end{cases}$$

where

$$R_1 = \left(\frac{\sqrt{2}-1}{\beta-\alpha+\sqrt{2}-1}\right)^{1/2},$$

$$R_2 = \frac{2(3-2\sqrt{2})}{(\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4(\beta-\alpha+2\sqrt{2}-3)(2\sqrt{2}-3)}}$$

and

$$R_3 = \frac{2}{(\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4(\beta-\alpha+1)}}.$$

Proof. We first consider the case where $R_2 \leqslant R_1$. Note that R_1 is the positive root of the equation $(\beta-\alpha+\sqrt{2}-1)r^2+1-\sqrt{2}=0$ or equivalently $a(r)=\sqrt{2}$ where $a(r)$ given by (2.8). For $r \leqslant R_2 \leqslant R_1$, we see that the increasing function $a(r)$ given by (2.8) satisfies

$$2(\sqrt{2}-1) < 1 \leqslant a(r) \leqslant a(R_2) = \frac{1+(\beta-\alpha-1)R_2^2}{1-R_2^2} \leqslant a(R_1) = \sqrt{2}.$$

Since R_2 is the smallest positive root of the equation $(\beta-\alpha+2\sqrt{2}-3)r^2-(\alpha+\beta)r-2\sqrt{2}+3=0$ in the interval $(0,1)$, it follows that $(\beta-\alpha+2\sqrt{2}-3)r^2-(\alpha+\beta)r-2\sqrt{2}+3 \leqslant 0$ for $0 \leqslant r \leqslant R_2$, and hence, upon rewriting, we have

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \frac{1+(\beta-\alpha-1)r^2}{1-r^2} - 2(\sqrt{2}-1), \quad r \leqslant R_2.$$

Therefore, for $r \leqslant R_2$, the disc in (2.7) becomes

$$\left|1+\frac{zf''(z)}{f'(z)}-a(r)\right| \leqslant a(r)-2(\sqrt{2}-1) \tag{3.8}$$

where $a(r)$ is given by (2.8). For $2(\sqrt{2}-1) < a < \sqrt{2}$, [10, Lemma 2.2] establishes that

$$\{w \in \mathbb{C} : |w-a| < a-2(\sqrt{2}-1)\} \subseteq \varphi_R(\mathbb{D}) = \Omega_R. \tag{3.9}$$

Using the inclusion in (3.9), it follows that the disc in (3.8) is contained in Ω_R, proving that the $\mathcal{CV}_R$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ is at least R_2.

To prove the sharpness of R_2, we consider the function f_0 defined by (3.2). For $z = -R_2$ in (3.3), we have

$$1+\frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R_2^2-(\alpha+\beta)R_2}{1-R_2^2} = 2\sqrt{2}-2 = \varphi_R(-1).$$

which proves that the radius R_2 is sharp for the function f_0 defined by (3.2) (See Fig.6(a)).

We now consider the case where $R_1 \leqslant R_2$. Clearly, R_3 is the smallest positive root of the equation $(\beta-\alpha+1)r^2-(\alpha+\beta)r-1=0$ in the interval $(0,1)$ and hence $(\beta-\alpha+1)r^2-(\alpha+\beta)r-1 \leqslant 0$ for $0 \leqslant r \leqslant R_3$. Rewriting this, we get

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant 2-\frac{1+(\beta-\alpha-1)r^2}{1-r^2}, \quad r \leqslant R_3.$$

Therefore, the disc in (2.7) becomes

$$\left|1+\frac{zf''(z)}{f'(z)}-a(r)\right| \leqslant 2-a(r) \tag{3.10}$$

where $a(r)$ is given by (2.8). For $\sqrt{2} < a < 2$, [10, Lemma 2.2] establishes the containment:

$$\{w \in \mathbb{C} : |w-a| < 2-a\} \subseteq \varphi_R(\mathbb{D}) = \Omega_R. \tag{3.11}$$

It is easy to see that $R_2 \leqslant R_3$. Since $R_1 \leqslant R_2$, it follows that $R_1 \leqslant R_3$. For $a(r)$ given by (2.8), we have

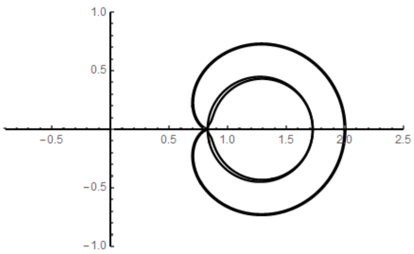
$$\sqrt{2} = a(R_1) \leqslant a(r) \leqslant a(R_3) = \frac{1+(\beta-\alpha-1)R_3^2}{1-R_3^2} < 2.$$

Using the inclusion in (3.11), it follows that the disc in (3.10) lies inside Ω_R proving that the $\mathcal{CV}_R$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ is at least R_3.

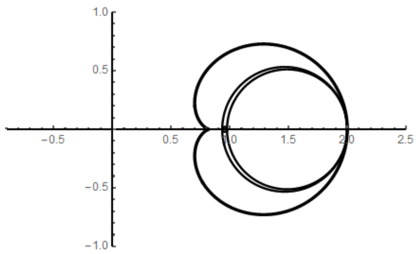
Consider the function f_0 defined by (3.2). For $z = R_3$ in (3.3), we have

$$1+\frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R_3^2+(\alpha+\beta)R_3}{1-R_3^2} = 2 = \varphi_R(1),$$

which proves the sharpness of the radius R_3 for the function f_0 defined by (3.2) (See Fig.6(b)). □



(a). Sharpness of $\mathcal{R}_{\mathcal{CV}_R} = R_2$ radius when $\alpha = 0.015$ and $\beta = 0.35$



(b). Sharpness of $\mathcal{R}_{\mathcal{CV}_R} = R_3$ radius when $\alpha = 0.002$ and $\beta = 0.05$

Figure 6. Sharpness $\mathcal{R}_{\mathcal{CV}_R}$ radius

Corollary 3.12. *Radius of convexity associated with the class $\mathcal{CV}_R$ for some special cases:*

(1) *The $\mathcal{CV}_R$ radius for the class $\mathcal{CC}(\sigma,0)$ is*

$$\mathcal{R}_{\mathcal{CV}_R} = \frac{3-2\sqrt{2}}{2(2-\sigma)+\sqrt{4((2-\sigma)^2+(6-4\sqrt{2})\sigma-8\sqrt{2}+11)}}.$$

(2) *The $\mathcal{CV}_R$ radius for the class V_k is*

$$\mathcal{R}_{\mathcal{CV}_R} = \frac{3-2\sqrt{2}}{k+\sqrt{k^2+44-32\sqrt{2}}}.$$

(3) *The $\mathcal{CV}_R$ radius for the class $\mathcal{SCC}(\alpha)$ is*

$$\mathcal{R}_{\mathcal{CV}_R} = \frac{3-2\sqrt{2}}{2(\alpha+1)+\sqrt{4((\alpha+1)^2+44-32\sqrt{2})}}.$$

Raina and Sokól [19] studied the class $\mathcal{CV}_{\leftmoon} = \mathcal{CV}(\varphi_{\leftmoon})$, where $\varphi_{\leftmoon}(z) = z+\sqrt{1+z^2}$ and proved that $f \in \mathcal{CV}_{\leftmoon}$ if and only if $1+(zf''(z)/f'(z)) \in \Omega_{\leftmoon}$, where $\Omega_{\leftmoon}$ is the interior of a lune given by $\Omega_{\leftmoon} := \{w \in \mathbb{C} : |w^2-1| < 2|w|\}$.

Theorem 3.13. *Let $0 \leqslant \alpha \leqslant \beta$. The sharp $\mathcal{CV}_{\leftmoon}$ radius for functions f such that $f' \in \mathcal{K}(\alpha,\beta)$ is given by*

$$\mathcal{R}_{\mathcal{CV}_{\leftmoon}} = \begin{cases} R_2, & R_2 \leqslant R_1 \\ R_3, & R_2 \geqslant R_1 \end{cases}$$

where

$$R_1 = \left(\frac{\sqrt{2}-1}{\beta-\alpha+\sqrt{2}-1}\right)^{1/2}, \quad R_2 = \frac{4-2\sqrt{2}}{(\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4(\sqrt{2}-2)(\beta-\alpha+\sqrt{2}-2)}}$$

and

$$R_3 = \frac{2\sqrt{2}}{(\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\sqrt{2}(\beta-\alpha+\sqrt{2})}}.$$

Proof. Firstly, we consider the case where $R_2 \leqslant R_1$. Note that R_1 is the positive root of the equation $(\beta-\alpha+\sqrt{2}-1)r^2+1-\sqrt{2}=0$ or equivalently $a(r)=\sqrt{2}$ where $a(r)$ is given by (2.8). For $r \leqslant R_2 \leqslant R_1$, we note that the increasing function $a(r)$ given by (2.8) satisfies

$$(\sqrt{2}-1) < 1 \leqslant a(r) \leqslant a(R_2) = \frac{1+(\beta-\alpha-1)R_2^2}{1-R_2^2} \leqslant a(R_1) = \sqrt{2}.$$

Since R_2 is the smallest positive root of the equation $(\beta-\alpha+\sqrt{2}-2)r^2-(\alpha+\beta)r-\sqrt{2}+2=0$ in the interval $(0,1)$, it follows that $(\beta-\alpha+\sqrt{2}-2)r^2-(\alpha+\beta)r-\sqrt{2}+2 \leqslant 0$ for $0 \leqslant r \leqslant R_2$ and hence upon rewriting, we have

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \frac{1+(\beta-\alpha-1)r^2}{1-r^2} - (\sqrt{2}-1), \quad r \leqslant R_2.$$

Therefore, for $r \leqslant R_2$ the disc in (2.7) becomes

$$\left|1+\frac{zf''(z)}{f'(z)} - a(r)\right| \leqslant a(r) - (\sqrt{2}-1). \tag{3.12}$$

where $a(r)$ is given by (2.8). For $2(\sqrt{2}-1) < a < \sqrt{2}$, [4, Lemma 2.1] provides the containment

$$\{w \in \mathbb{C} : |w-a| < a-(\sqrt{2}-1)\} \subseteq \varphi_{\leftmoon}(\mathbb{D}) = \Omega_{\leftmoon}. \tag{3.13}$$

Using the inclusion in (3.13), it follows that the disc in (3.12) lies within $\Omega_{\leftmoon}$ proving that the $\mathcal{CV}_{\leftmoon}$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ is at least R_2.

To examine the sharpness, we consider the function f_0 defined by (3.2). For $z=-R_2$ in (3.3), we have

$$1+\frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R_2^2-(\alpha+\beta)R_2}{1-R_2^2} = \sqrt{2}-1 = \varphi_{\leftmoon}(-1),$$

this proves the sharpness of the radius R_2 for the function f_0 defined by (3.2) (See Fig.7(a)).

We now consider the case where $R_1 \leqslant R_2$. Clearly, R_3 is the smallest positive root of the equation $(\beta-\alpha+1)r^2-(\alpha+\beta)r-1=0$ in the interval $(0,1)$ and hence $(\beta-\alpha+1)r^2-(\alpha+\beta)r-1 \leqslant 0$ for $0 \leqslant r \leqslant R_3$. Rewriting this, we get

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \sqrt{2}+1-\frac{1+(\beta-\alpha-1)r^2}{1-r^2}, \quad r \leqslant R_3.$$

Therefore, for $r \leqslant R_3$, the disc in (2.7) becomes

$$\left|1+\frac{zf''(z)}{f'(z)}-a(r)\right| \leqslant \sqrt{2}+1-a(r) \tag{3.14}$$

where $a(r)$ is given by (2.8). For $\sqrt{2} < a < \sqrt{2}+1$, [4, Lemma 2.1] establishes the inclusion:

$$\{w \in \mathbb{C} : |w-a| < \sqrt{2}+1-a\} \subseteq \varphi_{\leftmoon}(\mathbb{D}) = \Omega_{\leftmoon}. \tag{3.15}$$

It is easy to that $R_2 \leqslant R_3$. Since $R_1 \leqslant R_2$, it follows that $R_1 \leqslant R_3$. For $a(r)$ given by (2.8), we have

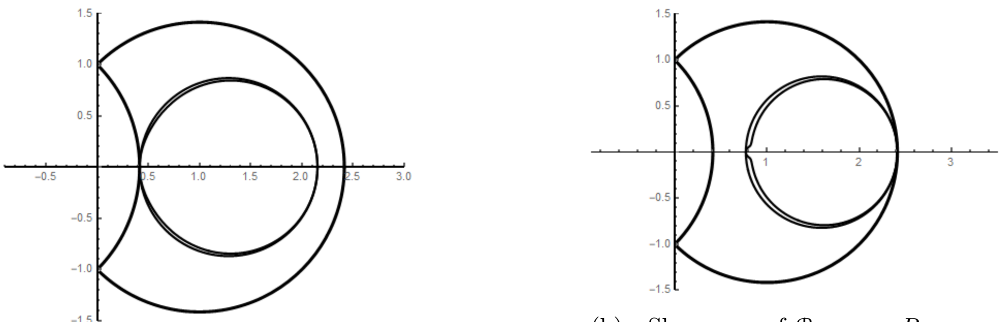
$$\sqrt{2} = a(R_1) \leqslant a(r) \leqslant a(R_3) = \frac{1+(\beta-\alpha-1)R_3^2}{1-R_3^2} < \sqrt{2}+1.$$

Using the inclusion (3.15), it follows that the disc in (3.14) lies inside $\Omega_{\leftmoon}$ proving that the $\mathcal{CV}_{\leftmoon}$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ is at least R_3.

Consider the function f_0 defined by (3.2) for proving the sharpness of the radius R_3. For $z = R_3$ in (3.3), we have

$$1+\frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R_3^2+(\alpha+\beta)R_3}{1-R_3^2} = \sqrt{2}+1 = \varphi_{\leftmoon}(1),$$

this establishes the sharpness of the radius R_3 for the function f_0 defined by (3.2) (See Fig.7(b)). □



(a). Sharpness of $\mathcal{R}_{\mathcal{CV}_{\leftmoon}} = R_2$ radius when $\alpha = 0.1$ and $\beta = 2$

(b). Sharpness of $\mathcal{R}_{\mathcal{CV}_{\leftmoon}} = R_3$ radius when $\alpha = 0.015$ and $\beta = 0.4$

Figure 7. Sharpness of $\mathcal{R}_{\mathcal{CV}_{\leftmoon}}$ radius

Corollary 3.14. *Radius of convexity associated with the class $\mathcal{CV}_{\leftmoon}$ for some special cases:*

(1) *The $\mathcal{CV}_{\leftmoon}$ radius for the class $\mathcal{CC}(\sigma,0)$ is*

$$\mathcal{R}_{\mathcal{CV}_{\leftmoon}} = \frac{(2-\sqrt{2})}{(2-\sigma)+\sqrt{(2-\sigma)^2+(\sqrt{2}-2\sigma)(\sqrt{2}-2)}}.$$

(2) *The* $\mathcal{CV}_{\leftmoon}$ *radius for the class* V_k *is*

$$\mathcal{R}_{\mathcal{CV}_{\leftmoon}} = \frac{2(2-\sqrt{2})}{k+\sqrt{k^2+8(1-\sqrt{2})}}.$$

(3) *The* $\mathcal{CV}_{\leftmoon}$ *radius for the class* $\mathcal{SCC}(\alpha)$ *is*

$$\mathcal{R}_{\mathcal{CV}_{\leftmoon}} = \frac{(2-\sqrt{2})}{(\alpha+1)+\sqrt{(\alpha+1)^2+2(2-\sqrt{2})}}.$$

Sharma et al. [24] studied the class $\mathcal{CV}_C = \mathcal{CV}(\varphi_C)$ which consists of convex functions associated with a cardiod, where $\varphi_C(z) = 1 + (4/3)z + (2/3)z^2$.

Theorem 3.15. *Let* $0 \leqslant \alpha \leqslant \beta$. *The sharp* $\mathcal{CV}_C$ *radius for functions* f *such that* $f' \in \mathcal{K}(\alpha, \beta)$ *is given by*

$$\mathcal{R}_{\mathcal{CV}_C} = \begin{cases} R_2, & R_2 \leqslant R_1 \\ R_3, & R_2 \geqslant R_1 \end{cases}$$

where

$$R_1 = \left(\frac{2}{3(\beta-\alpha)+2}\right)^{1/2},$$
$$R_2 = \frac{4}{3(\alpha+\beta)+\sqrt{9(\alpha+\beta)^2-8(3(\beta-\alpha)-2)}}$$

and

$$R_3 = \frac{4}{(\alpha+\beta)-\sqrt{(\alpha+\beta)^2+8(\beta-\alpha+2)}}.$$

Proof. We first consider the case where $R_2 \leqslant R_1$. Note that R_1 is the positive root of the equation $(3(\beta-\alpha)+2)r^2 - 2 = 0$ which is equivalent to setting $a(r) = 5/3$ where $a(r)$ is given by (2.8). For $r \leqslant R_2 \leqslant R_1$, we see that the increasing function $a(r)$ given by (2.8) satisfies

$$\frac{1}{3} < 1 \leqslant a(r) \leqslant a(R_2) \leqslant \frac{1+(\beta-\alpha-1)R_2^2}{1-R_2^2} \leqslant a(R_1) = \frac{5}{3}.$$

Since R_2 is the smallest positive root of the equation $(3(\beta-\alpha)-2)r^2 - 3(\alpha+\beta)r + 2 = 0$ in the interval $(0,1)$, it follows that $(3(\beta-\alpha)-2)r^2 - 3(\alpha+\beta)r + 2 \leqslant 0$ for $0 \leqslant r \leqslant R_2$ and hence upon rewriting, we have

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \frac{1+(\beta-\alpha-1)r^2}{1-r^2} - \frac{1}{3}, \quad r \leqslant R_2.$$

Therefore, for $r \leqslant R_2$, the disc in (2.7) becomes

$$\left|1 + \frac{zf''(z)}{f'(z)} - a(r)\right| \leqslant a(r) - \frac{1}{3} \tag{3.16}$$

where $a(r)$ is given by (2.8). For $1/3 < a < 5/3$, [24, Lemma 2.5] provides the inclusion:

$$\{w \in \mathbb{C} : |w-a| < a - 1/3\} \subseteq \varphi_C(\mathbb{D}) = \Omega_C. \tag{3.17}$$

Using the inclusion (3.17), it follows that the disc in (3.16) resides inside Ω_C proving that the $\mathcal{CV}_C$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ is at least R_2.

The function f_0 defined by (3.2) proves the sharpness of the radius R_2 . For $z = -R_2$ in (3.3), we have

$$1 + \frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R_2^2 - (\alpha+\beta)R_2}{1-R_2^2} = \frac{1}{3} = \varphi_C(-1),$$

which proves that the radius R_2 is sharp for the function f_0 defined by (3.2) (See Fig.8(a)).

We now consider the case where $R_1 \leqslant R_2$. Clearly, R_3 is the smallest positive root of the equation $(\beta - \alpha + 2)r^2 + (\alpha + \beta)r - 2 = 0$ in the interval $(0, 1)$ and hence $(\beta - \alpha + 2)r^2 + (\alpha + \beta)r - 2 \leqslant 0$ for $0 \leqslant r \leqslant R_3$. Upon rewriting, we get

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant 3 - \frac{1+(\beta-\alpha-1)r^2}{1-r^2}, \quad r \leqslant R_3.$$

Therefore, the disc in (2.7) becomes

$$\left|1 + \frac{zf''(z)}{f'(z)} - a(r)\right| \leqslant 3 - a(r) \tag{3.18}$$

where $a(r)$ is given by (2.8). For $5/3 < a < 3$, by [24, Lemma 2.5] the following inclusion relation holds:

$$\{w \in \mathbb{C} : |w - a| < 3 - a\} \subseteq \varphi_C(\mathbb{D}) = \Omega_C. \tag{3.19}$$

It is easy to see that $R_2 \leqslant R_3$. Since $R_1 \leqslant R_2$, it follows that $R_1 \leqslant R_3$. For $a(r)$ given by (2.8), we have

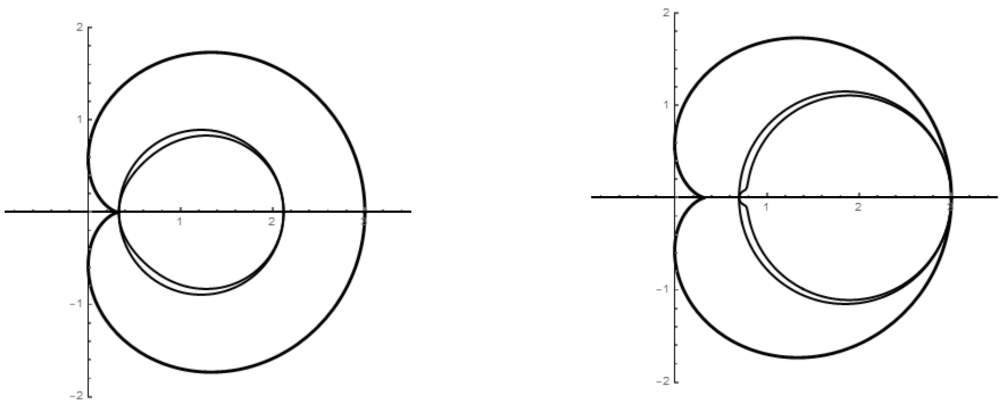
$$\frac{5}{3} = a(R_1) \leqslant a(r) \leqslant a(R_3) = \frac{1+(\beta-\alpha-1)R_3^2}{1-R_3^2} < 3.$$

Using the inclusion in (3.19), it follows that the disc in (3.18) lies inside Ω_C proving that the $\mathcal{CV}_C$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha, \beta)$ is at least R_3.

To prove the sharpness, we consider the function f_0 defined by (3.2). For $z = R_3$ in (3.3), we have

$$1 + \frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R_3^2 + (\alpha+\beta)R_3}{1-R_3^2} = 3 = \varphi_C(1).$$

which proves the sharpness of the radius R_3 for the function f_0 defined by (3.2) (See Fig.8(b)). ☐



(a). Sharpness of $\mathcal{R}_{\mathcal{CV}_C} = R_2$ radius when $\alpha = 0.3$ and $\beta = 2$

(b). Sharpness of $\mathcal{R}_{\mathcal{CV}_C} = R_3$ radius when $\alpha = 0.02$ and $\beta = 0.5$

Figure 8. Sharpness of $\mathcal{R}_{\mathcal{CV}_C}$ radius

Corollary 3.16. *Radius of convexity associated with the class $\mathcal{CV}_C$ for some special cases:*

(1) *The $\mathcal{CV}_C$ radius for the class $\mathcal{CC}(\sigma, 0)$ is*

$$\mathcal{R}_{\mathcal{CV}_C} = \frac{2}{3(2-\sigma) + \sqrt{9\sigma^2 - 24\sigma + 28}}.$$

(2) *The $\mathcal{CV}_C$ radius for the class V_k is*

$$\mathcal{R}_{\mathcal{CV}_C} = \frac{4}{3k + \sqrt{9k^2 - 32}}.$$

(3) *The $\mathcal{CV}_C$ radius for the class $\mathcal{SCC}(\alpha)$ is*

$$\mathcal{R}_{\mathcal{CV}_C} = \frac{2}{3(\alpha+1)+\sqrt{9(\alpha+1)^2-8}}.$$

Mendiratta et al. [16], introduced the class $\mathcal{CV}_e = \mathcal{CV}(\varphi_e)$ where $\varphi_e(z) = e^z$, which consists of all functions $f \in \mathcal{A}$ such that $1+(zf''(z)/f'(z)) \prec e^z$ or equivalently $|\log(1+(zf''(z)/f'(z)))| < 1$.

Theorem 3.17. *Let $0 \leqslant \alpha \leqslant \beta$. The sharp $\mathcal{CV}_e$ radius for functions f such that $f' \in \mathcal{K}(\alpha,\beta)$ is given by*

$$\mathcal{R}_{\mathcal{CV}_e} = \begin{cases} R_2, & R_2 \leqslant R_1 \\ R_3, & R_2 \geqslant R_1 \end{cases}$$

where

$$R_1 = \left(\frac{e+e^{-1}-2}{2(\beta-\alpha)+e+e^{-1}-2}\right)^{1/2},$$

$$R_2 = \frac{2(e-1)}{e(\alpha+\beta)+\sqrt{e^2(\alpha+\beta)^2-4(e(\beta-\alpha)-(e-1))(e-1)}}$$

and

$$R_3 = \frac{2(e-1)}{(\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4(\beta-\alpha+e-1)(e-1)}}.$$

Proof. We first consider the case where $R_2 \leqslant R_1$. Note that R_1 is the positive root of the equation $(2(\beta-\alpha)+e+e^{-1}-2)r^2-(e+e^{-1}-2)=0$ which is equivalent to setting $a(r) = (e+e^{-1})/2$ where $a(r)$ is given by (2.8). For $r \leqslant R_2 \leqslant R_1$, we see that the increasing function $a(r)$ satisfies

$$\frac{1}{e} < 1 \leqslant a(r) \leqslant a(R_2) = \frac{1+(\beta-\alpha-1)R_2^2}{1-R_2^2} \leqslant a(R_1) = \frac{e+e^{-1}}{2}.$$

Since R_2 is the smallest positive root of the equation $(e(\beta-\alpha)-e+1)r^2-e(\alpha+\beta)r+e-1=0$ in the interval $(0,1)$, it follows that $(e(\beta-\alpha)-e+1)r^2-e(\alpha+\beta)r+e-1 \leqslant 0$ for $0 \leqslant r \leqslant R_2$, and hence upon rewriting, we have

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \frac{1+(\beta-\alpha-1)r^2}{1-r^2}-\frac{1}{e}, \quad r \leqslant R_2.$$

Therefore, $r \leqslant R_2$, the disc in (2.7) becomes

$$\left|1+\frac{zf''(z)}{f'(z)}-a(r)\right| \leqslant a(r)-\frac{1}{e} \tag{3.20}$$

where $a(r)$ is given by (2.8). For $1/e < a \leqslant (e+e^{-1})/2$, [16, Lemma 2.5] provides the inclusion:

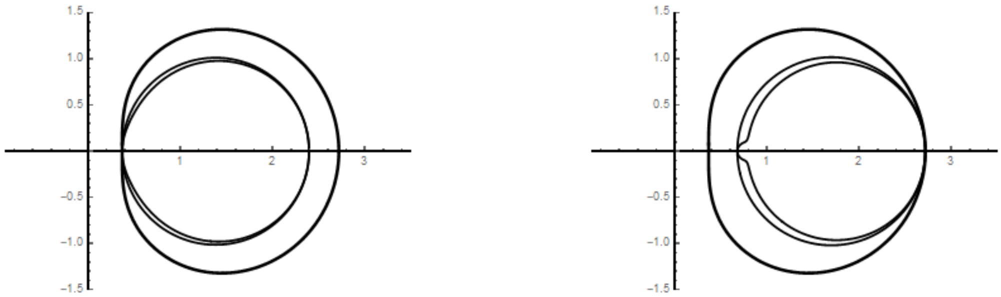
$$\{w \in \mathbb{C} : |w-a| < a-1/e\} \subseteq \varphi_e(\mathbb{D}) = \Omega_e. \tag{3.21}$$

Using the inclusion (3.21), it follows that the disc in (3.20) resides inside Ω_e proving that the $\mathcal{CV}_e$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ is at least R_2.

In order to prove the sharpness consider the function f_0 defined by (3.2). For $z = -R_2$ in (3.3), we have

$$\left|\log\left(1+\frac{zf_0''(z)}{f_0'(z)}\right)\right| = \left|\log\left(\frac{1+(\beta-\alpha-1)R_2^2-(\alpha+\beta)R_2}{1-R_2^2}\right)\right| = \left|\log\left(\frac{1}{e}\right)\right| = 1,$$

which proves the sharpness of the radius R_3 for the function f_0 defined by (3.2) (See Fig.9(a)).



(a). Sharpness of $\mathcal{R}_{\mathcal{CV}_e} = R_2$ radius when $\alpha = 0.1$ and $\beta = 1.9$

(b). Sharpness of $\mathcal{R}_{\mathcal{CV}_e} = R_3$ radius when $\alpha = 0.03$ and $\beta = 0.5$

Figure 9. Sharpness of $\mathcal{R}_{\mathcal{CV}_e}$ radius

We next consider the case where $R_1 \leqslant R_2$. Clearly, R_3 is the smallest positive root of the equation $(\beta - \alpha + e - 1)r^2 + (\alpha + \beta)r - (e - 1) = 0$ in the interval $(0, 1)$ and hence $(\beta - \alpha + e - 1)r^2 + (\alpha + \beta)r - (e - 1) \leqslant 0$ for $0 \leqslant r \leqslant R_3$. Rewriting this, we get

$$\frac{(\alpha + \beta)r}{1 - r^2} \leqslant e - \frac{1 + (\beta - \alpha - 1)r^2}{1 - r^2}, \quad r \leqslant R_3.$$

Therefore, the disc in (2.7) becomes

$$\left|1 + \frac{zf''(z)}{f'(z)} - a(r)\right| \leqslant e - a(r). \tag{3.22}$$

where $a(r)$ is given by (2.8). By [16, Lemma 2.5], for $(e + e^{-1})/2 < a < e$, the following inclusion relation holds:

$$\{w \in \mathbb{C} : |w - a| < e - a\} \subseteq \varphi_e(\mathbb{D}) = \Omega_e. \tag{3.23}$$

It is easy to see that $R_2 \leqslant R_3$. Since $R_1 \leqslant R_2$, it follows that $R_1 \leqslant R_3$. For $a(r)$ given by (2.8), we have

$$\frac{e + e^{-1}}{2} = a(R_1) \leqslant a(r) \leqslant a(R_3) = \frac{1 + (\beta - \alpha - 1)R_3^2}{1 - R_3^2} < e.$$

Using the inclusion in (3.23), it follows that the disc in (3.22) lies inside Ω_e proving that the $\mathcal{CV}_e$ radius is for functions whose derivatives belongs to $\mathcal{K}(\alpha, \beta)$ at least R_3.

To prove the sharpness, we consider the function f_0 defined by (3.2). For $z = R_3$ in (3.3), we have

$$\left|\log\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right)\right| = \left|\log\left(\frac{1 + (\beta - \alpha - 1)R_3^2 + (\alpha + \beta)R_3}{1 - R_3^2}\right)\right| = |\log e| = 1.$$

which proves the sharpness of the radius R_3 for the function f_0 defined by (3.2) (See Fig.9(b)). □

Corollary 3.18. *Radius of convexity associated with the class $\mathcal{CV}_e$ for some special cases:*

(1) *The $\mathcal{CV}_e$ radius for the class $\mathcal{CC}(\sigma, 0)$ is*

$$\mathcal{R}_{\mathcal{CV}_e} = \frac{e - 1}{e(2 - \sigma) + \sqrt{e^2(2 - \sigma)^2 - (e(1 - 2\sigma) + 1)(e - 1)}}.$$

(2) *The $\mathcal{CV}_e$ radius for the class V_k is*

$$\mathcal{R}_{\mathcal{CV}_e} = \frac{2(e - 1)}{ek + \sqrt{e^2(k^2 - 4) + 4}}.$$

(3) *The* $\mathcal{CV}_e$ *radius for the class* $\mathcal{SCC}(\alpha)$ *is*

$$\mathcal{R}_{\mathcal{CV}_e} = \frac{e-1}{e(\alpha+1)+\sqrt{e^2(\alpha+1)^2-(e^2-1)}}.$$

The class uniformly convex functions was introduced by Goodman [7]. A function $f \in \mathcal{A}$ is said to be uniformly convex if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|$$

for all $z \in \mathbb{D}$ and the class of all uniformly convex functions is denoted by $\mathcal{UCV}$. The following theorem was proved earlier by Ravichandran et al. [20] for the case $R_2 \leqslant R_1$. However, the proof given here is different. The result in the case $R_2 > R_1$ is not necessarily sharp.

Theorem 3.19. *Let* $0 < \gamma \leqslant 1$ *and* $0 \leqslant \alpha \leqslant \beta$. *The radius of uniform convexity for functions whose derivative belongs to* $\mathcal{K}(\alpha,\beta)$ *is given by* $\mathcal{R}_{\mathcal{UCV}} = R_2$ *if* $R_2 \leqslant R_1$ *and* $\mathcal{R}_{\mathcal{UCV}} \geqslant R_3$, *if* $R_2 \geqslant R_1$ *where*

$$R_1 = \left(\frac{1}{2(\beta-\alpha)+1}\right)^{1/2},$$
$$R_2 = \frac{1}{(\alpha+\beta)+\sqrt{(\alpha+\beta)^2-(2(\beta-\alpha)-1)}}$$

and

$$R_3 = \left(1-\frac{(\alpha+\beta)^2}{2(\beta-\alpha)}\right)^{1/2}.$$

Proof. We first consider the case where $R_2 \leqslant R_1$. Note that R_1 is the positive root of the equation $(2(\beta-\alpha)+1)r^2-1=0$ which is equivalent to setting $a(r)=3/2$ where $a(r)$ is given by (2.8). For $r \leqslant R_2 \leqslant R_1$, we see that the increasing function $a(r)$ satisfies

$$\frac{1}{2} < 1 \leqslant a(r) \leqslant a(R_2) = \frac{1+(\beta-\alpha-1)R_2^2}{1-R_2^2} \leqslant a(R_1) = \frac{3}{2}.$$

Since R_2 is the smallest positive root of the equation $(2(\beta-\alpha)-1)r^2-2(\alpha+\beta)r+1=0$ in the interval $(0,1)$, it follows that $(2(\beta-\alpha)-1)r^2-2(\alpha+\beta)r+1 \leqslant 0$ for $0 \leqslant r \leqslant R_2$, and hence upon rewriting, we have

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \frac{1+(\beta-\alpha-1)r^2}{1-r^2}-\frac{1}{2}, \quad r \leqslant R_2.$$

Therefore, $r \leqslant R_2$, the disc in (2.7) becomes

$$\left|1+\frac{zf''(z)}{f'(z)}-a(r)\right| \leqslant a(r)-\frac{1}{2} \tag{3.24}$$

where $a(r)$ is given by (2.8). For $1/2 < a \leqslant 3/2$, [23, Lemma 1] provides the inclusion:

$$\{w \in \mathbb{C} : |w-a| < a-1/2\} \subseteq \varphi(\mathbb{D}) = \Omega \tag{3.25}$$

where Ω is a parabolic region symmetric with respect to the real axis and $(1/2,0)$ is its vertex. Using the inclusion (3.25), it follows that the disc in (3.24) resides inside Ω proving that the $\mathcal{UCV}$ radius for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ is at least R_2.

In order to prove the sharpness consider the function f_0 defined by (3.2). For $z=-R_2$ in (3.3), we have

$$1+\frac{zf_0''(z)}{f_0'(z)} = \frac{1+(\beta-\alpha-1)R_2^2-(\alpha+\beta)R_2}{1-R_2^2} = \frac{1}{2},$$

which proves the sharpness of the radius R_2 for the function f_0 defined by (3.2).

We next consider the case where $R_1 \leqslant R_2$. Clearly, R_3 is the smallest positive root of the equation $2(\beta-\alpha)r^2 + (\alpha+\beta)^2 - 2(\beta-\alpha) = 0$ in the interval $(0,1)$ and hence $2(\beta-\alpha)r^2 + (\alpha+\beta)^2 - 2(\beta-\alpha) \leqslant 0$ for $0 \leqslant r \leqslant R_3$. Rewriting this, we get

$$\frac{(\alpha+\beta)r}{1-r^2} \leqslant \left(\frac{2(\beta-\alpha)r^2}{1-r^2}\right)^{1/2}, \quad r \leqslant R_3.$$

Therefore, the disc in (2.7) becomes

$$\left|1 + \frac{zf''(z)}{f'(z)} - a(r)\right| \leqslant (2a(r)-2)^{1/2}. \tag{3.26}$$

where $a(r)$ is given by (2.8). By [23, Lemma 1], for $3/2 \leqslant a$, the following inclusion relation holds:

$$\{w \in \mathbb{C} : |w-a| < \sqrt{2a-2}\} \subseteq \varphi(\mathbb{D}) = \Omega. \tag{3.27}$$

It is easy to see that $R_2 \leqslant R_3$. Since $R_1 \leqslant R_2$, it follows that $R_1 \leqslant R_3$. For $a(r)$ given by (2.8), we have

$$\frac{3}{2} = a(R_1) \leqslant a(r) \leqslant a(R_3) = \frac{1+(\beta-\alpha-1)R_3^2}{1-R_3^2}.$$

Using the inclusion in (3.27), it follows that the disc in (3.26) lies inside Ω proving that the $\mathcal{UCV}$ radius is for functions whose derivatives belongs to $\mathcal{K}(\alpha,\beta)$ at least R_3. □

A function $f \in \mathcal{A}$ is said to be strongly convex of order γ, $0 < \gamma \leqslant 1$, if

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| \leqslant \frac{\pi\gamma}{2}.$$

The set of all such functions is denoted by $\mathcal{SCV}(\gamma)$. We state the following theorem without proof. Finding the sharp radius constant is open.

Theorem 3.20. *Let* $0 < \gamma \leqslant 1$ *and* $0 \leqslant \alpha \leqslant \beta$. *The radius of strong convexity of order* γ *for functions whose derivative belongs to* $\mathcal{K}(\alpha,\beta)$ *is given by*

$$\mathcal{R}_{\mathcal{SCV}(\gamma)} \geqslant \frac{2\sin\frac{\pi\gamma}{2}}{(\alpha+\beta) + \sqrt{(\alpha+\beta)^2 - 4(\beta-\alpha-1)(\sin\frac{\pi\gamma}{2})^2}}.$$

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