Çankaya University Journal of Science and Engineering
Volume 9 (2012), No. 1, 59–67

The Modified (G'/G)-Expansion Method for Exact Solutions of the $(3+1)$-Dimensional Jimbo-Miwa Equation

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Özet. Bu makalede, (G'/G)-açılım metodunu başarıyla değiştirdik ve bir uygulama olarak (3+1)-boyutlu Jimbo-Miwa denkleminin kesin çözümlerini inşa etmek için önerdik. Elde edilen çözümlerin yani hiperbolik fonksiyon çözümlerinin, trigonometrik fonksiyon çözümlerinin ve kesirli çözümlerin her biri çalışılan denklemdeki değişkenlerin açık doğrusal bir fonksiyonunu içermektedir. Önerilen metodun, sembolik hesaplama yardımıyla, matematiksel fizikteki oluşum denklemlerinin çözümü için daha güçlü bir matematiksel araç olduğu gösterildi.[1]

Anahtar Kelimeler. Doğrusal olmayan oluşum denklemi, Jimbo-Miwa denklemi, (G'/G)-açılım metodu, hiperbolik fonksiyon çözümleri, trigonometrik fonksiyon çözümleri, rasyonel çözümler.

Abstract. In this paper, we successfully modified the (G'/G)-expansion method and as an application proposed to construct exact solutions of the (3+1)-dimensional Jimbo-Miwa equation. Each of the obtained solutions, namely the hyperbolic function solutions, the trigonometric function solutions and the rational solutions contain an explicit linear function of the variables in the equation in question. It is shown that the proposed method with the help of a symbolic computation provides a more powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

Keywords. Nonlinear evolution equation, Jimbo-Miwa equation, (G'/G)-expansion method, hyperbolic function solutions, trigonometric function solutions, rational solutions.

1. Introduction

Nonlinear evolution equations (NLEEs) have been a subject of study in various branches of mathematical-physical science such as physics, biology, chemistry, etc.

Received February 8, 2012; accepted May 5, 2012.

[1]Türkçe özet ve anahtar kelimeler, orijinal İngilizce metindeki ilgili kısmın doğrudan tercümesi olup *Çankaya University Journal of Science and Engineering* editörlüğü tarafından yazılmıştır. | Turkish abstract and the keywords are written by the editorial staff of *Çankaya University Journal of Science and Engineering* which are the direct translations of the related original English text.

ISSN 1309 - 6788

The analytical solutions of such equations are of fundamental importance since a lot of mathematical-physical models are described by NLEEs. Among the possible solutions to NLEEs, certain special form solutions may depend only on a single combination of variables such as traveling wave variables. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Some of these approaches are the Jacobi elliptic function method [1], the inverse scattering method [2], Hirota's bilinear method [3], the homogeneous balance method [4], the homotopy perturbation method [5], the Weierstrass function method [6], the symmetry method [7], the Adomian decomposition method [8], the differential transform method [9], the tanh/coth method [10], the Exp-function method [11, 12, 13] and so on. But, most of the methods may sometimes fail or can only lead to a kind of special solution and the solution procedures become very complex as the degree of nonlinearity increases.

Recently, the (G'/G)-expansion method, first introduced by Wang *et al.* [14], has become widely used to search for various exact solutions of NLEEs [15]–[23]. The value of the (G'/G)-expansion method is that one treats nonlinear problems by essentially linear methods. The method is based on the explicit linearization of NLEEs for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation.

The present paper is motivated by the desire to modify the (G'/G)-expansion method for constructing more general exact solutions of NLEEs. In order to illustrate the validity and advantages of the modified method, we would like to employ it to solve the (3+1)-dimensional Jimbo–Miwa equation:

$$2u_{yt} + 3u_y u_{xx} + 3u_x u_{xy} - 3u_{xz} + u_{xxxy} = 0.$$

2. Description of the (G'/G)-Expansion Method

The objective of this section is to outline the use of the (G'/G)-expansion method for solving certain nonlinear partial differential equations (PDEs). Suppose we have a nonlinear PDE for $u(x, y, z, t)$, in the form

$$P(u, u_t, u_x, u_{xy}, u_{yz}, u_{tt}, ...) = 0, \tag{1}$$

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives. The transformation $u(x, y, z, t) = U(\xi)$, $\xi = ax + by +$

$cz - \omega t$, reduces Eq. (1) to the ordinary differential equation (ODE)

$$P(U, -\omega U', aU', abU'', bcU'', \omega^2 U'', ...) = 0, \tag{2}$$

where $U = U(\xi)$, and prime denotes derivative with respect to ξ. We assume that the solution of Eq. (2) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi) = \sum_{i=1}^{m} \alpha_i \Big(\frac{G'}{G}\Big)^i + \alpha_0, \quad \alpha_m \neq 0, \tag{3}$$

where α_0 and α_i are constants to be determined later, $G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda \frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0, \tag{4}$$

where λ and μ are arbitrary constants. Using the general solutions of Eq. (4), we have

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0. \end{cases} \tag{5}$$

To determine u explicitly, we take the following four steps:

Step 1. Determine the integer m by substituting Eq. (3) along with Eq. (4) into Eq. (2), and balancing the highest order nonlinear term(s) and the highest order partial derivative.

Step 2. Substitute Eq. (3) give the value of m determined in *Step 1*, along with Eq. (4) into Eq. (2) and collect all terms with the same order of (G'/G) together, the left-hand side of Eq. (2) is converted into a polynomial in (G'/G). Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for $a, b, c, \omega, \alpha_0$ and α_i.

Step 3. Solve the system of algebraic equations obtained in *Step 2*, for $a, b, c, \omega, \alpha_0$ and α_i by the use of Maple.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions $v(\xi)$ of Eq. (2) depending on (G'/G), since the solutions of Eq. (4) are well known to us, then we can obtain exact solutions of Eq. (1) by integrating $v(\xi)$

with respect to ξ, r times:

$$u(\xi)=\int_0^{\xi}\int_0^{\xi_r}...\int_0^{\xi_2}v(\xi)d\xi_1 d\xi_2...d\xi_{r-1}d\xi_r+\sum_{j=0}^{r}d_j\xi^{r-j}, \tag{6}$$

where d_j are arbitrary constants. If $r=1$, there is only the last defined integral over the interval $[0,\xi]$. Otherwise, the obtained solutions will definitely contain a polynomial part in ξ.

3. Application on the (3+1)-Dimensional Jimbo-Miwa Equation

In this section, we would like to use our method to obtain new and more general exact solutions of the (3+1)-dimensional Jimbo-Miwa equation:

$$2u_{yt}+3u_y u_{xx}+3u_x u_{xy}-3u_{xz}+u_{xxxy}=0, \tag{7}$$

which passes the Painleve test only for a subclass of solutions and its symmetry algebra does not have a Kac-Moody-Virasoro structure.

Using the transformation $\xi=ax+by+cz-\omega t$, we reduce Eq. (7) into an ODE of the form:

$$-(2b\omega+3ac)u''+6a^2bu'u''+a^3bu''''=0. \tag{8}$$

Integrating Eq. (8) once with respect to ξ and setting the integration constant as zero yields

$$-(2b\omega+3ac)u'+3a^2b(u')^2+a^3bu'''=0, \tag{9}$$

further letting $r=1$, and $u'=v$, we have

$$-(2b\omega+3ac)v+3a^2b(v)^2+a^3bv''=0. \tag{10}$$

According to Step 1, we get $m+2=2m$, hence $m=2$. We then suppose that Eq. (10) has the following formal solutions:

$$v=\alpha_2\left(\frac{G'}{G}\right)^2+\alpha_1\left(\frac{G'}{G}\right)+\alpha_0, \qquad \alpha_2\neq 0. \tag{11}$$

Substituting Eq. (11) along with Eq. (4) into Eq. (10) and collecting all terms with the same order of (G'/G), together, the left-hand sides of Eq. (10) are converted into a polynomial in (G'/G).

Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for $a, b, c, \omega, \alpha_0, \alpha_1$, and α_2, as follows:

$$
\begin{aligned}
&\left(\frac{G'}{G}\right)^0 : -3ac\alpha_0 - 2b\omega\alpha_0 + 3a^2b\alpha_0^2 + a^3b\alpha_1\lambda\mu + 2a^3b\alpha_2\mu^2 = 0, \\
&\left(\frac{G'}{G}\right)^1 : -3ac\alpha_1 - 2b\omega\alpha_1 + 6a^2b\alpha_0\alpha_1 + a^3b\alpha_1\lambda^2 + 2a^3b\alpha_1\mu + 6a^3b\alpha_2\lambda\mu = 0, \\
&\left(\frac{G'}{G}\right)^2 : 3a^2b\alpha_1^2 - 3ac\alpha_2 - 2b\omega\alpha_2 + 6a^2b\alpha_0\alpha_2 + \\
&\qquad\qquad + 3a^3b\alpha_1\lambda + 4a^3b\alpha_2\lambda^2 + 8a^3b\alpha^2\mu = 0, \quad (12) \\
&\left(\frac{G'}{G}\right)^3 : 2a^3b\alpha_1 + 6a^2b\alpha_1\alpha_2 + 10a^3b\alpha_2\lambda = 0, \\
&\left(\frac{G'}{G}\right)^4 : 6a^3b\alpha_2 + 3a^2b\alpha_2^2 = 0.
\end{aligned}
$$

Solving the set of algebraic equations by use of Maple, we get the following results:

$$\alpha_2 = -2a,\ \alpha_1 = -2a\lambda,\ \alpha_0 = -\frac{1}{3}a(\lambda^2 + 2\mu),\ \omega = \frac{a(4a^2b\mu - a^2b\lambda^2 - 3c)}{2b}, \quad (13)$$

and

$$\alpha_2 = -2a,\ \alpha_1 = -2a\lambda,\ \alpha_0 = -2a\mu,\ \omega = -\frac{a(4a^2b\mu - a^2b\lambda^2 + 3c)}{2b}. \quad (14)$$

Substituting the above two sets in (11), we get

$$v = -2a\left(\frac{G'}{G}\right)^2 - 2a\lambda\left(\frac{G'}{G}\right) - \frac{1}{3}a(\lambda^2 + 2\mu),\ \omega = \frac{a(4a^2b\mu - a^2b\lambda^2 - 3c)}{2b}, \quad (15)$$

and

$$v = -2a\left(\frac{G'}{G}\right)^2 - 2a\lambda\left(\frac{G'}{G}\right) - 2a\mu,\ \omega = -\frac{a(4a^2b\mu - a^2b\lambda^2 + 3c)}{2b}. \quad (16)$$

Substituting the general solutions of Eq. (4) into Eqs. (15) and (16), respectively, we obtain three types of traveling wave solutions of Eq. (7):

When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solutions:

$$u = -\frac{1}{2}a(\lambda^2 - 4\mu)\int_0^{\xi}\left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right)^2 d\xi_1 + \frac{1}{6}a(\lambda^2 - 4\mu)\xi + d, \quad (17)$$

where $\xi = ax + by + cz - (a(4a^2b\mu - a^2b\lambda^2 - 3c)/2b)t,\ C_1,\ C_2$, and d are arbitrary constants,

$$u = -\frac{1}{2}a(\lambda^2 - 4\mu)\int_0^{\xi}\left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right)^2 d\xi_1 + \frac{1}{2}a(\lambda^2 - 4\mu)\xi + d, \quad (18)$$

where $\xi = ax + by + cz + (a(4a^2b\mu - a^2b\lambda^2 + 3c)/2b)t,\ C_1,\ C_2$, and d are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we obtain trigonometric function solutions:

$$u = -\frac{1}{2}a(4\mu - \lambda^2)\int_0^{\xi}\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right)^2 d\xi_1 + \frac{1}{6}a(\lambda^2 - 4\mu)\xi + d, \quad (19)$$

where $\xi = ax + by + cz - (a(4a^2b\mu - a^2b\lambda^2 - 3c)/2b)t,\ C_1,\ C_2$, and d are arbitrary constants,

$$u = -\frac{1}{2}a(4\mu - \lambda^2)\int_0^{\xi}\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right)^2 d\xi_1 + \frac{1}{2}a(\lambda^2 - 4\mu)\xi + d, \quad (20)$$

where $\xi = ax + by + cz + (a(4a^2b\mu - a^2b\lambda^2 + 3c)/2b)t,\ C_1,\ C_2$, and d are arbitrary constants.

And finally, when $\lambda^2 - 4\mu = 0$, we obtain the rational solution:

$$u = \frac{2aC_2}{C_1 + C_2\xi} + d, \tag{21}$$

where $\xi = ax + by + cz + (3ac/2b)t,\ C_1,\ C_2$, and d are arbitrary constants.

To obtain some special cases of the solutions obtained above, we set $C_2 = 0$, then Eq. (17) becomes

$$u = a\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) - \frac{1}{3}(\lambda^2 - 4\mu)\xi + d, \tag{22}$$

where $\xi = ax + by + cz - (a(4a^2b\mu - a^2b\lambda^2 - 3c)/2b)t$, and d is an arbitrary constant.

In view of the relation between the kink-type solution and the kink-bell-type solution [24], from Eq. (22) we also have

$$u = a\sqrt{\lambda^2 - 4\mu}\big[\tanh(\sqrt{\lambda^2 - 4\mu}\ \xi) + i\mathrm{sech}(\sqrt{\lambda^2 - 4\mu}\ \xi)\big] - \frac{1}{3}(\lambda^2 - 4\mu)\xi + d, \tag{23}$$

where $\xi = ax + by + cz - (a(4a^2b\mu - a^2b\lambda^2 - 3c)/2b)t$, and d is an arbitrary constant.

If we set again $C_1 = 0$, then Eq. (22) becomes

$$u = a\sqrt{\lambda^2 - 4\mu}\coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) - \frac{1}{3}(\lambda^2 - 4\mu)\xi + d, \tag{24}$$

where $\xi = ax + by + cz - (a(4a^2b\mu - a^2b\lambda^2 - 3c)/2b)t$, and d is an arbitrary constant.

Similarly, setting $C_2 = 0$, and using Eq. (19) and the relation [24] we have

$$u = a\sqrt{4\mu - \lambda^2}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) - \frac{1}{3}(\lambda^2 - 4\mu)\xi + d, \tag{25}$$

where $\xi = ax + by + cz - (a(4a^2b\mu - a^2b\lambda^2 - 3c)/2b)t$, and d is an arbitrary constant, and

$$u = a\sqrt{4\mu - \lambda^2}\big[\tan(\sqrt{\lambda^2 - 4\mu}\ \xi) \pm i\sec(\sqrt{\lambda^2 - 4\mu}\ \xi)\big] - \frac{1}{3}(\lambda^2 - 4\mu)\xi + d, \tag{26}$$

where $\xi = ax + by + cz - (a(4a^2b\mu - a^2b\lambda^2 - 3c)/2b)t$, and d is an arbitrary constant.

We would like to note here that solutions (17)-(26) with an explicit linear function in ξ can't be obtained by the (G'/G)-expansion method [17, 18, 20, 23, 24], and that they have been checked with Maple by putting them back into the original Eq. (7).

4. Conclusions

In this article, the modified (G'/G)-expansion method is developed to solve the (3+1)-dimensional Jimbo-Miwa equation, and we successfully obtained more general traveling wave solutions of this equation. As a result, hyperbolic function solutions and trigonometric function solutions with parameters are obtained, from which some known solutions, including the kink-type solitary wave solution and the singular traveling wave solution, are recovered by setting the parameters as special values. These obtained solutions with free parameters may be important to explain some physical phenomena. The paper shows that the modified algorithm is effective and can be used for many other NLDDEs in mathematical physics.

Acknowledgment. The author would like to thank from the Young Researchers Club, Islamic Azad University, Ardabil Branch for its financial support.

References

[1] G. T. Liu and T. Y. Fan, New applications of developed Jacobi elliptic function expansion methods, *Physics Letters A* **345** (2005), 161–166.

[2] M. J. Ablowitz and H. Segur, *Solitons and Inverse Scattering Transform*, SIAM, Philadelphia 1981.

[3] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge 2004.

[4] M. L. Wang, Exact solutions for a compound KdV-Burgers equation, *Physics Letters A* **213** (1996), 279–287.

[5] J. H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, *Applied Mathematics and Computation* **151** (2004), 287–292.

[6] Z. Y. Yan, An improved algebra method and its applications in nonlinear wave equations, *Chaos, Solitons & Fractals* **21** (2004), 1013–1021.

[7] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York 1989.

[8] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer, Boston 1994.

[9] R. Abazari and M. Abazari, Numerical simulation of generalized Hirota-Satsuma coupled KdV equation by RDTM and comparison with DTM, *Communications in Nonlinear Science and Numerical Simulation* **17** (2012), 619–629.

[10] H. Jafari, A. Borhanifar and S. A. Karimi, New solitary wave solutions for the bad Boussinesq and good Boussinesq equations, *Numerical Methods for Partial Differential Equations* **25** (2009), 1231–1237.

[11] A. Borhanifar, M. M. Kabir and L. Maryam Vahdat, New periodic and soliton wave solutions for the generalized Zakharov system and (2+1)-dimensional Nizhnik-Novikov-Veselov system, *Chaos, Solitons & Fractals* **42** (2009), 1646–1654.

[12] A. Borhanifar and M. M. Kabir, New periodic and soliton solutions by application of Exp-function method for nonlinear evolution equations, *Journal of Computational and Applied Mathematics* **229** (2009), 158–167.

[13] T. Öziş and İ. Aslan, Exact and explicit solutions to the (3 + 1)-dimensional Jimbo-Miwa equation via the Exp-function method, *Physics Letters A* **372** (2008), 7011–7015.

[14] M. Wang, X. Li and J. Zhang, The $(\frac{G'}{G})$–expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics, *Physics Letters A* **372** (2008), 417–423.

[15] A. Bekir, Application of the $(\frac{G'}{G})$–expansion method for nonlinear evolution equations, *Physics Letters A* **372** (2008), 3400–3406.

[16] İ. E. İnan, (G'/G)-Expansion method for traveling wave solutions of the sixth-order Ramani equation, *Çankaya University Journal of Science and Engineering* **7** (2010), 51–57.

[17] R. Abazari, The $(\frac{G'}{G})$-expansion method for the coupled Boussinesq equation, *Procedia Engineering* **10** (2011), 2845–2850.

[18] R. Abazari, Application of $(\frac{G'}{G})$–expansion method to travelling wave solutions of three nonlinear evolution equation, *Computers & Fluids* **39** (2010), 1957–1963.

[19] R. Abazari, The $(\frac{G'}{G})$–expansion method for Tzitzéica type nonlinear evolution equations, *Mathematical and Computer Modelling* **52** (2010), 1834–1845.

[20] R. Abazari and R. Abazari, Hyperbolic, trigonometric and rational function solutions of Hirota–Ramani equation via $(\frac{G'}{G})$-expansion method, *Mathematical Problems in Engineering* **2011** (2011), Article ID 424801, doi:10.1155/2011/424801. `http://downloads.hindawi.com/journals/mpe/2011/424801.pdf`, 2010. Online; accessed 05-May-2012.

[21] R. Abazari, General travelling wave solutions of quintic nonlinearity of Klein-Gordon equation, *Australian Journal of Basic and Applied Sciences* **5** (2011), 197–205.

[22] R. Abazari, The solitary wave solutions of Zoomeron equation, *Applied Mathematical Sciences* **5** (2011), 2943–2949.

[23] M. M. Kabir, A. Borhanifar and R. Abazari, Application of $(\frac{G'}{G})$–expansion method to Regularized Long Wave (RLW) equation, *Computers & Mathematics with Applications* **61** (2011), 2044–2047.

[24] C. P. Liu, The relation between the kink-type solution and the kink-bell-type solution of nonlinear evolution equations, *Physics Letters A* **312** (2003), 41–48.