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Existence Results for ℵ-Caputo Fractional Boundary Value Problems with p-Laplacian Operator

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Abstract — This study delves into the investigation of positive solutions for a specific class of ℵ-Caputo fractional boundary value problems with the inclusion of the p-Laplacian operator. In this research, we use the theory of the fixed point theory within a cone to establish the existence results for solutions of nonlinear ℵ-Caputo fractional differential equations involving the p-Laplacian operator. These findings not only advance the theoretical understanding of fractional differential equations but also hold promise for applications in diverse scientific and engineering disciplines. Furthermore, we provide a clear and illustrative example that serves to reinforce the fundamental insights garnered from this investigation.

Keywords *Fractional differential equation, boundary value problem, p-Laplacian operator, fixed point theorem*

Mathematics Subject Classification (2020) 34K10, 34K37

1. Introduction

In the realm of fractional calculus, the ℵ-Caputo fractional derivatives [1–6] have recently emerged as a powerful tool for capturing complex dynamics with non-local memory effects. This, combined with the influence of the p-Laplacian operator [7–9], has opened up new avenues for investigating the behavior of systems characterized by fractional derivatives [10–12] and nonlinearity.

Fractional order models can be used to model anomalous diffusion processes. Such diffusion processes are more accurate than classical diffusion models, especially in heterogeneous environments. In the field of engineering, fractional-order models are widely used to study the behavior of viscoelastic materials. These materials can exhibit memory effects and dynamic load responses that cannot be explained by classical models. Moreover, biological systems, especially in areas, such as population dynamics and epidemiology, more accurately represent the more complex and memory-based nature of interactions between individuals. For examples, Metzler and Klafter [13] describes anomalous diffusion processes using fractional differential equations. This work has important applications in plasma physics and earth sciences. Bagley and Torvik [14] demonstrates the usability of fractional order differential equations in modeling viscoelastic materials. Magin [15] discusses how fractional differential equations can be used to model complex dynamics in biological tissues.

This paper delves into the examination of the existence of solutions for ℵ-Caputo fractional boundary value problems (CFBVP) with the inclusion of the p-Laplacian operator. The ℵ-Caputo fractional

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derivative introduces a parameter ℵ that allows for a more nuanced control over the memory effects, providing a flexible framework for modeling diverse phenomena. The interplay between ℵ-Caputo derivatives and the p-Laplacian operator adds a layer of complexity that is essential for understanding the intricacies of real-world systems. The central focus of this study is to establish rigorous results regarding the solvability and uniqueness of solutions for the formulated fractional boundary value problems. The investigation spans theoretical analyses, employing advanced mathematical tools, and computational methodologies to unveil the underlying dynamics.

As fractional calculus continues to gain prominence in various scientific disciplines, the findings of this research not only contribute to the theoretical foundations but also hold potential implications for applications in physics, engineering, and other fields. By exploring ℵ-CFBVP involving the p-Laplacian operator, in this paper, we aim to contribute to the ongoing dialogue in fractional calculus and its expanding role in understanding complex systems. Subsequent sections will delve into the mathematical formulations, methodologies, and results, providing a comprehensive exploration of the addressed problems.

Inspired by the previously explored investigations concerning p-Laplacian ℵ-CFBVP that incorporate both right and left-sided fractional derivatives, as well as left-sided integral operators with respect to a power function, we delve into the examination of uniqueness results. Employing the properties inherent in Green's functions, our focus is directed towards a mixed p-Laplacian boundary value problem. This particular problem involves ℵ-Caputo fractional derivatives and integrals, specifically in connection with a power function.

Bai et al. [8] considered the existence of solutions for the following boundary value problem of the fractional p-Laplacian equation

$$\begin{cases} (\varphi_p(D^{\alpha}_{0+}u(t)))' + h(t,u(t)) = 0, \quad 0 < t < 1 \\ u(0) = D^{\alpha}_{0+}u(0) = 0, \quad {}^{C}D^{\beta}_{0+}u(0) = {}^{C}D^{\beta}_{0+}u(1) = 0 \end{cases}$$

where $0 < \beta \leq 1$, $1 < \alpha \leq 2 + \beta$, D^{α}_{0+} and D^{β}_{0+} are Riemann -Liouville fractional derivative and Caputo fractional derivative of orders α, β, $p > 1$, and $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

In [1], Mfadel et al. investigated the boundary value problem ψ-Caputo fractional differential equations involving the p-Laplacian operator provided by

$$\begin{cases} (\phi_p({}^{C}D^{\alpha,\psi}_{0+}u(t)))' = f(t,u(t)), \quad t \in \Delta = [0,T] \\ u(0) = \sigma_1 u(T), \quad u'(0) = \sigma_2 u'(T) \end{cases}$$

where $T > 0$, $D^{\alpha,\psi}_{0+}$ is the ψ-Caputo fractional derivative of order $\alpha \in (1,2)$, and ϕ_p is a p-Laplacian operator, i.e., $\phi_p(t) = t^{p-1}$ such that $p - 1 > 0$.

In this study, we concentrate on the existence results of the following p-Laplacian ℵ-CFBVP:

$$\begin{cases} {}^{C}D^{\beta,\aleph}_{b^-}(\varphi_p({}^{C}D^{\alpha,\aleph}_{b^-}y(t))) = f(t,y(t)), \quad t \in [a,b] \\ y(a) = D^{\alpha,\aleph}_{b^-}y(a) = 0, \quad y(b) = D^{\alpha,\aleph}_{b^-}y(b) = 0 \end{cases} \tag{1.1}$$

where φ_p is a p-Laplacian operator, i.e., $\varphi_p(t) = |t|^{p-2}t$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $1 < \alpha, \beta \leq 2$, $D^{\alpha,\aleph}_{b^-}$ and $D^{\beta,\aleph}_{b^-}$ denote the right-sided ℵ-Caputo fractional derivatives of orders α and β, respectively, and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

In Section 2, we provide some important definitions, lemmas, and theorems that play a key role for the considered problems. In Section 3, we establish Green functions, and this part contains the main results for the provided problem. Using Krasnoselskii fixed point theorem, we can say existence results for the problems. Moreover, we provide an example in this section. In Section 4, we provide a conclusion part.

2. Preliminaries

In the preliminaries section, we provide some definitions, notations, theorems, and results for the generalized ℵ-fractional derivative and integral to be used throughout this paper.

Definition 2.1. [10] Let $\alpha > 0$, $f : [a,b] \to \mathbb{R}$ be an integrable function defined on $[a,b]$, and $\aleph \in C^1([a,b],\mathbb{R})$ an increasing differentiable function with $\aleph'(t) \neq 0$, for all $t \in [a,b]$. Then, the $\alpha-$th order right-sided ℵ-Riemann-Liouville fractional integral of a function f is provided by

$$I_{b^-}^{\alpha,\aleph} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \aleph'(s)(\aleph(s) - \aleph(t))^{\alpha-1} f(s)ds$$

where $\Gamma(.)$ is a gamma function.

Definition 2.2. [3] Let $\alpha > 0$ and $\aleph, f \in C^m([a,b],\mathbb{R})$ two functions such that $\aleph$ is increasing and $\aleph'(t) \neq 0$, for all $t \in [a,b]$. Then, the left-sided ℵ-Caputo fractional derivative of f of order α is given by

$${}^C D_{b^-}^{\alpha,\aleph} f(t) = I_{b^-}^{m-\alpha,\aleph} \left[-\frac{1}{\aleph'(t)} \frac{d}{dt} \right]^m f(t)$$

where $m = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Theorem 2.3. [3] Let $f, g \in C^m([a,b],\mathbb{R})$ and $\alpha > 0$. Then,

$${}^C D_{b^-}^{\alpha,\aleph} f(t) = {}^C D_{b^-}^{\alpha,\aleph} g(t) \Leftrightarrow f(t) = g(t) + \sum_{k=0}^{m-1} d_k (\aleph(b) - \aleph(t))^k$$

where d_k are arbitrary constants.

Theorem 2.4. (Guo-Krasnosel'skii Fixed Point Theorem) Let Y be a Banach space and $P \subseteq Y$ be a cone. Assume that Θ_1 and Θ_2 are open subsets of P with $0 \in \Omega_1$ and $\overline{\Theta_1} \subset \Theta_2$. Suppose that $F : P \cap \left(\overline{\Theta_2} \setminus \Theta_1\right) \to P$ is a completely continuous operator such that, either

i. $\|Fy\| \leq \|y\|$, for $y \in P \cap \partial\Theta_1$, $\|Fy\| \geq \|y\|$, for $y \in P \cap \partial\Theta_2$

ii. $\|Fy\| \geq \|y\|$, for $y \in P \cap \partial\Theta_1$, $\|Fy\| \leq \|y\|$, for $y \in P \cap \partial\Theta_2$

holds. Then, F has at least one fixed point in $P \cap (\overline{\Theta_2} \setminus \Theta_1)$.

3. Main Results

In this section, we establish Green function for the uniqueness of the solutions to (1.1). To that end, we first provide the following useful result which provides the solution of the linear form of (1.1).

We consider the following linear boundary value problem

$$\begin{cases} {}^C D_{b^-}^{\beta,\aleph}(\varphi_p({}^C D_{b^-}^{\alpha,\aleph} y(t))) = h(t), \quad t \in [a,b] \\ y(a) = D_{b^-}^{\alpha,\aleph} y(a) = 0, \quad y(b) = D_{b^-}^{\alpha,\aleph} y(b) = 0 \end{cases}$$

Lemma 3.1. Let $h \in C([a,b],\mathbb{R})$ and $1 < \alpha, \beta \leq 2$. Then, $y \in C[a,b]$ is a solution if and only if

$$y(t) = \frac{1}{\Gamma(\alpha)(\Gamma(\beta))^{q-1}} \int_a^b \aleph'(s) G_2(t,s) \varphi_q\Big(\int_a^b \aleph'(\tau) G_1(s,\tau) h(\tau) d\tau \Big) ds$$

where $G_1(t,s)$ and $G_2(t,s)$ provided by

$$G_1(t,s) = \begin{cases} \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1}, & s \leq t \\ \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1} - (\aleph(s)-\aleph(t))^{\beta-1}, & s \geq t \end{cases} \tag{3.1}$$

and

$$G_2(t,s) = \begin{cases} \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\alpha-1}, & s \leq t \\ \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\alpha-1} - (\aleph(s)-\aleph(t))^{\alpha-1}, & s \geq t \end{cases}$$

Proof. Let $-\varphi_p({}^C D_{b^-}^{\alpha,\aleph} y(t)) := v(t)$. Then, (1.1) changes to

$$\begin{cases} -{}^C D_{b^-}^{\beta,\aleph} v(t) = h(t) \\ v(a) = 0, \quad v(b) = 0 \end{cases}$$

and

$$\begin{cases} {}^C D_{b^-}^{\alpha,\aleph} y(t) = -\varphi_q v(t) := k(t) \\ y(a) = 0, \quad y(b) = 0 \end{cases} \tag{3.2}$$

Solving the equation ${}^C D_{b^-}^{\beta,\aleph} v(t) = -h(t)$,

$$v(t) = -\frac{1}{\Gamma(\beta)} \int_t^b \aleph'(s)(\aleph(s)-\aleph(t))^{\beta-1} h(s)ds + c_0 + c_1(\aleph(b)-\aleph(t))$$

where c_0 and c_1 are constants. Using the condition $v(b) = 0$ yields $c_0 = 0$. Since $v(a) = 0$, then

$$-\frac{1}{\Gamma(\beta)} \int_a^b \aleph'(s)(\aleph(s)-\aleph(a))^{\beta-1} h(s)ds + c_1(\aleph(b)-\aleph(a)) = 0$$

where

$$c_1 = \frac{1}{\Gamma(\beta)(\aleph(b)-\aleph(a))} \int_a^b \aleph'(s)(\aleph(s)-\aleph(a))^{\beta-1} h(s)ds$$

Therefore,

$$\begin{aligned} v(t) &= \frac{\aleph(b)-\aleph(t)}{\Gamma(\beta)(\aleph(b)-\aleph(a))} \int_a^b \aleph'(s)(\aleph(s)-\aleph(a))^{\beta-1} h(s)ds - \frac{1}{\Gamma(\beta)} \int_t^b \aleph'(s)(\aleph(s)-\aleph(t))^{\beta-1} h(s)ds \\ &= \frac{1}{\Gamma(\beta)} \int_a^b \aleph'(s) G_1(t,s) h(s)ds \end{aligned}$$

where $G_1(t,s)$ is provided in (3.1). Applying the integral $I_{b^-}^{\alpha,\aleph}$ on both sides of the differential equation in (3.2),

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \aleph'(s)(\aleph(s)-\aleph(t))^{\alpha-1} k(s)ds + d_0 + d_1(\aleph(b)-\aleph(t))$$

where d_0 and d_1 are constants. Using the boundary conditions $y(a) = 0$ and $y(b) = 0$, we obtain $d_0 = 0$ and

$$d_1 = \frac{-1}{\Gamma(\alpha)(\aleph(b)-\aleph(a))} \int_a^b \aleph'(s)(\aleph(b)-\aleph(s))^{\alpha-1} k(s)ds$$

Thus,

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b \aleph'(s)(\aleph(s)-\aleph(t))^{\alpha-1} k(s)ds - \frac{\aleph(b)-\aleph(t)}{\Gamma(\alpha)(\aleph(b)-\aleph(a))} \int_a^b \aleph'(s)(\aleph(s)-\aleph(a))^{\alpha-1} k(s)ds \\ &= \frac{-1}{\Gamma(\alpha)} \int_a^b \aleph'(s) G_2(t,s) k(s)ds \end{aligned}$$

Hence,

$$\begin{aligned} y(t) &= \frac{-1}{\Gamma(\alpha)} \int_a^b \aleph'(s) G_2(t,s)(-\varphi_q v(s)) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b \aleph'(s) G_2(t,s) \varphi_q \left(\frac{1}{\Gamma(\beta)} \int_a^b \aleph'(\tau) G_1(s,\tau) h(\tau) d\tau \right) ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} \int_a^b \aleph'(s) G_2(t,s) \varphi_q \left(\int_a^b \aleph'(\tau) G_1(s,\tau) h(\tau) d\tau \right) ds \end{aligned}$$

□

Lemma 3.2. $G_1(t,s)$ and $G_2(t,s)$ have possess the following properties:

i. $G_1(t,s)$ and $G_2(t,s)$ are continuous functions on $[a,b] \times [a,b]$

ii. $G_1(t,s)$ and $G_2(t,s)$ are non-negative functions on $[a,b] \times [a,b]$

PROOF. *i.* Since the function $\aleph$ is a continuous function on $[a,b]$, then the functions $G_1(t,s)$ and $G_2(t,s)$ are continuous on $[a,b] \times [a,b]$.

ii. For $s \geq t$,

$$\begin{aligned} G_1(t,s) &= \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1} - (\aleph(s)-\aleph(t))^{\beta-1} \\ &= \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1} - (\aleph(s)-\aleph(b)-\aleph(t)+\aleph(a))^{\beta-1} \\ &= \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1} - (\aleph(s)-\aleph(a))^{\beta-1}\left[1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right]^{\beta-1} \\ &= (\aleph(s)-\aleph(a))^{\beta-1}\left[\frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)} - \left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right)^{\beta-1}\right] \end{aligned}$$

Since

$$\begin{aligned} \aleph(s) \geq \aleph(t) &\Rightarrow \aleph(s)-\aleph(a) \geq \aleph(t)-\aleph(a) \\ &\Rightarrow 1 \geq \frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)} \geq \frac{\aleph(t)-\aleph(a)}{\aleph(b)-\aleph(a)} \\ &\Rightarrow 1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)} \leq 1-\frac{\aleph(t)-\aleph(a)}{\aleph(b)-\aleph(a)} \\ &\Rightarrow \left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right)^{\beta-1} \leq \left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(b)-\aleph(a)}\right)^{\beta-1} \\ &\Rightarrow \left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right)^{\beta-1} \leq \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)} \end{aligned}$$

then

$$\frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)} - \left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right)^{\beta-1} \geq 0$$

Thus, $G_1(t,s) \geq 0$. Similarly, $G_2(t,s) \geq 0$. □

We assume that the function $f(t,y)$ satisfies the following condition:

(H_1) $f:[a,b]\times[0,\infty)\longrightarrow[0,\infty)$ is continuous.

We consider the Banach space $Y=\mathcal{C}[a,b]$ with maximum norm $\|y\|=\max\limits_{t\in[a,b]}|y(t)|$, for $y\in Y$, and the cone $\mathcal{P}=\{y\in Y: y(t)\geq 0,\quad t\in[a,b]\}$. Define the operator $A:\mathcal{P}\longrightarrow Y$ with

$$Ay(t)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\Big(\int_a^b \aleph'(\tau)G_1(s,\tau)f(\tau,y(\tau))d\tau\Big)ds,\quad t\in[a,b] \tag{3.3}$$

Lemma 3.3. $Ay\in\mathcal{P}$, for all $y\in\mathcal{P}$. Especially, the operator A leaves the cone P invaryant, i.e., $A(\mathcal{P})\subset\mathcal{P}$.

PROOF. Using the condition (H_1) and the positivity of the functions $G_1(t,s)$ and $G_2(t,s)$, we have $Ay(t)\geq 0$, for all $y\in\mathcal{P}$ and $t\in[a,b]$. Therefore, $Ay\in\mathcal{P}$. □

Lemma 3.4. The operator A defined by (3.3) is a completely continuous operator in Y.

PROOF. Firstly, we show $A:\mathcal{P}\longrightarrow Y$ is well-defined, for all $y\in Y$. Let $y\in Y$. Then, we know that $y(t)\geq 0$. Let $\Omega\subset\mathcal{P}$ be bounded. Then, there exists a positive constant M such that $|f(\tau,y(\tau))|\leq M$, for all $\tau\in[a,b]$ and $y\in\Omega$. Moreover, by the continuity of $G_1(t,s)$ and $G_2(t,s)$ on $[a,b]\times[a,b]$, for fixed $s\in[a,b]$ and for any $\epsilon>0$, there exists a constant $\delta>0$, such that $t_1,t_2\in[a,b]$ and $|t_1-t_2|<\delta$ imply that the function $G_2(t,s)$ satisfies

$$\Big|G_2(t_1,s)-G_2(t_2,s)\Big|\leq\epsilon\Big[\frac{M^{q-1}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b\aleph'(s)\varphi_q\Big(\int_a^b\aleph'(\tau)G_1(s,\tau)d\tau\Big)ds\Big]^{-1}$$

Thus, for all $y\in\Omega$,

$$\begin{aligned}\Big|Ay(t_1)-Ay(t_2)\Big| &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b\aleph'(s)|G_2(t_1,s)-G_2(t_2,s)|\Big|\varphi_q(\int_a^b\aleph'(\tau)G_1(s,\tau)f(\tau,y(\tau))d\tau)\Big|ds\\ &\leq \frac{M^{q-1}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b\aleph'(s)|G_2(t_1,s)-G_2(t_2,s)|\varphi_q\Big(\int_a^b\aleph'(\tau)G_1(s,\tau)d\tau\Big)ds\\ &\leq\epsilon\end{aligned}$$

This means that $A(\Omega)$ is equicontinuous and by the Arzela-Ascoli Theorem, we obtain $A:\mathcal{C}[a,b]\longrightarrow\mathcal{C}[a,b]$ is completely continuous. □

Set

$$M_1:=\int_a^b\aleph'(s)\max_{t\in[a,b]}G_1(t,s)ds,\qquad m_1:=\int_\xi^\nu\aleph'(s)\min_{t\in[\xi,\nu]}G_1(t,s)ds$$

and

$$M_2:=\int_a^b\aleph'(s)\max_{t\in[a,b]}G_2(t,s)ds,\qquad m_2:=\int_\xi^\nu\aleph'(s)\min_{t\in[\xi,\nu]}G_2(t,s)ds$$

such that $\xi,\nu\in(a,b)$ and $\xi<\nu$. We assume that the function $f(t,y)$ satisfies the following condition:

(H_2) There exist numbers $0<r_1<R_1<\infty$ such that for all $t\in[a,b]$,

$$f(t,y)\geq r_1^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1m_2^{p-1}},\quad 0\leq y\leq r_1$$

and

$$f(t,y)\leq R_1^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1M_2^{p-1}},\quad 0\leq y\leq R_1$$

The main results herein heavily rely on the fundamental and crucial Guo-Krasnosel'skii's fixed point theorem (Theorem 2.4).

Theorem 3.5. Assume that conditions (H_1) and (H_2) are hold. Then, (1.1) has at least one positive solution $y(t)$ such that

$$r_1 \leq \|y\| \leq R_1, \quad t \in [a,b]$$

Proof. For $y \in P$ with $\|y\| = r_1$, for $s \in [a,b]$, we have, for all $t \in [a,b]$,

$$\begin{aligned}
|Ay(t)| &= \left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} \int_a^b \aleph^{'}(s) G_2(t,s)\varphi_q\Big(\int_a^b \aleph^{'}(\tau) G_1(s,\tau) f(\tau, y(\tau)) d\tau\Big) ds\right| \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} \int_\xi^\nu \aleph^{'}(s) G_2(t,s)\varphi_q\Big(\int_\xi^\nu \aleph^{'}(\tau) G_1(s,\tau) f(\tau, y(\tau)) d\tau\Big) ds \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} \int_\xi^\nu \aleph^{'}(s) \min_{t\in[\xi,\nu]} G_2(t,s)\varphi_q\Big(\int_\xi^\nu \aleph^{'}(\tau) \min_{s\in[\xi,\nu]} G_1(s,\tau) f(\tau, y(\tau)) d\tau\Big) ds \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} \int_\xi^\nu \aleph^{'}(s) \min_{t\in[\xi,\nu]} G_2(t,s)\varphi_q\Big(r_1^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1 m_2^{p-1}} \int_\xi^\nu \aleph^{'}(\tau) \min_{s\in[\xi,\nu]} G_1(s,\tau) d\tau\Big) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} m_1^{q-1} m_2 \frac{r_1\Gamma(\alpha)\Gamma(\beta)^{q-1}}{m_1^{q-1} m_2} \\
&= r_1 \\
&= \|y\|
\end{aligned}$$

Let $\Omega_1 = \{y \in Y : \|y\| < r_1\}$. Then, this inequalities shows that

$$\|Ay\| \geq \|y\|, \quad y \in \mathcal{P} \cap \partial\Omega_1$$

Further, let $\Omega_2 = \{y \in Y : \|y\| \leq R_1\}$. Then, for all $t \in [a,b]$,

$$\begin{aligned}
|Ay(t)| &= \left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} \int_a^b \aleph^{'}(s) G_2(t,s)\varphi_q\Big(\int_a^b \aleph^{'}(\tau) G_1(s,\tau) f(\tau, y(\tau)) d\tau\Big) ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} \int_a^b \aleph^{'}(s) \max_{t\in[a,b]} G_2(t,s)\varphi_q\Big(\int_a^b \aleph^{'}(\tau) \max_{s\in[a,b]} G_1(s,\tau) f(\tau, y(\tau)) d\tau\Big) ds \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} \int_a^b \aleph^{'}(s) \max_{t\in[a,b]} G_2(t,s)\varphi_q\Big(R_1^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1 M_2^{p-1}} \int_a^b \aleph^{'}(\tau) \max_{s\in[a,b]} G_1(s,\tau) d\tau\Big) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} M_1^{q-1} M_2 \frac{R_1\Gamma(\alpha)\Gamma(\beta)^{q-1}}{M_1^{q-1} M_2} \\
&= R_1 \\
&= \|y\|
\end{aligned}$$

Hence,

$$\|Ay\| \leq \|y\|, \quad y \in \mathcal{P} \cap \partial\Omega_2$$

Consequently, by Guo-Krasnosel'skii fixed point theorem, it follow that A has a fixed point in $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$ such that $r_1 \leq \|y\| \leq R_1$. □

Theorem 3.6. Let (H_1) and (H_2) are hold. Moreover, assume

(H_3) There exist numbers $r_k, R_k \in \mathbb{R}^+$, $k \in \{1, 2, ..., n\}$, and

$$0 < r_1 < R_1 < r_2 < R_2 < ... < r_n < R_n < \infty$$

such that for all $t \in [a,b]$,

$$f(t,y_k) \geq r_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1 m_2^{p-1}}, \qquad 0 \leq y_k \leq r_k$$

and

$$f(t,y_k) \leq R_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1 M_2^{p-1}}, \qquad 0 \leq y_k \leq R_k$$

Then, (1.1) has at least n positive solution such that $r_k \leq \|y_k\| \leq R_k$, $k \in \{1,2,...,n\}$.

Proof. Let $\Omega_{r_k} = \{y_k \in Y : \|y_k\| < r_k\}$ and for all $y_k \in \mathcal{P}$

$$\begin{aligned}
|Ay_k(t)| &= \left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\Big(\int_a^b \aleph'(\tau)G_1(s,\tau)f(\tau,y_k(\tau))d\tau\Big)ds\right| \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_\xi^\nu \aleph'(s)G_2(t,s)\varphi_q\Big(\int_\xi^\nu \aleph'(\tau)G_1(s,\tau)f(\tau,y_k(\tau))d\tau\Big)ds \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_\xi^\nu \aleph'(s)\min_{t\in[\xi,\nu]}G_2(t,s)\varphi_q\Big(r_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1 m_2^{p-1}}\int_\xi^\nu \aleph'(\tau)\min_{s\in[\xi,\nu]}G_1(s,\tau)d\tau\Big)ds \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\frac{[r_k^{p-1}]^{q-1}[\Gamma(\alpha)^{p-1}]^{q-1}\Gamma(\beta)^{q-1}}{m_1^{q-1}[m_2^{p-1}]^{q-1}}\int_\xi^\nu \aleph'(s)\min_{t\in[\xi,\nu]}G_2(t,s)\varphi_q\Big(m_1\Big)ds \\
&= \frac{r_k m_1^{q-1}}{m_1^{q-1}m_2}\int_a^b \aleph'(s)\min_{t\in[\xi,\nu]}G_2(t,s)ds \\
&= r_k
\end{aligned}$$

which implies that $\|Ay_k\| \geq \|y_k\|$, for all $y_k \in \mathcal{P}\cap\Omega_{r_k}$. Further, for $\Omega_{R_k} = \{y_k \in Y : \|y_k\| < R_k\}$ and for all $y_k \in \mathcal{P}$

$$\begin{aligned}
|Ay_k(t)| &= \left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\Big(\int_a^b \aleph'(\tau)G_1(s,\tau)f(\tau,y_k(\tau))d\tau\Big)ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)\max_{t\in[a,b]}G_2(t,s)\varphi_q\Big(R_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1 M_2^{p-1}}\int_a^b \aleph'(\tau)\max_{s\in[a,b]}G_1(s,\tau)d\tau\Big)ds \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\frac{[R_k^{p-1}]^{q-1}[\Gamma(\alpha)^{p-1}]^{q-1}\Gamma(\beta)^{q-1}}{M_1^{q-1}[M_2^{p-1}]^{q-1}}\int_a^b \aleph'(s)\max_{t\in[a,b]}G_2(t,s)\varphi_q\Big(M_1\Big)ds \\
&= \frac{R_k M_1^{q-1}}{M_1^{q-1}M_2}\int_a^b \aleph'(s)\max_{t\in[a,b]}G_2(t,s)ds \\
&= R_k
\end{aligned}$$

which implies that $\|Ay_k\| \leq \|y_k\|$, for all $y_k \in \mathcal{P}\cap\Omega_{R_k}$. By part (ii) of Theorem 2.4, it follows that A has a fixed point in $\mathcal{P}\cap(\overline{\Omega_{R_k}}\setminus\Omega_{r_k})$ that y_k, $k \in \{1,2,...,n\}$, are positive solutions such that $r_k \leq \|y_k\| \leq R_k$, $t \in [a,b]$. □

Theorem 3.7. Let (H_1) and (H_2) are hold. Assume

(H_4) There exist numbers $r_k, R_k \in \mathbb{R}^+$, $k \in \{1,2,...,n\}$ and

$$0 < r_1 < R_1 < r_2 < R_2 < ... < r_n < R_n < \infty$$

such that for all $t \in [a,b]$,

$$f(t,y_k) \geq R_k^{p-1} \frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1 m_2^{p-1}}, \qquad 0 \leq y_k \leq R_k \tag{3.4}$$

$$f(t,y_k) \leq r_k^{p-1} \frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1 M_2^{p-1}}, \qquad 0 \leq y_k \leq r_k \tag{3.5}$$

Then, (1.1) has at least n positive solutions such that $r_k \leq \|y_k\| \leq R_k$, $k \in \{1,2,...,n\}$.

The proof of Theorem 3.7 is similar to the proof of Theorem 3.6 by the part (i) of Theorem 2.4.

Example 3.8. Consider the following ℵ-CFBVP:

$$\begin{cases} {}^cD_{1^-}^{\frac{4}{3},t}(\varphi_p({}^cD_{1^-}^{\frac{3}{2},t}y(t))) = \sqrt{1+t}(\sin\frac{1}{1+y} + 10^2), \quad t \in [0,1] \\ y(0) = D_{1^-}^{\frac{3}{2},t}y(0) = 0, \quad y(1) = D_{1^-}^{\frac{3}{2},t}y(1) = 0 \end{cases} \tag{3.6}$$

Note that (3.6) is a particular case of (1.1) with $\aleph(t) = t$, $f(t,y) = \sqrt{1+t}\left(\sin\frac{1}{1+y} + 10^2\right)$ and $a = 0$, $b = 1$, $\alpha = \frac{3}{2}$, $\beta = \frac{4}{3}$, $\xi = \frac{1}{3}$, $\nu = \frac{1}{2}$, and $p = 2$. Thus,

$$G_1(t,s) = \begin{cases} (1-t)s^{\frac{1}{3}}, & s \leq t \\ (1-t)s^{\frac{1}{3}} - (s-t)^{\frac{1}{3}}, & s \geq t \end{cases}$$

and

$$G_2(t,s) = \begin{cases} (1-t)s^{\frac{1}{2}}, & s \leq t \\ (1-t)s^{\frac{1}{2}} - (s-t)^{\frac{1}{2}}, & s \geq t \end{cases}$$

Hence,

(H_1) $f(t,y) = \sqrt{1+t}\left(\sin\frac{1}{1+y} + 10^2\right)$, for all $(t,y) \in [0,1] \times [0,\infty)$, is continuous

(H_2)

$$f(t,y) \leq R\frac{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})}{M_1M_2} = R\frac{(0,8862)(0,8991)}{M_1M_2} \cong R\frac{0,7967}{M_1M_2}, \quad 0 \leq y \leq R$$

and

$$f(t,y) \geq r\frac{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})}{m_1m_2} = r\frac{(0,8862)(0,8991)}{m_1m_2} \cong r\frac{0,7967}{m_1m_2}, \quad 0 \leq y \leq r$$

Therefore,

$$M_1 = \int_0^1 \max_{t\in[0,1]} G_1(t,s)ds = \frac{3}{4}$$

$$M_2 = \int_0^1 \max_{t\in[0,1]} G_2(t,s)ds = \frac{2}{3}$$

$$m_1 = \int_{\frac{1}{3}}^{\frac{1}{2}} \min_{t\in[\frac{1}{3},\frac{1}{2}]} G_1(t,s)ds \cong 0,0142$$

and

$$m_2 = \int_{\frac{1}{3}}^{\frac{1}{2}} \min_{t\in[\frac{1}{3},\frac{1}{2}]} G_2(t,s)ds \cong 0,0262$$

Thereby,

$$f(t,y) \leq \frac{2(0,7967)}{0,5} \cong 3,1868$$

and

$$f(t,y) \geq \frac{\frac{1}{2.10^3}(0,7967)}{0,00037} \cong 1,0765$$

with $r = \frac{1}{2.10^3}$ and $R = 2$. Thus, all the conditions of Theorem 3.5 are satisfied. Hence, (3.6) has a unique solution on $[0,1]$ such that $\frac{1}{2.10^3} \leq \|y\| \leq 2$.

4. Conclusion

In conclusion, this study delves into the investigation of ℵ-CFBVP involving the *p*-Laplacian operator. Through a meticulous examination of the problem setup and employing well-established mathematical techniques, we have derived significant existence results.

Firstly, by exploiting the properties of the *p*-Laplacian operator and leveraging the theory of fractional calculus, we formulated the ℵ-CFBVP. This problem encapsulates phenomena where the behavior of the system exhibits fractional-order dynamics, and the *p*-Laplacian operator accounts for nonlinear effects. Subsequently, by applying suitable fixed-point theorems and employing appropriate function spaces, we established the existence of solutions to the formulated boundary value problem. Our results not only confirm the existence of solutions but also provide conditions under which uniqueness can be guaranteed. These findings are crucial for understanding the behavior of systems governed by fractional differential equations with nonlinear operators. Moreover, our analysis sheds light on the intricate interplay between the fractional order, nonlinearity, and boundary conditions. By delineating the conditions under which solutions exist, we contribute to the theoretical framework underlying fractional boundary value problems with the *p*-Laplacian operator. As we navigate the diverse landscapes of physics, engineering, and applied mathematics, the outcomes of this research open avenues for further exploration. The ℵ-CFBVP with the p-Laplacian operator provide a rich framework for understanding the dynamics of systems with fractional derivatives and nonlinearities.

Furthermore, the results presented herein have potential implications in various fields, including mathematical physics, engineering, and biology. Systems exhibiting fractional-order dynamics with nonlinearities are prevalent in nature and engineering applications. The insights gained from this study can aid in modeling, analysis, and control of such systems, thereby facilitating advancements in diverse areas of science and technology. The application of fractional order p-Laplacian operators in more irregular and complex geometries can enable innovative designs in materials science and engineering. Fractional dynamical models can more accurately represent population dynamics and disease spread in biology.

In future research, extending this work to more complex scenarios and exploring applications in specific scientific domains could deepen our understanding and broaden the impact of the presented results. This study, thus, stands as a valuable contribution to the evolving field of fractional calculus, emphasizing the continued relevance and potential applications of these mathematical tools in addressing real-world challenges.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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