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The Notion of Topological Entropy in Fuzzy Metric Spaces

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Özet. Bu makalenin amacı, bir bulanık metrik uzayının dönüşüm fonksiyonu tarafından türetilen bulanık yarıdinamik sistemler için topolojik entropi kavramını genişletmektir. Eğer bir metrik uzayının iki düzgün denk metriği varsa o halde bulanık entropi bu iki metriğe bağlı bir değişmezdir. Rasgele büyüklükte bulanık entropili kaotik bulanık yarıdinamik sistemlerin inşası için bir metot sunuyoruz. Ayrıca, bulanık entropinin bulanık düzgün topolojik denklik bağıntısı altında kalıcı olduğunu ispatlıyoruz.[†]

Anahtar Kelimeler. Bulanık entropi, bulanık metrik uzayı, yarıtıkız, bulanık yarıtıkız.

Abstract. The aim of this paper is to extend the notion of topological entropy for fuzzy semidynamical systems created by a self-map on a fuzzy metric space. We show that if a metric space has two uniformly equivalent metrics, then fuzzy entropy is a constant up to these two metrics. We present a method to construct chaotic fuzzy semidynamical systems with arbitrary large fuzzy entropy. We also prove that fuzzy entropy is a persistent object under a fuzzy uniformly topological equivalent relation.

Keywords. Fuzzy entropy, fuzzy metric space, semicompact, fuzzy semicompact.

1. Introduction

Stability of physical and engineering systems can be considered from geometrical [1], and topological viewpoints. In both of them, topological entropy is one of the main tools to determine the complexity of a system. Also, it is an essential invariant in application [12, 14, 16]. The positive topological entropy of a map implies to its chaotic behavior [4]. Topological entropy for continuous maps first has been studied by Bowen and Dinaburg [2, 5, 17]. This notion has been extended for discontinuous maps in [3]. In fuzzy metric spaces [6, 7, 8, 9, 10, 13] the notion of metric has been

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extended to the rate of nearness. So we need a new concept of topological entropy to explain the complexity of systems created by the self-maps of fuzzy metric spaces. This notion must determine the complexity and as given in [15] it does not change under synchronization and fuzzy topological conjugate relations. In Section 3 we introduce the notion of fuzzy entropy and in Theorem 3.2 we show that it is an extension of the notion of topological entropy. In Theorem 3.3 we prove that it is a persistent object up to uniformly equivalent metrics. We prove that the set of topological entropies is not bounded from above and this is good news for engineers who theoretically can construct fuzzy systems with an arbitrarily large complexity. In fact Theorem 3.5 implies that it is possible to construct security systems with an arbitrarily large security. As a final result we show that fuzzy entropy is a constant object up to fuzzy uniformly topological equivalent relations.

2. Preliminaries

Let us recall the notion of topological entropy for discontinuous maps. We assume (X, d) is a compact metric space, $T : X \to X$ is a mapping and T^i is the composition of T, i times with itself, where i is a natural number. The mapping T may not be continuous.

For a natural number n we define:

$$d_n(x, y) = \max\{d(T^i(x), T^i(y)) : x, y \in X \text{ and } i \in \{0, 1, 2, ..., n-1\}\}.$$

If $F \subseteq X$, $\epsilon > 0$ and $n \in \mathbb{N}$, then F is called an (n, ϵ) spanning subset of X with respect to T if for given $x \in X$ there is $y \in F$ such that $d_n(x, y) \leq \epsilon$. A subset E of X is called an (n, ϵ) separated if $d_n(x, y) > \epsilon$ when x and y are different points in E. $r_n(\epsilon, X, T)$ denotes the number of elements of an (n, ϵ) spanning set for X with respect to T with the smallest cardinality. Also, $s_n(\epsilon, X, T)$ denotes the number of elements of an (n, ϵ) separated set for X with respect to T with the largest cardinality. We define $r(\epsilon, X, T) = \lim_{n\to\infty}(1/n)(\log(r_n(\epsilon, X, T)))$ and $s(\epsilon, X, T) = \lim_{n\to\infty}(1/n)(\log(s_n(\epsilon, X, T)))$. The entropy of T is denoted by $h(T)$ and it defined by $h(T) := \lim_{\epsilon\to 0} r(\epsilon, X, T)$. To extend this notion for fuzzy dynamical systems we recall the concept of continuous triangular norm [13].

A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if it satisfies the following condition;

i) $*$ is an associative and commutative operation;

ii) $a * 1 = a$ for all $a \in [0, 1]$;
iii) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$, where $a, b, c, d \in [0, 1]$.

A fuzzy metric space is a tripe $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and $M : X \times X \times (0, \infty) \longrightarrow [0, 1]$ is a mapping which has the following properties:

For every $x, y, z \in X$ and $t, s > 0$;

1) $M(x, y, t) > 0$;
2) $M(x, y, t) = 1$ if and only if $x = y$;
3) $M(x, y, t) = M(y, x, t)$;
4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is a continuous map.

Definition 2.1. A fuzzy metric space $(X, M, *)$ is called semicompact if for every $t > 0$ and $\epsilon > 0$ there is $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup_{i=1}^{n} B(x_i, \epsilon, t)$, where $B(x_i, \epsilon, t) = \{x : M(x, x_i, t) \geq 1 - \epsilon\}$.

In this paper, we assume that $T : X \to X$ is a mapping and $(X, M, *)$ is a semicompact fuzzy metric space.

For a natural number n we define:

$$M_n(x, y, t) = \min\{M(T^i(x), T^i(y), t) : x, y \in X \text{ and } i \in \{0, 1, 2, ..., n - 1\}\}.$$

If $F \subseteq X$, $\epsilon > 0$, $t > 0$ and $n \in \mathbb{N}$, then F is called an (n, ϵ, t) fuzzy spanning subset of X with respect to T if for given $x \in X$ there is $y \in F$ such that $M_n(x, y, t) \geq 1-\epsilon$. Subset E of X is called an (n, ϵ, t) fuzzy separated if $M_n(x, y, t) \leq 1 - \epsilon$, when x and y are different points in E. $r_n(\epsilon, t, X, T)$ denotes the number of elements of an (n, ϵ, t) fuzzy spanning set for X with respect to T, with the smallest cardinality. Also, $s_n(\epsilon, t, X, T)$ denotes the number of elements of an (n, ϵ, t) fuzzy separated set for X with respect to T, with the largest cardinality. We define

$$r(\epsilon, t, X, T) = \lim_{n\to\infty} \frac{1}{n}(\log r_n(\epsilon, t, X, T)), \text{ and}$$
$$s(\epsilon, t, X, T) = \lim_{n\to\infty} \frac{1}{n}(\log s_n(\epsilon, t, X, T)).$$

3. Fuzzy Entropy

Let us to begin this section with the following theorem.

Theorem 3.1. *Let $(X, M, *)$ be a semicompact fuzzy metric space and $T : X \longrightarrow X$ be a mapping. Then $r_n(\epsilon, t, X, T)$ and $r(\epsilon, t, X, T)$ are natural numbers.*

Proof. Let $\epsilon > 0$, $t > 0$ and $n \in \mathbb{N}$ be given. Since $*$ is a continuous t-norm and $1 * 1 = 1$ then there is $1 > \lambda > 0$ such that $(1-\lambda) * (1-\lambda) \geq (1-\epsilon)$. Moreover the semi compactness of X implies $X \subseteq \bigcup_{i=1}^{k} B(x_i, \lambda, t/2)$, for some $x_i \in X$, and $1 \leq i \leq k$.

We define

$$A_{(j_0,j_1,\dots,j_{n-1})} := \left\{ x : T^i(x) \in B\left(x_{j_i}, \lambda, \frac{t}{2}\right) \right\} \text{ where } 1 \leq j_i \leq k.$$

For each non empty set $A_{(j_0,j_1,\dots,j_{n-1})}$ we choose a unique $y_{(j_0,j_1,\dots,j_{n-1})} \in A_{(j_0,j_1,\dots,j_{n-1})}$ and we define $A := \{y_{(j_0,j_1,\dots,j_{n-1})} : 1 \leq j_i \leq k\}$. Since for every $0 \leq i \leq n-1$, j_i is between 1 and k, then $|A| \leq k^n$, where $|A|$ denotes the cardinality of A. We show that A is an (n, ϵ, t) fuzzy spanning set for X with respect to T. Let x be an arbitrary member of X. Since $\bigcup_{1 \leq j_i \leq k} A_{(j_0,j_1,\dots,j_{n-1})} = X$, then for every $0 \leq i \leq n-1$ there is $1 \leq j_i \leq k$ such that $x \in A_{(j_0,j_1,\dots,j_{n-1})}$. Therefore $y_{(j_0,j_1,\dots,j_{n-1})}, x \in A_{(j_0,j_1,\dots,j_{n-1})}$. Now the definition of $A_{(j_0,j_1,\dots,j_{n-1})}$ implies $T^i(x), T^i(y_{(j_0,j_1,\dots,j_{n-1})}) \in B(x_{j_i}, \lambda, t/2)$ for each $0 \leq i \leq n-1$. So

$$\begin{aligned} M(T^i(x), T^i(y_{(j_0,j_1,\dots,j_{n-1})}, t) &= M\left(T^i(x), T^i(y_{(j_0,j_1,\dots,j_{n-1})}), \frac{t}{2} + \frac{t}{2}\right) \\ &\geq M\left(T^i(x), x_{j_i}, \frac{t}{2}\right) * M\left(x_{j_i}, T^i(y_{(j_0,j_1,\dots,j_{n-1})}), \frac{t}{2}\right) \\ &\geq (1-\lambda) * (1-\lambda) \geq (1-\epsilon). \end{aligned}$$

Thus

$$M_n(x, y_{(j_0,j_1,\dots,j_{n-1})}, t) \geq 1 - \epsilon \text{ for every } 0 \leq i \leq n-1.$$

Since $y_{(j_0,j_1,\dots,j_{n-1})}$ is a member of A, then A is an (n, ϵ, t) fuzzy spanning set for X with respect to T. So $r_n(\epsilon, t, X, T) \leq k^n$. Thus

$$r(\epsilon, t, X, T) = \lim_{n \to \infty} \frac{1}{n} \log r_n(\epsilon, t, X, T) \leq k.$$

Hence $r_n(\epsilon, t, X, T)$ and $r(\epsilon, t, X, T)$ are natural numbers. □

Remark 3.1.

(i) If $\epsilon_1 < \epsilon_2$ then $r_n(\epsilon_1, t, X, T) \geq r_n(\epsilon_2, t, X, T)$ and $r(\epsilon_1, t, X, T) \geq r(\epsilon_2, t, X, T)$.

(ii) If $\epsilon_1 < \epsilon_2$ then $s_n(\epsilon_1, t, X, T) \geq s_n(\epsilon_2, t, X, T)$ and $s(\epsilon_1, t, X, T) \geq s(\epsilon_2, t, X, T)$.

Now we define fuzzy entropy for a map T on a semicompact fuzzy metric space $(X, M, *)$.

Definition 3.1. Let $(X, M, *)$ be semicompact and $T : X \to X$ be a mapping. Then we define the entropy of T by $h_{M,t}(T) := \lim_{\epsilon\to 0}(r(\epsilon, t, X, T))$.

Since the fuzzy metric M has essential role in the above definition we denote the fuzzy entropy by $h_{t,M}(T)$.

The next theorem implies that fuzzy entropy is an extension of topological entropy.

Theorem 3.2. *Let (X, d) be a compact metric space. Also, let $(X, M, *)$ be a fuzzy semicompact such that $M(x, y, t) = t/(t + d(x, y))$, and $*$ be an arbitrary t-norm. Now let $T : X \to X$ be a maping. Then $h(T) = h_{M,t}(T)$.*

Proof. Let $t > 0$ be given. Also consider F be (n, ϵ) spanning set for X with respect to T. If x is a member of X, then there is $y \in F$ such that $d_n(x, y) \leq \epsilon$. Therefore

$$M_n(x, y, t) \geq \frac{t}{t+\epsilon} = 1 - \frac{\epsilon}{t+\epsilon}.$$

So F is $(n, \epsilon/(t + \epsilon), t)$ fuzzy spanning set for X with respect to T. Thus

$$r_n\left(\frac{\epsilon}{t+\epsilon}, t, X, T\right) \leq r_n(\epsilon, X, T)$$

and so

$$r\left(\frac{\epsilon}{t+\epsilon}, t, X, T\right) \leq r(\epsilon, X, T).$$

Then $h_{M,t}(T) \leq h(T)$. Now Let F be (n, ϵ, t) fuzzy spanning set for X with respect to T. If x is a member of X, then there is $y \in F$ such that $M_n(x, y, t) \geq 1 - \epsilon$. Therefore $d_n(x, y) \leq t\epsilon/(1 - \epsilon)$. So F is $(n, t\epsilon/(1 - \epsilon))$ spanning set for X with respect to T. Thus

$$r_n\left(\frac{t\epsilon}{1-\epsilon}, X, T\right) \leq r_n(\epsilon, t, X, T)$$

and so

$$r\left(\frac{t\epsilon}{1-\epsilon}, X, T\right) \leq r(\epsilon, t, X, T).$$

Then $h(T) \leq h_{M,t}(T)$. Therefore $h_{M,t}(T) = h(T)$. □

Let $(X_1, M_1, *_1)$, $(X_2, M_2, *_2)$ be two fuzzy metric spaces. A function $T : X_1 \longrightarrow X_2$ is called fuzzy continuous. If for every $x \in X_1$, $t > 0$ and $\epsilon > 0$ there is $\delta > 0$ such that, $M_1(x, y, t) \geq (1 - \delta)$ implies $M_2(T(x), T(y), t) \geq (1 - \epsilon)$.

Now we define uniformly fuzzy continuity for a mapping T.

Definition 3.2. Let $(X_1, M_1, *_1)$, $(X_2, M_2, *_2)$ be fuzzy metric spaces. A mapping $T : X_1 \longrightarrow X_2$ is called uniformly fuzzy continuous. If for every $t > 0$ and $\epsilon > 0$ there is $\delta > 0$ such that, $M_1(x, y, t) \geq (1 - \delta)$ implies $M_2(T(x), T(y), t) \geq (1 - \epsilon)$, for every $x, y \in X_1$.

Also, two fuzzy metrics $(X, M, *)$ and $(X, M_1, *_1)$ on X are called uniformly fuzzy equivalent if two mappings $\mathrm{id} : (X, M, *) \to (X, M_1, *_1)$ and $\mathrm{id} : (X, M_1, *_1) \to (X, M, *)$ are uniformly fuzzy continuous.

Theorem 3.3.

(i) *Let $n \in \mathbb{N}$, $t > 0$ and $\epsilon > 0$. Also, let $\lambda > 0$ be a number such that $(1-\lambda)*(1-\lambda) \geq (1-\epsilon)$. Then $r_n(\epsilon, t, X, T) \leq s_n(\epsilon, t, X, T) \leq r_n(\lambda, t/2, X, T)$.*

(ii) *Let $(X, M_1, *_1)$ and $(X, M_2, *_2)$ be uniformly fuzzy equivalent and $T : X \to X$ be a mapping. Then $h_{M,t}(T) = h_{M_1,t}(T)$.*

Proof.

(i) If E is an (n, ϵ, t) fuzzy separated subset of X with the maximum cardinality then E is an (n, ϵ, t) fuzzy spanning set for X with respect to T, to prove this, let $x \in X$. Since E have maximum cardinality then there is $y \in E$ such that $M_n(x, y, t) \geq 1 - \epsilon$. Therefore $r_n(\epsilon, t, X, T) \leq s_n(\epsilon, t, X, T)$. To show the other inequality suppose E is an (n, ϵ, t) fuzzy separated subset of X with respect to T and F is an $(n, \lambda, t/2)$ fuzzy spanning set for X with respect to T. We define $\phi : E \to F$ as follows. For $x \in E$, we define $\phi(x) \in F$ such that $M_n(x, \phi(x), t/2) \geq (1 - \lambda)$. ϕ is injective, because if x_1 and x_2 are two members of E such that $\phi(x_1) = \phi(x_2)$, then $M_n(x_1, \phi(x_1), t/2) \geq 1 - \lambda$ and $M_n(x_2, \phi(x_2), t/2) \geq 1 - \lambda$. So

$$M_n(x_1, x_2, t) \geq M_n(x_1, \phi(x_1), t/2) * M_n(\phi(x_1), x_2, t/2) \geq 1 - \epsilon.$$

Since $x_1, x_2 \in E$ then $x_1 = x_2$. So the cardinality of E is not greater than of F. Hence $s_n(\epsilon, t, X, T) \leq r_n(\lambda, t/2, X, T)$.

(ii) Let $\epsilon_1 > 0$ be given, choose $\epsilon_2 > 0$ such that

$$M_1(x, y, t) \geq (1 - \epsilon_2) \Rightarrow M(x, y, t) \geq (1 - \epsilon_1)$$

and choose $\epsilon_3 > 0$ such that

$$M(x, y, t) \geq (1 - \epsilon_3) \Rightarrow M_1(x, y, t) \geq (1 - \epsilon_2).$$

Then

$$r_n(\epsilon_1, t, (X, M, *), T) \leq r_n(\epsilon_2, t, (X, M_1, *), T)$$

and

$$r_n(\epsilon_2, t, (X, M_1, *), T) \leq r_n(\epsilon_3, t, (X, M, *), T).$$

Hence

$$r(\epsilon_1, (X, M, *), T) \leq r(\epsilon_2, (X, M_1, *), T) \leq r(\epsilon_3, (X, M, *), T).$$

If $\epsilon_1 \to 0$, then $\epsilon_2 \to 0$, and $\epsilon_3 \to 0$ then

$$h_{M,t}(T) = h_{M_1,t}(T).$$

□

Corollary 3.1. $r(\epsilon, t, X, T) \leq s(\epsilon, t, X, T) \leq r(\lambda, t/2, X, T)$, *where* $(1-\lambda)*(1-\lambda) \geq (1-\epsilon)$.

Theorem 3.4. *Let* $(X, M, *)$ *be a semicompact fuzzy metric space and* $T : X \to X$ *be a mapping. Then* $h_{M,t}(T^m) \leq m h_{M,t}(T)$, *where* m *is a natural number.*

Proof. Let $n \in \mathbb{N}$ and $\epsilon > 0$ be given. If F is an (nm, ϵ, t) fuzzy spaning set for X with respect to T, then F is an (n, ϵ, t) fuzzy separated set for X with respect to T^m. Therefore $r_n(\epsilon, t, X, T^m) \leq r_{nm}(\epsilon, t, X, T)$. So $h_{M,t}(T^m) \leq m h_{M,t}(T)$. □

In the next theorem we present a condition which implies the equality instead of inequality mentioned in Theorem 3.4.

Theorem 3.5. *Let* $T : X \to X$ *be a mapping such that* $M(x, y, t) \geq M(T(x), T(y), t)$ *for every* $x, y \in X$. *Then* $h_{M,t}(T^m) = m h_{M,t}(T)$, *where* m *is a natural number.*

Proof. Let $n \in \mathbb{N}$ and $\epsilon > 0$ be given. Since $M(x, y, t) \geq M(T(x), T(y), t)$ for every $x, y \in X$. Then $M_n(x, y, t) = M(T^{n-1}(x), T^{n-1}(y), t)$. Therefore $r_{nm}(\epsilon, t, X, T) \leq r_n(\epsilon, t, X, T^m)$. So $h_{M,t}(T^m) \leq m h_{M,t}(T)$. □

Let $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ be two semicompact fuzzy metric spaces. We define a fuzzy metric space $(X_1 \times X_2, M, *)$, by defining $a * b = \min\{a *_1 b, a *_2 b\}$ and $M((x, y), (x', y'), t) = \min\{M_1(x, x', t), M_2(y, y', t)\}$. Let $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ be two semicompact fuzzy metric spaces. Moreover let $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ be two mappings . We define $T : X_1 \times X_2 \to X_1 \times X_2$ by $T((x, y)) = (T_1(x), T_2(y))$. Then we have the next theorem.

Theorem 3.6. $h_{M,t}(T) \leq h_{M_1,t}(T_1) + h_{M_2,t}(T_2)$.

Proof. Let $n \in \mathbb{N}$ and $\epsilon > 0$ be given. Let F_1 and F_2 be (n, ϵ, t) fuzzy spanning sets for X_1 and X_2 with respect to T_1 and T_2 respectively. Then $F_1 \times F_2$ is an (n, ϵ, t) fuzzy spanning set for X with respect to T. So

$$r_n(\epsilon, t, X, T) \leq r_n(\epsilon, t, X, T_1)\, r_n(\epsilon, t, X, T_2),$$

and as a result $h_{M,t}(T) \leq h_{M_1,t}(T_1) + h_{M_2,t}(T_2)$. □

Definition 3.3. Let $T : (X, M, *) \to (X, M, *)$ and $S : (Y, M', *') \to (Y, M', *')$ be two mappings. Then T and S are called fuzzy topological equivalent if there is a fuzzy homeomorphism $g : (X, M, *) \to (Y, M', *')$ such that $g \circ T = S \circ g$.

Theorem 3.7. *In Definition 3.3, let g and g^{-1} be two fuzzy uniformly continuous maps. Then $h_{M,t}(T) = h_{M',t}(S)$.*

Proof. Let $\epsilon > 0$, $t > 0$ be given and n be a natural number. Since g is fuzzy uniformly continuous, then there is $\delta > 0$ such that $M(x, x', t) \geq 1 - \delta$ implies $M'(g(x), g(x'), t) \geq 1 - \epsilon$. Now let F be an (n, δ, t) fuzzy spanning set for X with respect to T. We show that $g(F)$ is (n, ϵ, t) fuzzy spanning set for Y with respect to S. Let $y \in Y$ be given. So there is $x \in X$ such that $g(x) = y$. Therefore there is $x' \in F$ that $M_n(x, x', t) \geq 1 - \delta$. Now we have $M(T^i(x), T^i(x'), t) \geq 1 - \delta$ for $0 \leq i \leq n - 1$. So $M'(g(T^i(x)), g(T^i(x')), t) \geq 1 - \epsilon$ for $0 \leq i \leq n - 1$. Since $g \circ T = S \circ g$, we have $M'(S^i(g(x)), S^i(g(x')), t) \geq 1 - \epsilon$. Therefore $g(F)$ is (n, ϵ, t) fuzzy spanning set for Y with respect to S. So $r_n(\epsilon, t, Y, S) \leq r_n(\delta, t, X, T)$. Thus $h_{M',t}(S) \leq h_{M,t}(T)$. By a similar method, we can deduce $h_{M,t}(T) \leq h_{M',t}(S)$. So

$$h_{M,t}(T) = h_{M',t}(S).$$

□

Now let us to present an example.

Example 3.1. Let $(X, M, *)$ be fuzzy semicompact such that

$$X = \{(x_i)_{i=1}^{\infty} : x_i \in \{0, 1\}\}, \qquad d((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty}) = \sum_{x_i \neq y_i} (1/2^i),$$

$$M((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty}, t) = \frac{t}{t + d((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty})},$$

and $*$ be an arbitrary t-norm. For every $x = (x_i)_{i=1}^{\infty} \in X$, κ_x denotes the largest natural number i such that $x_j = x_1$ for every $j \leq i$. Also, if for each $i \in \mathbb{N}$, $x_i = 1$

or, for each $i \in \mathbb{N}$, $x_i = 0$ we put $\kappa_x = \infty$. We define $T : X \to X$ by

$$T(x = (x_i)_{i=1}^{\infty}) = \begin{cases} (x_{i+\kappa_x})_{i=1}^{\infty} & \text{if } \kappa_x \text{ is an odd number} \\ (x_{i+\kappa_x})_{i=0}^{\infty} & \text{if } \kappa_x \text{ is an even number} \\ x & \text{if } \kappa_x \text{ is infinity} \end{cases}$$

We show that the fuzzy entropy of T is positive. We define $g : X \to X$ by

$$g(x = (x_i)_{i=1}^{\infty}) = \begin{cases} (x_{i+\kappa_x})_{i=1}^{\infty} & \text{if } \kappa_x \text{ is finite} \\ x & \text{if } \kappa_x \text{ is infinity} \end{cases}$$

Let $t > 0$ be given. Moreover let $\lambda = 1/(1 + 32t)$, $\epsilon = 1/32$ and $n \in \mathbb{N}$ be an even number. Then there is $k \in \mathbb{N}$ such that $n = 2k$. We say that $x \sim y$ if $x_1 = y_1$ and $\kappa_{g^i(x)} = \kappa_{g^i(y)}$ for $0 \leq i \leq (k-1)$. It is clear that $\sim$ is a equivalence relation on X. If

$$A = \{[x] : x \in X, \kappa_{g^i(x)} \in \{1, 2\}, 0 \leq i \leq (k-1)\},$$

then the number of members of A is 2^{k+1}. For $[x] \in A$, we choose a unique $y_{[x]} \in [x]$. If $B = \{y_{[x]} : [x] \text{ is a member of } A\}$, then $|B| = |A| = 2^{k+1}$. We take $y, y' \in B$ such that $y = (y_i)_{i=1}^{\infty} \neq y' = (y'_i)_{i=1}^{\infty}$. If $y_1 \neq y'_1$, then $d(y, y') \geq \epsilon$. If $y_1 = y'_1$, then there is $0 \leq t \leq (k-1)$ such that $\kappa_{g^t(y)} \neq \kappa_{g^t(y')}$ and $\kappa_{g^j(y)} = \kappa_{g^j(y')}$ for $0 \leq j < t$. We know that $\kappa_{g^j(y)} = \kappa_{g^j(y')}$, for $0 \leq j < t$. So there is $m \leq 2t$ such that $g^t(y) = (y_{i+m})_{i=1}^{\infty}$ and $g^t(y') = (y'_{i+m})_{i=1}^{\infty}$. Hence $d(T^m(y), T^m(y')) \geq \epsilon$. Thus

$$M_n(y, y', t) = \min\left\{\frac{t}{t + d(T^i(y), T^i(y'))} : 0 \leq i \leq (n-1)\right\} \leq 1 - \frac{1}{1 + 32t} = 1 - \frac{1}{\lambda}.$$

So $s_n(\lambda, t, X, T) \geq |B| = 2^{k+1}$. Let λ_0 be a number such that $(1 - \lambda_0) * (1 - \lambda_0) \geq 1 - \lambda$. Theorem 3.3 implies, $r_n(\lambda_0, t/2, X, T) \geq s_n(\lambda, t, X, T) \geq 2^{k+1}$ So $h_{M,t/2}(T) \geq (\log 2)/2$.

4. Conclusion

Uncertainty is a special property of human made means. Thus fuzzy systems are more compatible with human made means or natural description of phenomena. In this direction fuzzy entropy is a good means to describe the complexity of a fuzzy system. In Theorem 3.5 we present a method to construct systems with an arbitrarily large security. We suggest another method to construct complex systems with finding a condition for the equality instead of inequality presented in Theorem 3.6, and this is a topic for further research.

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