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Some New Difference Sequence Spaces Defined By Modulus Function

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ABSTRACT

In this paper, we define some new paranormed difference sequence spaces. We also show some inclusion theorems between these new sequence spaces.

Keywords: *Modulus function;* ***Difference seq****uence; paranormed space; matrix transformation; Riesz mean*

1. Introduction and Background

By w, we shall denote the space of all real valued sequences. Any vector subspace of w is called as a *sequence space*. We shall write ℓ_∞, c, and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively.

The definiton of a modulus function was introduced in the year 1953 by Nakano [1].We recall that a modulus f is a function f: $[0,\infty)\to[0,\infty)$ such that (i) f(x) = 0 if and only if x= 0, (ii)f(x+y)≤f(x) +f(y) for all x≥0, y≥0,(iii) f is increasing,(iv) f is continuous from the right at zero. It follows that f must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take f(x)= x x+1, then f(x) is bounded. If $f(x)=x^p$, 0<p<1 then the modulus f(x) is unbounded. Let X be a linear metric space. A function p : X → R is called paranorm, if a. p(x) ≥ 0, for all x ∈ X, b. p(−x)=p(x), for all x ∈ X, c. p(x+y) ≤ p(x)+p(y), for all x,y ∈ X, d. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n\to\infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to\infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n\to\infty$.

The definition of difference sequence spaces was introduced by Kizmaz [2] as follows

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\}$$

for Z=c,c_o and ℓ_∞ where $\Delta x_k=x_k-x_{k+1}$ for all k∈N. Tripathy and Esi [3] defined the concept of Δ_m-difference sequence such that $\Delta_m x=(\Delta_m x_k)=(x_k-x_{k+m})$ for all m∈N (the set of all natural numbers) and defined related sequence spaces $l_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_o(\Delta_m)$. Later, the difference sequence spaces were studied by many authors, (see, [4],[5],[6],[7])

Consider the sequence of positive numbers (q_k) and write $S_n=\sum_{k=0}^{n} q_k$ for n∈N. Then the matrix $S^q=(s^q_{nk})$ of Riesz mean (S,q_n) is given by.

$$S^q_{nk}=\begin{cases}\frac{q_k}{s_n} & if\ 0\le k\le n;\ \ 0\ if\ k>n.\end{cases}$$

The Riesz mean (S,q_n) is regular if and only if $S_n\to\infty$ as $n\to\infty$ (see, [8] and [9]).

For any set A of sequences, the space of multipliers of A, denoted by M(A), given by

$M(A)=\{a\in w: ax\in A\ for\ all\ x\in A\}$.

2. Main Results

In this section, we shall introduce the difference spaces $S^q_\infty(\Delta,f,p,m)$, $S^q(\Delta,f,p,m)$ and $S^q_o(\Delta,f,p,m)$ of Riesz type. We also show some inclusion theorems between these new sequence spaces.

Definition 2.1. Let $p=(p_k)$ be bounded sequence with $0<p_k\le \sup_k p_k<\infty$.

Now we define the following sequence spaces as

follows. Let f be an modulus function, then

$$S^q_\infty(\Delta,f,p,m)=\left\{x=(x_k): sup_k\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i)\right|^{p_i}<\infty\right\},$$

$$S^q(\Delta,f,p,m)=\left\{x=(x_k): lim_k\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i - l)\right|^{p_i}=0\right\}$$ for some l∈R and

$$S^q_o(\Delta,f,p,m)=\left\{x=(x_k): lim_k\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i)\right|^{p_i}=0\right\}$$

where $\Delta_m x_k=x_k-x_{k+m}$ for all k∈N and $Q_k=\sum_{i=0}^{k} q_i$

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Theorem 2.1. The difference spaces $S^q_\infty(\Delta,f,p,m)$, $S^q(\Delta,f,p,m)$ and $S^q_o(\Delta,f,p,m)$ are complete linear spaces paranormed by G defined by

$$G(x)=\sum_{r=1}^{m}|x_r| + sup_k\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i)\right|^{p_i/M}$$

where $M=(1, sup_k p_k)$.

Proof. We only prove for the difference space $S^q_\infty(\Delta,f,p,m)$. Others can be proved similarly. The linearity of the difference space $S^q_\infty(\Delta,f,p,m)$ with respect to co-ordinate wise addition follows from following inequalities

$$G(x+y)=\sum_{r=1}^{m}|x_r+y_r| + sup_k\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m(x_i+y_i))\right|^{p_i/M} \leq \sum_{r=1}^{m}|x_r| + sup_k\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i)\right|^{p_i/M} + \sum_{r=1}^{m}|y_r| + sup_k\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m y_i)\right|^{p_i/M}$$

$=G(x+y)$

and scalar multiplication follows from using the inequality $|\gamma|^{p_k}\leq max\{1,|\gamma|^M\}$ for any $\gamma\in R$, see [4]. It is clear that $G(0)=0$ and $G(-x)=-G(x)$ for all $x=(x_k)\in S^q_\infty(\Delta,f,p,m)$. Subadditivity of G follows from the inequality

$G(\gamma x)\leq max(1, |\gamma|)G(x)$. Hence G is paranorm on the difference space $S^q_\infty(\Delta,f,p,m)$. Now, let (x^n) be any Cauchy sequence of the difference space $S^q_\infty(\Delta,f,p,m)$ where $x^n=(x^1_n, x^2_n, x^3_n,...)$. Then, for a given $\varepsilon>0$ there exists a positive integer $n_o(\varepsilon)$ such that $G(x^s-x^t)<\varepsilon$ for all $s,t\geq n_o(\varepsilon)$. Using definition of G and for each fixed $k\in N$ and

$$\sum_{r=1}^{m}|x_r^s+x_r^t|<\varepsilon$$

and $sup_k\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m(x_i^s+x_i^t))\right|^{p_i/M}<\varepsilon$ ...(*)

for all $s,t\geq n_o(\varepsilon)$. Hence $|x_r^s+x_r^t|<\varepsilon$ for $s,t\geq n_o(\varepsilon)$ and $r=1,2,...,m$. Then (x_r^s) is a Cauchy sequence in R for $r=1,2,...,m$. Let $\lim_{s\to\infty} x_r^s=x_r$ say for $r=1,2,...,k$. Also from (*) we have for $\varepsilon>0$, there exists a natural number $n_o(\varepsilon)$ such that

$$\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m(x_i^s+x_i^t))\right|^{p_i}<\varepsilon^M$$

for $s,t> n_o(\varepsilon)$. From last inequality, by taking $t\to\infty$ we obtain

$$\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m(x_i^s+x_i))\right|^{p_i}<\varepsilon^M$$

for $s> n_o(\varepsilon)$. Thus, $G(x^s-x)\leq\varepsilon$ for $s\geq n_o(\varepsilon)$. Finally, taking $\varepsilon=1$ in below inequality and letting $s\geq n_o(1)$, we have by Minkowski's inequality

$G(x^s)\leq G(x)+G(x^s-x)\leq 1+G(x)$

which implies that $x\in S^q_\infty(\Delta,p,m)$. Since , $G(x^s-x)\leq\varepsilon$ for all $i\geq n_o(\varepsilon)$, it follows that

$x^s\to x$ as $s\to\infty$, hence we obtain that the difference sequence space $S^q_\infty(\Delta,f,p,m)$ is complete.

Theorem 2.2. If $p=(p_k)$ and $t=(t_k)$ are bounded sequences of positive numbers with $0<p_k\leq t_k<\infty$ for each k, then for any modulus function f, we have

(a) $S^q_\infty(\Delta,f,p,m)\subset S^q_\infty(\Delta,f,t,m)$,

(b) $S^q(\Delta,f,p,m)\subset S^q(\Delta,f,t,m)$,

(c) $S^q_o(\Delta,f,p,m)\subset S^q_o(\Delta,f,t,m)$.

Proof. We only prove (a) and the rest can be proved similarly. Let $x=(x_k)\in S^q_\infty(\Delta,f,p,m)$. Then

$$sup_k\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i)\right|^{p_i}<\infty$$

Now, for sufficiently large values of k, say $k\geq k_o$ for some fixed k_o, we have

$$\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i)\right|^{p_i}<\infty$$

but $p_k\leq t_k$ for each k, we have

$$\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i)\right|^{p_i}\leq\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i)\right|^{t_i}<\infty.$$

The result follows from this inequality.

The proof of the following result is routine verification.

Theorem 2.3. If $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers with $0<p_k, q_k<\infty$ and $r_k=min(p_k,q_k)$, then for any modulus function f, $S^q(\Delta,f,r,m)= S^q(\Delta,f,p,m)\cap S^q(\Delta,f,q,m)$,

Proof. It follows from Theorem 2.2.(b) that $S^q(\Delta,f,r,m)\subset S^q(\Delta,f,p,m)\cap S^q(\Delta,f,q,m)$. For any complex number γ, $|\gamma||\gamma|^{r_k}\leq max\{|\gamma|^{p_i}, |\gamma|^{q_i}\}$, therefore $S^q(\Delta,f,p,m)\cap S^q(\Delta,f,q,m)\subset S^q(\Delta,f,r,m)$ and the proof is complete.

Theorem 2.4. If $H=sup p_k<\infty$, then for any modulus function f, we have $\ell_\infty\subset M(S^q_o(\Delta,f,p,m))$.

Proof. Let $a=(a_k)\in\ell_\infty$, then there exist some $T>0$ and all k such that $|a_k|<1+[|T|]$. Hence $x=(x_k)\in S^q_o(\Delta,f,p,m)$ implies

$$\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m a_i x_i)\right|^{p_i}\leq$$

$(1+[|T|])H\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\, f(\Delta_m x_i)\right|^{p_i}$ which gives the result.

Theorem 2.5. For any modulus function f,

(a) $S^q(\Delta,f,p,m) \subset S^q(\Delta,p,m)$ if the limit $\lim_{t\to\infty} \frac{f(t)}{t}=\beta$ exists.

(b) $S^q(\Delta,p,m) \subset S^q(\Delta,f,p,m)$ if there exists

a positive constant λ such that $f(t)\leq\lambda t$ for all $t\geq 0$.

Proof. (a) From Proposition 1 of Maddox [10], we have $\beta = inf\left\{\frac{f(t)}{t} : t > 0\right\}$, so that $0\leq\beta\leq f(1)$. Let $x=(x_k)\in S^q(\Delta,f,p,m)$ and $\beta>0$. Then definition of β, $\beta t\leq f(t)$ for all $t\geq 0$ and

$$\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\,(\Delta_m x_i - l)\right|^{p_i}$$

$$\leq \max\left(1,\beta^{-H}\right)\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\,f(\Delta_m x_i - l)\right|^{p_i}$$

and hence $x=(x_k)\in S^q(\Delta,p,m)$.

(b) Let $f(t)\leq\lambda t$ for all $t\geq 0$ for $\lambda>0$.

Suppose that $x=(x_k)\in S^q(\Delta,p,m)$, then we have

$$\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\,f(\Delta_m x_i - l)\right|^{p_i}$$

$$\leq \max\left(1,\lambda^{-H}\right)\left|\frac{1}{Q_k}\sum_{i=0}^{k} q_i\,(\Delta_m x_i - l)\right|^{p_i}$$

and hence $x=(x_k)\in S^q(\Delta,f,p,m)$.

The proof of following proposition is easy, so we omit it.

Proposition 2.5. The inclusions $S^q_o(\Delta,f,p,m) \subset S^q(\Delta,f,p,m) \subset S^q_\infty(\Delta,f,p,m)$ is valid.

3. Discussion

This this study, we define some new difference sequence spaces defined by modulus function and establish some properties of these properties of these spaces. If the readers wish, the spaces given in this study can be converted into new types of sequence spaces by changing the difference operator.

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