Çankaya University Journal of Science and Engineering
Volume 9 (2012), No. 1, 1–8

On Solving First Order Linear Complex Partial Derivative Equations by the Two Dimensional Differential Transform Method

Murat Düz

Karabük University, Department of Mathematics, 78050 Karabük, Turkey
mduz@karabuk.edu.tr

Özet. Bu çalışmada birinci mertebeden lineer kompleks denklemler çözüldü. İlk olarak bu denklemleri lineer ve imajiner kısımlarına ayırdık. Böylece iki denklem elde edildi. İki boyutlu diferensiyel dönüşüm metodu uygulanarak çözümün reel ve imajiner kısımları elde edildi.

Anahtar Kelimeler. Diferensiyel dönüşüm, kompleks denklem.

Abstract. In this study, first order linear complex equations are solved. Firstly we separate the real and imaginary parts of these equations. Thus two equalities are obtained. The real and imaginary parts of the solution are obtained by using the two dimensional differential transform method.

Keywords. Differential transform, complex equation.

1. Introduction

The concept of differential transform (in one dimension) was first proposed and applied to solve linear and non linear initial value problems in electric circuit analysis by Zhou [1]. By using the one dimensional differential transform method, nonlinear differential equations were solved in [3]. The differential transform method gives a solution in the form of a polynomial. The differential transform is an iterative procedure. Solving partial differential equations by the two dimensional differential transform method was proposed by Cha'o Kuang Chen and Shing Huei Ho in [2]. Two dimensional differential transform is derived from two dimensional Taylor series expansion. In this paper, linear complex partial differential equations are solved by using the two dimensional differential transform method. Let $w = w(z, \bar{z})$ be a

Received June 7, 2011; accepted December 16, 2011.

ISSN 1309 - 6788

complex function. Here $z = x + iy$, $w(z, \bar{z}) = u(x, y) + iv(x, y)$. The derivative according to z and $\bar{z}$ of $w = w(z, \bar{z})$ is defined as follows:

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right), \tag{1}$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right), \tag{2}$$

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{3}$$

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}, \tag{4}$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{5}$$

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{6}$$

2. Two Dimensional Differential Transform

Definition 2.1. The two dimensional differential transform of function $f(x, y)$ is defined as follows

$$F(k, h) = \frac{1}{k!\,h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial y^h}\right]_{x=0,y=0}. \tag{7}$$

In Equation (1), $f(x, y)$ is an original function and $F(k, h)$ is a transformed function, which is called T-function briefly.

Definition 2.2. The differential inverse transform of $F(k, h)$ is defined as follows

$$f(x, y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} F(k, h) x^k y^h \tag{8}$$

$$= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!\,h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial y^h} f(x, y)\right]_{x=0,y=0} x^k y^h. \tag{9}$$

Equation (9) implies that the concept of two dimensional differential transform is derived from two dimensional Taylor series expansion.

Theorem 2.1. [2] *If* $w(x, y) = u(x, y) + v(x, y)$ *then* $W(k, h) = U(k, h) + V(k, h)$.

Theorem 2.2. [2] *If* $w(x, y) = \lambda u(x, y)$ *then* $W(k, h) = \lambda U(k, h)$.

Theorem 2.3. [2] *If* $w(x, y) = \dfrac{\partial u(x, y)}{\partial x}$ *then* $W(k, h) = (k + 1)U(k + 1, h)$.

Theorem 2.4. [2] *If* $w(x,y) = \dfrac{\partial u(x,y)}{\partial y}$ *then* $W(k,h) = (h+1)U(k,h+1)$.

Theorem 2.5. [2] *If* $w(x,y) = \dfrac{\partial^{r+s} u(x,y)}{\partial x^r \partial y^s}$ *then*

$$W(k,h) = (k+1)(k+2)\cdots(k+r)(h+1)(h+2)\cdots(h+s)U(k+r,h+s).$$

Theorem 2.6. [2] *If* $w(x,y) = u(x,y)v(x,y)$ *then*

$$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s).$$

Theorem 2.7. [2] *If* $w(x,y) = x^m y^n$ *then*

$$W(k,h) = \delta(k-m,h-n) = \delta(k-m)\,\delta(h-n),$$

where $\delta(k-m) = \begin{cases} 1, & k=m, \\ 0, & k \neq m, \end{cases} \qquad \delta(h-n) = \begin{cases} 1, & h=n, \\ 0, & h \neq n. \end{cases}$

In real applications, the function $w(x,y)$ is expressed by a finite series and the equation (8) can be written as

$$w(x,y) = \sum_{k=0}^{m}\sum_{h=0}^{n} W(k,h)x^k y^h. \tag{10}$$

The equation (10) implies that

$$\sum_{k=m+1}^{\infty}\sum_{h=n+1}^{\infty} W(k,h)x^k y^h$$

is negligibly small.

3. Using the Two-Dimensional Differential Transform to Solve Complex Partial Differential Equations

To demonstrate how to use the two-dimensional transform to solve complex partial equations examples are given here.

Example 3.1. Solve the following boundary value problem

$$2\frac{\partial w}{\partial \bar{z}} + \frac{\partial w}{\partial z} = 2z + 7 \tag{11}$$

with the boundary conditions

$$w(x,0) = x^2 + 5x, \tag{12}$$

$$\frac{\partial w}{\partial y}(x,0) = i(2x+1). \tag{13}$$

From equalities (5), (6) and (11)

$$\begin{aligned} 2\frac{\partial w}{\partial \bar{z}} + \frac{\partial w}{\partial z} &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &= \frac{3}{2}\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} + \frac{3i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} = 2(x+iy) + 7, \end{aligned}$$

and therefore we get the following equations

$$3\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 4x + 14, \tag{14}$$

$$\frac{\partial u}{\partial y} + 3\frac{\partial v}{\partial x} = 4y. \tag{15}$$

From differential transform of (14) and (15) we obtain

$$3(k+1)U(k+1,h) - (h+1)V(k,h+1) = 4\,\delta(k-1,h) + 14\,\delta(k,h), \tag{16}$$

$$(h+1)U(k,h+1) + 3(k+1)V(k+1,h) = 4\,\delta(k,h-1), \tag{17}$$

and

$$w(x,y) = \sum_{k=0}^{m}\sum_{h=0}^{n} W(k,h)x^k y^h = \sum_{k=0}^{m}\sum_{h=0}^{n}[U(k,h) + iV(k,h)]x^k y^h. \tag{18}$$

From (18) and (12) we get the following equalities

$$\begin{aligned} &U(0,0) = 0,\ U(1,0) = 5,\ U(2,0) = 1,\ U(i,0) = 0\ (i = 3,4,5,\ldots), \\ &V(i,0) = 0\ (i = 0,1,2,\ldots), \end{aligned} \tag{19}$$

$$\frac{\partial w}{\partial y} = \sum_{k=0}^{m}\sum_{h=0}^{n} h[U(k,h) + iV(k,h)]x^k y^{h-1}. \tag{20}$$

From (20) and (13) we obtain

$$\begin{aligned} &V(0,1) = 1,\ V(1,1) = 2,\ V(i,1) = 0\ (i = 2,3,4,\ldots), \\ &U(i,1) = 0\ (i = 0,1,2,\ldots). \end{aligned} \tag{21}$$

Substituting equations (19) and (21) into equations (16) and (17) and by using recursive method

$$U(0,2) = -1 \tag{22}$$

and all the others are zero. Thus from equalities (19), (21) and (22) we find that

$$u(x,y) = x^2 - y^2 + 5x, \tag{23}$$

$$v(x,y) = 2xy + y. \tag{24}$$

From equalities (23) and (24) we obtain

$$\begin{aligned} w(x,y) &= x^2 - y^2 + 5x + i(2xy + y) \\ &= x^2 + 2ixy - y^2 + 3(x+iy) + 2(x-iy) \\ &= z^2 + 3z + 2\bar{z}. \end{aligned}$$

Example 3.2. Solve the following boundary value problem

$$z\frac{\partial w}{\partial z} - \bar{z}\frac{\partial w}{\partial \bar{z}} = 2z^2 + 3\bar{z} \tag{25}$$

with the boundary conditions

$$w(x,0) = x^2 - 3x, \tag{26}$$

$$\frac{\partial w}{\partial y}(x,0) = i(2x+3). \tag{27}$$

From equalities (5), (6) and (25) we get the following equality:

$$\begin{aligned} &(x+iy)\left[\frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \\ &\quad - (x-iy)\left[\frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] = 2(x+iy)^2 + 3(x-iy). \end{aligned} \tag{28}$$

Equality (28) is equivalent to the following equations:

$$x\frac{\partial v}{\partial y} - y\frac{\partial v}{\partial x} = 2x^2 - 2y^2 + 3x, \tag{29}$$

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 4xy - 3y. \tag{30}$$

From the differential transform of equations (29) and (30) we obtain the following equations:

$$\begin{aligned} \sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-1,h-s)(s+1)V(k-r,s+1) - \sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r,h-s-1)(k-r+1)V(k-r+1,s) \\ = 2\,\delta(k-2,h) - 2\,\delta(k,h-2) + 3\,\delta(k-1,h), \end{aligned} \tag{31}$$

$$\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r,h-s-1)(k-r+1)U(k-r+1,s)-\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-1,h-s)(s+1)U(k-r,s+1)$$
$$=4\sum_{r=0}^{k}\sum_{s=0}^{h}[\delta(r-1,s)\,\delta(k-r,h-s-1)-3\,\delta(k,h-1)]. \quad (32)$$

From (26) we obtain

$$\begin{aligned}&U(0,0)=0,\ U(1,0)=-3,\ U(2,0)=1,\ U(i,0)=0\ (i=3,4,5,\ldots),\\&V(i,0)=0\ (i=0,1,2,\ldots).\end{aligned} \quad (33)$$

From (27) we obtain

$$\begin{aligned}&U(i,1)=0\ (i=0,1,2,\ldots),\\&V(0,1)=3,\ V(1,1)=2,\ V(i,1)=0\ (i=2,3,4,\ldots).\end{aligned} \quad (34)$$

Substituting equations (33) - (34) into equations (31) - (32) and by recursive method

$$U(0,2)=-1$$

and all the others are zero. Thus we find that

$$u(x,y)=x^2-y^2-3x \quad (35)$$

and

$$v(x,y)=2xy+3y. \quad (36)$$

From (35) and (36) equalities we get

$$\begin{aligned}w(x,y)&=x^2-y^2-3x+i(2xy+3y)\\&=x^2-y^2+2xyi-3(x-iy)\\&=z^2-3\bar{z}.\end{aligned}$$

Example 3.3. Solve the following boundary value problem

$$6z\frac{\partial w}{\partial \bar{z}}+5\frac{\partial w}{\partial z}=54z-25\bar{z} \quad (37)$$

with the boundary conditions

$$w(x,0)=2x^3-2x^2+4x+5, \quad (38)$$
$$\frac{\partial w}{\partial y}(x,0)=i(6x^2+6x-4). \quad (39)$$

From (37) we can write that

$$6(x+iy)\left[\frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + 5\left[\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]=54(x+iy)-25(x-iy).$$

Hence

$$3x\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)-3y\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{5}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=29x, \tag{40}$$

$$3x\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+3y\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{5}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=79y. \tag{41}$$

From differential transforms of equations (40) and (41), we obtain the following equations:

$$\sum_{r=0}^{k}\sum_{s=0}^{h}3\,\delta(r-1,h-s)(k-r+1)U(k-r+1,s)-\sum_{r=0}^{k}\sum_{s=0}^{h}3\,\delta(r,h-s-1)(s+1)U(k-r,s+1) -\sum_{r=0}^{k}\sum_{s=0}^{h}3\,\delta(r-1,h-s)(s+1)V(k-r,s+1)-\sum_{r=0}^{k}\sum_{s=0}^{h}3\,\delta(r,h-s-1)(k-r+1)V(k-r+1,s) +\frac{5}{2}(k+1)U(k+1,h)+\frac{5}{2}(h+1)V(k,h+1)=29\,\delta(k-1,h), \tag{42}$$

$$\sum_{r=0}^{k}\sum_{s=0}^{h}3\,\delta(r,h-s-1)(k-r+1)U(k-r+1,s)+\sum_{r=0}^{k}\sum_{s=0}^{h}3\,\delta(r-1,h-s)(s+1)U(k-r,s+1) +\sum_{r=0}^{k}\sum_{s=0}^{h}3\,\delta(r-1,h-s)(k-r+1)V(k-r+1,s)-\sum_{r=0}^{k}\sum_{s=0}^{h}3\,\delta(r,h-s-1)(s+1)V(k-r,s+1) +\frac{5}{2}(k+1)V(k+1,h)-\frac{5}{2}(h+1)U(k,h+1)=79\,\delta(k,h-1). \tag{43}$$

From (38) we obtain

$$U(0,0)=5,\ U(1,0)=4,\ U(2,0)=-2,\ U(3,0)=2,\ U(i,0)=0\ (i=4,5,6,\ldots), \quad V(i,0)=0\ (i=0,1,2,\ldots). \tag{44}$$

From (39) we obtain

$$U(i,1)=0\ (i=0,1,2,\ldots), \quad V(0,1)=-4,\ V(1,1)=6,\ V(2,1)=6,\ V(i,1)=0\ (i=3,4,5,\ldots). \tag{45}$$

If we write in Equation (42) $k=0,\ h=2$ then

$$\frac{5}{2}U(1,2)+\frac{15}{2}V(0,3)-6U(0,2)-3V(1,1)=0. \tag{46}$$

If we write (43) $k=h=1$ then

$$6U(0,2)+18-5U(1,2)=0. \tag{47}$$

Similarly if we write (43) $k=0,\ h=1$ then

$$3U(1,0)-3V(0,1)+\frac{5}{2}V(1,1)-5U(0,2)=79. \tag{48}$$

From (46), (47) and (48) and also by using (44) and (45) we get the following equalities:

$$U(1,2)=-6,\ U(0,2)=-8,\ V(0,3)=-2. \tag{49}$$

Substituting equations (44), (45) and (49) into equations (42) - (43) and by recursive method the others are zero. From equations (44), (45) and (49) we obtain

$$u(x,y)=2x^3-6xy^2-2x^2-8y^2+4x+5, \tag{50}$$

$$v(x,y)=6x^2y-2y^3+6xy-4y. \tag{51}$$

Equations (50) and (51) imply

$$\begin{aligned} w(x,y) &= 2x^3-6xy^2-2x^2-8y^2+4x+5+i(6x^2y-2y^3+6xy-4y) \\ &= 2(x^3+3ix^2y-3xy^2-iy^3)+3(x^2+2ixy-y^2)-5(x^2+y^2)+4(x-iy)+5 \\ &= 2z^3+3z^2-5|z|^2+4\bar{z}+5. \end{aligned}$$

4. Conclusion

In this study, we solved first order linear complex equations by using the differential transform method. We aim to solve the higher order linear and nonlinear complex equations with differential transform method.

References

[1] J. K. Zhou, *Differential Transformation and its Application for Electrical Circuits*, Huazhang University Press, Wuhan, China 1986.

[2] C. K. Chen and S. H. Ho, Solving partial differential equations by two dimensional differential transform, *Applied Mathematics and Computation* **106** (1999), 171–179.

[3] Y. Keskin and G. Oturanç, The differential transform methods for nonlinear functions and its applications, *Selçuk Journal of Applied Mathematics* **9** (2008), 69–76.