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Stochastic Stability in Fuzzy Dynamical Systems

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Özet. Bu makalede bulanık dinamik sistemler için gölgeleme özelliği kavramını ortaya atıyor ve bu özelliğin bulanık topolojik eşleniklik altında değişmez olduğunu kanıtlıyoruz. Gölgeleme özellikli dinamik sistemleri kullanarak bulanık gölgeleme özellikli bulanık dinamik sistemlerin inşası için bir metot sunuyoruz.[†]

Anahtar Kelimeler. Bulanık metrik uzayı, bulanık dinamik sistem, gölgeleme özelliği, limit gölgeleme özelliği.

Abstract. In this paper we introduce the notion of the shadowing property for fuzzy dynamical systems and prove that this property is invariant under fuzzy topological conjugacy. By using of dynamical systems with the shadowing property we present a method to construct fuzzy dynamical systems with the fuzzy shadowing property.

Keywords. Fuzzy metric space, fuzzy dynamical system, shadowing property, limit shadowing property.

1. Introduction

Qualitative theory of dynamical systems is the study of the long-term behavior of evolving systems under perturbations [1, 2, 8, 10, 11]. The evolution of a particular state of a dynamical system is related to its orbits. For this purpose we need to know the structure of these orbits by solving a system of differential equations. If we cannot solve a system of differential equations analytically, then pseudo orbits are suggested as replacements for the orbits. Let us to explain this theory more precisely by considering a set X and a map $f : X \to X$. In numerical computation of f with the initial value $x_0 \in M$ we can approximate $f(x_0)$ by x_1. To continue this algorithm we can compute the value x_2 close to $f(x_1)$ and so on. The sequence

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$\{x_k\}$ is called a pseudo orbit and sometimes it is an approximation for the orbit $\mathcal{O}(x, f) = \{f^n(x)\}_{n\geq 0}$. In this paper we are going to use of fuzzy distances to approximate the pseudo orbits and their shadowing properties (stochastic stability). In the next section we introduce the notion of the fuzzy limit shadowing property (FLmSP), the fuzzy pseudo orbit tracing property (FPOTP) and the notion of the unique fuzzy pseudo orbit tracing property (UFPOTP) for a fuzzy continuous map. We prove that these properties are invariant under fuzzy topological conjugacy (Theorem 3.2). Moreover we prove that FPOTP is a fuzzy topological property i.e. it is invariant under a fuzzy homeomorphism. Moreover we introduce a method to construct new fuzzy dynamical systems with the above properties by the given dynamical systems which have those properties (Theorem 3.3).

2. Basic Notions

Let us to recall the definition of fuzzy metric space [9]. A binary operation $* : (0,1] \times (0,1] \longrightarrow (0,1]$ is called a *continuous triangular norm* (t-norm) if $*$ satisfies the following conditions:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$ for all $a \in (0,1]$;
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
5. If $a * b = a * c$ then $b = c$.

Properties 4 and 5 of a continuous t-norm imply that if $a * b \leq a * c$ then $b \leq c$.

Definition 2.1 (Kramosil and Michalek [9])**.** A *fuzzy metric space* is a triple $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and $M : X \times X \times [0,\infty) \longrightarrow [0,1]$ is a mapping which has the following properties:

For every $x, y, z \in X$ and $t, s > 0$:

1. $M(x, y, t) > 0$;
2. $M(x, y, t) = 1$ if and only if $x = y$;
3. $M(x, y, t) = M(y, x, t)$;
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
5. $M(x, y, \cdot) : (0,\infty) \longrightarrow [0,1]$ is a continuous map.

Let $(X, M, *)$ be a fuzzy metric space. A set $A \subset X$ is called a *fuzzy open set* if for any $x \in A$ there exist $0 < r < 1$ and $T_0 \in (0,\ \infty)$ so that if $M(x, y, t) > 1 - r$ then

$y \in A$ for all $t > T_0$. Let $(X, M, *)$ be a fuzzy metric space. An *open ball* $B(x, r, t)$ with center $x \in X$ and radius r, $0 < r < 1$, $t > 0$ is defined by

$$B(x, r, t) = \{y \in X : \; M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Let

$$\tau_M = \{A \subset X : x \in A \Leftrightarrow \text{there exist } t > 0 \text{ and } r \in (0, 1) \text{ s.t } \; B(x, r, t) \subset A\}.$$

Then τ_M is a topology on X [4, 5]. A fuzzy metric space $(X, M, *)$ is called a *compact fuzzy metric space* if (X, τ_M) is a compact space.

Remark 2.1. Let (X, d) be a metric space. Then

$$M(x, y, t) = M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

with the t-norm $a * b = ab$ is a fuzzy metric defined on X (called standard fuzzy metric space [3]) and the topology τ_d induced by the metric d and the topology τ_M are the same.

A fuzzy map $f : X \to X$ is said to be *fuzzy continuous* at x_0, if for each $\epsilon \in (0, 1)$ and each $t > 0$ there is $\delta \in (0, 1)$ so that for each x with $M(x, x_0, t) > 1 - \delta$, we deduce $M(f(x), f(x_0), t) > 1 - \epsilon$. Also f is called *uniformly fuzzy continuous* if for any $\epsilon \in (0, 1)$ there is $\delta \in (0, 1)$ so that for each x and y with $M(x, y, t) > 1 - \delta$, we deduce $M(f(x), f(y), t) > 1 - \epsilon$. As the classical mathematical analysis one can prove that each continuous fuzzy map on a compact fuzzy metric space is a uniformly fuzzy continuous map [4].

In the theory of structural stability it is very important to know that a property is invariant under topological conjugacy or not.

Definition 2.2. We say that two fuzzy maps $f : X \to X$ and $g : Y \to Y$ on fuzzy metric spaces $(X, M, *)$ and $(Y, M', \star)$ are *topologically conjugate* if there is a fuzzy homeomorphism $\phi : X \to Y$ (a fuzzy continuous bijection map with fuzzy continuous inverse) so that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \phi \downarrow & & \downarrow \phi \\ Y & \xrightarrow[g]{} & Y \end{array}$$

Now we are ready to introduce the concept of the shadowing property for fuzzy dynamical systems.

Definition 2.3. Let $(X, M, *)$ be a fuzzy metric space. We say that a fuzzy homeomorphism $f : X \to X$ has the *fuzzy pseudo-orbit tracing property* (FPOTP) on X, if for each $\epsilon \in (0, 1)$ and $t > 0$, there exists $\delta \in (0, 1)$ so that for a given sequence $\xi = \{x_k\}$ with

$$M(f(x_k), x_{k+1}, t) > 1 - \delta \quad \text{for } k \in \mathbb{Z}$$

(called δ-pseudo orbit) there exists a point $p \in X$ such that

$$M(f^k(p), x_k, t) > 1 - \epsilon \quad \text{for } k \in \mathbb{Z}$$

(in this case we say $p \in X$ ϵ-shadowed ξ). We say that f has the *unique fuzzy pseudo orbit tracing property* (UFPOTP), if there exists $\epsilon' \in (0, 1)$ so that for each $\epsilon \in (0, \epsilon')$ and each $t > 0$ there exists $\delta \in (0, 1)$ so that any δ-pseudo orbit is ϵ-shadowed with a unique $p \in X$.

Definition 2.4. Let $(X, M, *)$ be a fuzzy metric space. We say that a fuzzy homeomorphism $f : X \to X$ has the *fuzzy limit shadowing property* (FLmSP) on X, if for each $t > 0$ and each sequence $\xi = \{x_k\}$ with

$$\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1$$

there exists a point $p \in X$ so that

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1.$$

Example 2.1. Every constant map on a compact fuzzy metric space has both the FPOTP and the FLmSP.

3. Main Results

We assume that $(X, M, *)$ is a compact fuzzy metric space.

Theorem 3.1. *If $f : X \to X$ is a fuzzy homeomorphism then f has the FPOTP if and only if so does f^{-1}.*

Proof. Let $\epsilon \in (0, 1)$ and $t > 0$ be given. If f has the FPOTP, then there is $\delta \in (0, 1)$ so that for each sequence $\{x_k\}_{k\in\mathbb{Z}} \subset X$ with

$$M(f(x_k), x_{k+1}, t) > 1 - \delta \text{ for } k \in \mathbb{Z}, \tag{1}$$

there exists a point $p \in X$ such that

$$M(f^k(p), x_k, t) > 1 - \epsilon \text{ for } k \in \mathbb{Z}. \tag{2}$$

We choose $\delta' \in (0,1)$ so that the inequality $M(x,y,t) > 1-\delta'$ implies $M(f(x), f(y), t) > 1 - \delta$. Let $g = f^{-1}$ and the sequence $\{y_k\}_{k\in\mathbb{Z}} \subset X$ satisfies the inequality

$$M(g(y_k), y_{k+1}, t) > 1 - \delta' \text{ for } k \in \mathbb{Z}.$$

Then

$$M(y_k, f(y_{k+1}), t) = M(f(g(y_k)), f(y_{k+1}), t) > 1 - \delta \text{ for } k \in \mathbb{Z}.$$

Thus the sequence $\{x_k : \ x_k = y_{-k}\}_{k\in\mathbb{Z}}$ satisfies the relation (1), so there exists a point $p \in X$ such that the relation (2) holds. Therefore

$$M(g^k(p), y_k, t) = M(f^{-k}(p), x_{-k}, t) > 1 - \epsilon \text{ for } k \in \mathbb{Z}.$$

For the proof of the other side it is enough to replace f^{-1} by f. □

Theorem 3.2. *Let $(X, M, *)$ and $(X, M', \star)$ be compact fuzzy metric spaces and let $f : X \to X$ and $g : Y \to Y$ be two fuzzy homeomorphisms. If f and g are topologically conjugate then*

(i) *f has the FPOTP if and only if so does g.*
(ii) *f has the FLmSP if and only if so does g.*

Proof. (i) Let $\phi : X \to Y$ be a fuzzy homeomorphism so that $g \circ \phi = \phi \circ f$ and f has the FPOTP. If $\epsilon \in (0,1)$ and $t > 0$ there is $\epsilon' \in (0,1)$ so that the inequality $M(x,y,t) > 1-\epsilon'$ implies $M'(\phi(x), \phi(y), t) > 1-\epsilon$. Since f has the FPOTP there is $\delta' \in (0,1)$ so that for any sequence $\{x_k\}_{k\in\mathbb{Z}} \subset X$ with

$$M(f(x_k), x_{k+1}, t) > 1 - \delta' \text{ for } k \in \mathbb{Z} \tag{3}$$

there exists a point $p \in X$ such that the inequality

$$M(f^k(p), x_k, t) > 1 - \epsilon' \text{ for } k \in \mathbb{Z}$$

holds. We choose $\delta \in (0,1)$ so that the inequality $M'(x,y,t) > 1-\delta$ implies $M(\phi^{-1}(x), \phi^{-1}(y), t) > 1 - \delta'$. Given a sequence $\{y_k\}_{k\in\mathbb{Z}} \subset Y$ with

$$M'(f(y_k), y_{k+1}, t) > 1 - \delta \text{ for } k \in \mathbb{Z},$$

then for the sequence $\{x_k: \ x_k = \phi^{-1}(y_k)\}_{k\in\mathbb{Z}} \subset X$ we deduce

$$\begin{aligned} M(f(x_k), x_{k+1}, t) &= M(f(\phi^{-1}(y_k)), \phi^{-1}(y_{k+1}), t) \\ &= M(\phi^{-1}(g(y_k)), \phi^{-1}(y_{k+1}), t) \\ &> 1-\delta' \end{aligned}$$

for $k \in \mathbb{Z}$. Thus $\{x_k\}_{k\in\mathbb{Z}}$ satisfies the relation (3). So there exists a point $p \in X$ such that

$$M(f^k(p), x_k, t) > 1-\epsilon' \text{ for } k \in \mathbb{Z}.$$

Hence

$$M'(g^k(\phi(p)), x_k, t) = M'(\phi(f^k(p)), \phi(x_k), t) > 1-\epsilon \text{ for } k \in \mathbb{Z}.$$

(ii) Let $h : X \to Y$ be a fuzzy homeomorphism so that $g \circ \phi = \phi \circ f$ and f has the FLmSP. Given $t > 0$ and a sequence $\{y_k\}_{k\in\mathbb{Z}} \subset Y$ so that

$$\lim_{k\to\infty} M'(g(y_k), y_{k+1}, t) = 1. \tag{4}$$

If $\epsilon \in (0,1)$ then there is $\epsilon' \in (0,1)$ so that the inequality $M(x,y,t) > 1-\epsilon$ implies $M'(\phi(x), \phi(y), t) > 1-\epsilon$. So there is $\delta \in (0,1)$ such that the inequality $M'(x,y,t) > 1-\delta$ implies $M(\phi^{-1}(x), \phi^{-1}(y), t) > 1-\epsilon$. By the relation (4) there is $k_0 \in \mathbb{N}$ so that

$$M'(g(y_k), y_{k+1}, t) > 1-\delta \text{ for } k \geq k_0.$$

Hence for the sequence $\{x_k: \ x_k = \phi^{-1}(y_k)\}_{k\geq k_0} \subset X$ we deduce

$$\begin{aligned} M(f(x_k), x_{k+1}, t) &= M(f(\phi^{-1}(y_k)), \phi^{-1}(y_{k+1}), t) \\ &= M(\phi^{-1}(g(y_k)), \phi^{-1}(y_{k+1}), t) > 1-\delta' \quad \text{for} \quad k \geq k_0. \end{aligned}$$

Thus

$$\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1.$$

So there exists a point $p \in X$ such that

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1.$$

Therefore there is $k_1 \in \mathbb{N}$ so that $M(f^k(p), x_k, t) > 1-\delta$ for $k \geq k_1$. Thus

$$M'(g^k(\phi(p)), x_k, t) = M'(\phi(f^k(p)), \phi(x_k), t) > 1-\epsilon \text{ for } k \geq k_1.$$

Hence

$$\lim_{k\to\infty} M'(g^k(\phi(p)), y_k, t) = 1.$$

□

Theorem 3.3. *Let* $(X, M, *)$ *and* $(Y, M', *)$ *be two compact fuzzy metric spaces and let* $f : X \to X$ *and* $g : Y \to Y$ *be continuous maps. If* $f \times g : X \times Y \to X \times Y$ *is the map defined by*

$$f \times g(x, y) = (f(x), g(y)) \text{ for } (x, y) \in X \times Y,$$

then

(i) $f \times g$ *has the FPOTP if and only if both* f *and* g *have the FPOTP;*
(ii) $f \times g$ *has the FLmSP if and only if both* f *and* g *have the FLmSP.*

To prove this theorem we first prove the following lemma.

Lemma 3.1. *Let* $(X, M, *)$ *and* $(Y, M', *)$ *be two compact fuzzy metric spaces. Then* $(X \times Y, \widetilde{M}, *)$ *is a compact fuzzy metric space where*

$$\widetilde{M}((x_1, y_1), (x_2, y_2), t) = \min\{M(x_1, x_2, t), M'(y_1, y_2, t)\}.$$

Proof. If $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$ and $s, t > 0$, then

$$\begin{aligned}
&\widetilde{M}((x_1, y_1), (x_2, y_2), t + s) \\
&\qquad = \min\{M(x_1, x_2, t + s), M'(y_1, y_2, t + s)\} \\
&\qquad \geq \min\{M(x_1, x_3, t) * M(x_3, x_2, s), M'(y_1, y_3, t) * M'(y_3, y_2, s)\} \\
&\qquad \geq \min\{M(x_1, x_3, t), M'(y_1, y_3, t)\} * \min\{M(x_3, x_2, s), M'(y_3, y_2, s)\} \\
&\qquad = \widetilde{M}((x_1, y_1), (x_3, y_3), t) * \widetilde{M}((x_3, y_3), (x_2, y_2), s).
\end{aligned}$$

Clearly other properties of fuzzy metric are hold, so $(X \times Y, \widetilde{M}, *)$ is a fuzzy metric space. If $U \times V \in \tau_{\widetilde{M}}$, $x \in U$ and $y \in V$, then $x \times y \in U \times V$, so there exist $r \in (0, 1)$ and $t > 0$ such that $B_{\widetilde{M}}((x, y), r, t) \subset U \times V$. Now we show that $B_M(x, r, t) \subset U$ and $B_{M'}(y, r, t) \subset V$. Let $x' \in B_M(x, r, t)$ and $y' \in B_{M'}(y, r, t)$, then $M(x, x', t) > 1 - r$ and $M'(y, y', t) > 1 - r$. Hence $(x', y') \in B_{\widetilde{M}}((x, y), r, t) \subset U \times V$. So $x' \in U$ and $y' \in V$. These imply that $U \in \tau_M$ and $V \in \tau_{M'}$. Therefore compactness of X and Y implies that $(X \times Y, \widetilde{M}, *)$ is compact. □

Proof of Theorem 3.3. (i) Suppose that $f \times g$ has the FPOTP. Given $\epsilon \in (0, 1)$ and $t > 0$, there is $\delta \in (0, 1)$ with the properties of Definition 2.3. Let two sequences $\{x_k\}_{k \in \mathbb{Z}} \subset X$ and $\{y_k\}_{k \in \mathbb{Z}} \subset Y$ satisfy in the inequalities

$$M(f(x_k), x_{k+1}, t) > 1 - \delta \text{ for } k \in \mathbb{Z},$$

$$M(g(y_k), y_{k+1}, t) > 1 - \delta \text{ for } k \in \mathbb{Z}.$$

Then for the sequence $\{(x_k, y_k)\}_{k\in\mathbb{Z}} \subset X \times Y$ we deduce

$$M((f \times g)^k(x_k, y_k), (x_{k+1}, y_{k+1}), t) > 1 - \delta \text{ for } k \in \mathbb{Z}.$$

Thus there exists $(p, q) \in X \times Y$ so that

$$\widetilde{M}((f \times g)^k(p, q), (x_k, y_k), t) > 1 - \epsilon \text{ for } k \in \mathbb{Z}.$$

Hence

$$M(f^k(p), x_k, t) > 1 - \epsilon \text{ for } k \in \mathbb{Z},$$
$$M(g^k(q), y_k, t) > 1 - \epsilon \text{ for } k \in \mathbb{Z}.$$

Therefore f and g have the FPOTP. Now let both f and g have the FPOTP. If the sequence $\{(x_k, y_k)\}_{k\in\mathbb{Z}} \subset X \times Y$ satisfies in the inequality

$$\widetilde{M}((f \times g)^k(x_k, y_k), (x_{k+1}, y_{k+1}), t) > 1 - \delta \text{ for } k \in \mathbb{Z},$$

then

$$M(f(x_k), x_{k+1}, t) > 1 - \delta \text{ for } k \in \mathbb{Z},$$
$$M(g(y_k), y_{k+1}, t) > 1 - \delta \text{ for } k \in \mathbb{Z}.$$

Hence there exist $p \in X$ and $q \in Y$ so that

$$M(f^k(p), x_k, t) > 1 - \epsilon \text{ for } k \in \mathbb{Z},$$

$$M(g^k(q), y_k, t) > 1 - \epsilon \text{ for } k \in \mathbb{Z}.$$

Thus

$$\widetilde{M}((f \times g)^k(p, q), (x_k, y_k), t) > 1 - \epsilon \text{ for } k \in \mathbb{Z}.$$

(ii) Suppose that $f \times g$ has the FLmSP. If $\{x_k\}_{k\in\mathbb{Z}} \subset X$ and $\{y_k\}_{k\in\mathbb{Z}} \subset Y$ are two sequences satisfying the relations

$$\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1 \quad \text{and} \quad \lim_{k\to\infty} M(g(y_k), y_{k+1}, t) = 1,$$

then

$$\lim_{k\to\infty} \widetilde{M}((f \times g)(x_k, y_k), (x_{k+1}, y_{k+1}), t) = 1.$$

Thus there exists $(p, q) \in X \times Y$ so that

$$\lim_{k\to\infty} \widetilde{M}((f \times g)^k(p, q), (x_k, y_k), t) = 1.$$

So

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1 \text{ and } \lim_{k\to\infty} M(g^k(q), y_k, t) = 1.$$

The proof of the other side is similar to the above proof. □

Let (X, d) be a metric space and $f : X \to X$ be a homeomorphism. The notion of POTP was first introduced by Anosov as follows. A dynamical system f has the *pseudo-orbit tracing property* (POTP) on X if for each $\epsilon > 0$ there is $\delta > 0$ so that for a given sequence $\xi = \{x_k\}_{n\in\mathbb{Z}}$ with

$$d(f(x_k), x_{k+1}) < \delta \quad \text{for } k \in \mathbb{Z}$$

there exists a point $x \in X$ such that

$$d(f^k(x), x_k) < \epsilon \quad \text{for } k \in \mathbb{Z}.$$

(In this case we say $p \in X$ ϵ-shadowed ξ). A dynamical system f has the *limit shadowing property* (LmSP) on X [11] if for any sequence $\xi = \{x_k\}_{k\geq 0}$ with

$$r(f(x_k), x_{k+1}) \to 0, \quad k \to \infty$$

there exists a point $x \in X$ so that

$$r(f^k(x), x_k) \to 0, \quad k \to \infty.$$

Proposition 3.1. *Let $f : (X, d) \to (X, d)$ be a homeomorphism, then $f : (X, M_d, *) \to (X, M_d, *)$ is a fuzzy homeomorphism and we deduce*

(i) *f has the POTP if and only if it has the FPOTP;*
(ii) *f has the LmSP if and only if it has the FLmSP.*

Proof. (i) Assume that f has the POTP. Given $\epsilon \in (0, 1)$, $t > 0$ and choose $\epsilon' \in (0, t\epsilon/(1-\epsilon)) \cap (0, 1)$. Therefore there is $\delta' \in (0, 1)$ so that for any sequence $\{x_k\}_{k\in\mathbb{Z}}$ with

$$d(f(x_k), x_{k+1}) < \delta' \quad \text{for } k \in \mathbb{Z}$$

there exists a point $p \in X$ such that

$$d(f^k(p), x_k) < \epsilon' \quad \text{for } k \in \mathbb{Z}.$$

If we choose $\delta \in (0, \delta'/(t + \delta'))$ then for each sequence $\{x_k\}_{k\in\mathbb{Z}}$ with

$$M_d(f(x_k), x_{k+1}, t) > 1 - \delta \quad \text{for } k \in \mathbb{Z}$$

we deduce

$$d(f(x_k), x_{k+1}) < \delta' \quad \text{for } k \in \mathbb{Z}.$$

So there exists a point $p \in X$ such that

$$d(f^k(p), x_k) < \epsilon' < \frac{t\epsilon}{1-\epsilon} \quad \text{for } k \in \mathbb{Z}.$$

Thus

$$M_d(f^k(p), x_k, t) > 1 - \epsilon \quad \text{for } k \in \mathbb{Z}.$$

Now we assume that f has the FPOTP as a fuzzy homeomorphism on $(X, M_d, *)$. If $\epsilon \in (0,1)$ and $\epsilon' \in (0, \epsilon/(1+\epsilon))$, then there is $\delta' \in (0,1)$ so that for each sequence $\{x_k\}_{k\in\mathbb{Z}}$ with

$$M_d(f(x_k), x_{k+1}, 1) > 1 - \delta' \quad \text{for } k \in \mathbb{Z}$$

there exists a point $p \in X$ such that

$$M_d(f^k(p), x_k, 1) > 1 - \epsilon' \quad \text{for } k \in \mathbb{Z}.$$

Now if $\delta \in (0, \delta'/(1-\delta')) \cap (0,1)$ then for each sequence with

$$d(f(x_k), x_{k+1}) < \delta \quad \text{for } k \in \mathbb{Z}$$

we have

$$M_d(f(x_k), x_{k+1}, 1) > 1 - \delta' \quad \text{for } k \in \mathbb{Z}.$$

Hence there is $p \in X$ so that

$$M_d(f^k(p), x_k, 1) > 1 - \epsilon' > \frac{1}{1+\epsilon} \quad \text{for } k \in \mathbb{Z}.$$

Thus

$$d(f^k(p), x_k) < \epsilon \quad \text{for } k \in \mathbb{Z}.$$

(ii) This part is clear because for each two sequences x_k and y_k we deduce $\lim_{k\to\infty} d(x_k, y_k) = 0$ if and only if for each $t > 0$ $\lim_{k\to\infty} M_d(x_k, y_k, t) = 1$. □

Definition 3.1. We say that a homeomorphism $f : X \to X$ is *expansive* if there exists $e \in (0,1)$ (called *expansive constant*) so that for each $t > 0$, if $M(f^k(x), f^k(y), t) > 1 - e$ for all $k \in \mathbb{Z}$, then $x = y$.

Proposition 3.2. *Let $f : X \to X$ be an expansive homeomorphism. If f has the FPOTP then it has the UFPOTP.*

Proof. Let f be expansive with expansive constant $e \in (0,1)$. By continuity of $*$ there exists $\epsilon' \in (0,1)$ so that $(1-\epsilon') * (1-\epsilon') \geq (1-e)$. If $\epsilon \in (0, \epsilon')$ then there exists $\delta \in (0,1)$ so that any δ-pseudo-orbit is ϵ-shadowed with a point $p \in X$. Let δ-pseudo-orbit $\{x_k\}_{k\in\mathbb{Z}}$, ϵ-shadowed with two points $p, q \in X$. So for all $k \in \mathbb{Z}$ we deduce

$$M(f^k(p), f^k(q), 2t) \geq M(f^k(p), x_k, t) * M(f^k(q), x_k, t) > (1-\epsilon) * (1-\epsilon) > 1 - e.$$

Hence $p = q$. □

4. Examples

Example 4.1. Let $X = S^1$ be the unit circle and $a * b = ab$. If $d(x, y)$ is the distance between x and y as follows then (X, d) is a compact metric space.

$$d(x = (\cos 2\pi t, \sin 2\pi t), y = (\cos 2\pi s, \sin 2\pi s)) = |t - s| \ ; \ t, s \in [0, 1).$$

So $(X, M, *)$ where $M(x, y, t) = t/(t + d(x, y))$ is a compact fuzzy metric space. We consider the homeomorphism $f(x) = x$ on X. Let $\delta \in (0, \frac{1}{3})$ and fixed $x_0 \in X$. We can choose x_1 so that $M(x_1, x_0, 1) = 1 - \frac{\delta}{2}$ and $x_2 \in X$ so that $M(x_2, x_1, 1) = 1 - \frac{\delta}{2}$. To continue the algorithm we find the sequence $\{x_k\}_{k \geq 0}$ so that

$$M(f(x_k), x_{k+1}, 1) = 1 - \frac{\delta}{2} \text{ for } k \geq 0.$$

Clearly there is no point $p \in X$ such that

$$M(f^k(p), x_k, 1) > 1 - \frac{1}{3} \text{ for } k \geq 0.$$

Hence f does not have the FPOTP.

Example 4.2. In [11, Example 1.19] is presented a homeomorphism on a compact metric space so that it has the LmSP but does not have the POTP. Hence by Proposition 3.1 it has the FLmSP but does not have the FPOTP (with respect to the standard fuzzy metric).

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