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Research article

From Ricci Soliton to Almost Contact Metric Structures

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Abstract

In this paper, we construct almost contact metric structures on a three-dimensional Riemannian manifold equipped with an almost Ricci soliton. Then, we give the techniques necessary to define the nature of such structures. Concrete examples are given.

Keywords: *Almost contact metric structure, trans-Sasakian manifolds, C_{12}-manifolds, Ricci soliton.*

AMS Subject Classification (2020): *Primary 53C15; Secondary 53C55.*

1. Introduction

In the last few years, the hypothesis of geometric flows has become the most fascinating mathematical tools for describing geometric structures in Riemannian geometry.

The investigation of singularities of the flows is significantly affected by the specific section of solutions where the metric evolves through diffeomorphisms, as they appear as possible singularity models. They are frequently called soliton solutions.

In 1982, Hamilton [10] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool to study Riemannian manifolds, especially for manifolds with positive curvatures.

The vector field V generates the Ricci soliton that is viewed as a special solution of the Ricci flow, and is called the generating vector field. A Ricci soliton is said to be a gradient Ricci soliton if the generating vector field V is the gradient of a potential function.

The first that studied the existence of Ricci soliton on contact geometry is Sharma [15], and after that, many works emerged in this direction that deal with different types of almost contact metric manifolds in particular, the normal ones. See, for example, [8, 9, 11, 16].

Recently, in [1], the authors investigated Ricci soliton and generalized Ricci soliton on a 3-dimensional C_{12}-manifold, which is a non-normal almost contact metric manifold.

By reflecting on these works, we have the right to ask about the study of the reverse direction, i.e. the construction of almost contact metric structures starting from an almost Ricci soliton.

One of the goals of this paper is to explore other ways for constructing almost contact metric structures on a three-dimensional Riemannian manifold equipped with an almost Ricci soliton. Results in our paper can be divided in two parts. In the first part, we construct almost contact metric structures starting from only a unit vector field. In the second part, we study the nature of these structures Based on the presence of an almost Ricci soliton on a Riemannian manifold.

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2. Preliminaries

2.1. Almost contact metric manifolds

For more background on almost contact metric manifolds, we recommend the references [5, 17].
An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$-tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2(X) = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \end{cases} \tag{2.1}$$

for any vector fields X, Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

The fundamental 2-form ϕ is defined by $\phi(X,Y) = g(X, \varphi Y)$. It is known that the almost contact structure (φ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X,Y) = N_\varphi(X,Y) + 2d\eta(X,Y)\xi = 0, \tag{2.2}$$

for any X, Y on M, where N_φ denotes the Nijenhuis torsion of φ, given by

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.3}$$

In [13], the author proves that (φ, ξ, η, g) is trans-Sasakian structure of type (α, β) if and only if it is normal and

$$\mathrm{d}\eta = \alpha\phi, \qquad \mathrm{d}\phi = 2\beta\eta \wedge \phi, \tag{2.4}$$

where d denotes the exterior derivative, $\alpha = \frac{1}{2n}\delta\phi(\xi)$, $\beta = \frac{1}{2n}div\xi$ and δ is the codifferential of g.
It is well known that the trans-Sasakian condition may be expressed as an almost contact metric structure satisfying

$$(\nabla_X\varphi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big). \tag{2.5}$$

In this paper we focus on the case of three-dimensional almost contact metric manifold. So, in this case, Olszak [12], proved that any almost contact metris structure is trans-Sasakian of type (α, β) if and only if

$$\nabla_X\xi = -\alpha\varphi X - \beta\varphi^2 X, \tag{2.6}$$

where $2\alpha = tr_g(\varphi\nabla\xi)$ and $2\beta = \mathrm{div}\xi$.

It is clear that a trans-Sasakian manifold of type $(1,0)$ is a Sasakian manifold and a trans-Sasakian manifold of type $(0,1)$ is a Kenmotsu manifold. A trans-Sasakian manifold of type $(0,0)$ is called a cosymplectic manifold.

In [7], the Ricci tensor S and Riemannian curvature tensor R for 3-dimensional trans-Sasakian manifolds type (α, β) are studied and their explicit formulas are given by

$$\begin{aligned} S(X,Y) &= \Big(\frac{r}{2} + \xi(\beta) - (\alpha^2 - \beta^2)\Big)g(X,Y) \\ &\quad -\Big(\frac{r}{2} + \xi(\beta) - 3(\alpha^2 - \beta^2)\Big)\eta(X)\eta(Y) \\ &\quad -\eta(X)\big(\varphi Y(\alpha) + Y(\beta)\big) - \big(X(\beta) + \varphi X(\alpha)\big)\eta(Y), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} R(X,Y)Z &= \Big(\frac{r}{2} + 2\xi(\beta) - 2(\alpha^2 - \beta^2)\Big)\big(g(Y,Z)X - g(X,Z)Y\big) \\ &\quad -g(Y,Z)\Big(\big(\frac{r}{2} + \xi(\beta) - 3(\alpha^2 - \beta^2)\big)\eta(X)\xi \\ &\quad -\eta(X)\big(\varphi\mathrm{grad}\alpha - \mathrm{grad}\beta\big) + \big(X(\beta) + \varphi X(\alpha)\big)\xi\Big) \\ &\quad +g(X,Z)\Big(\big(\frac{r}{2} + \xi(\beta) - 3(\alpha^2 - \beta^2)\big)\eta(Y)\xi \\ &\quad -\eta(Y)\big(\varphi\mathrm{grad}\alpha - \mathrm{grad}\beta\big) + \big(Y(\beta) + \varphi Y(\alpha)\big)\xi\Big) \\ &\quad -\Big(\big(Z(\beta) + \varphi Z(\alpha)\big)\eta(Y) + \big(Y(\beta) + \varphi Y(\alpha)\big)\eta(Z) \\ &\quad +\big(\frac{r}{2} + \xi(\beta) - 3(\alpha^2 - \beta^2)\big)\eta(Y)\eta(Z)\Big)X, \end{aligned} \tag{2.8}$$

where r is the scalar curvature of the manifold M.

Another important elementary almost contact metric structure has been studied recently in several works, it is the class C_{12}. See for example, [1, 2, 3, 6].

Remark 2.1. Unlike our previous papers related to C_{12}-manifold (see [1, 2, 3, 4, 6]), the existence of three non-zero global vector fields ξ, $\psi = \nabla_\xi \xi \neq 0$ and $\varphi\psi$ which are naturally pairwise orthogonal, encourages us to suggest the name "**corner manifold**" instead of C_{12}-manifold.

The authors proves that (φ, ξ, η, g) is a corner structure if and only if it is integrable i.e., $N_\varphi = 0$ and

$$\mathrm{d}\eta = \eta \wedge \omega \qquad \mathit{and} \qquad \mathrm{d}\phi = 0, \tag{2.9}$$

where $\omega = \nabla_\xi \eta$ and $\omega^\sharp = \psi = \nabla_\xi \xi$, that is, $\omega(X) = g(\psi, X)$ for all X vector field on M.

It is well known that the corner condition may be expressed as an almost contact metric structure satisfying

$$(\nabla_X \varphi)Y = -\eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big). \tag{2.10}$$

In dimension three, there are two nice characterizations for corner manifolds given in [2, 4] by:

Theorem 2.1. *Let $(M^3, \varphi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. M is a corner manifold if and only if*

$$\nabla_X \xi = \eta(X)\psi, \tag{2.11}$$

where $\psi = \nabla_\xi \xi$. Or, equivalently

$$\nabla_{\varphi X} \xi = 0. \tag{2.12}$$

In general, the vector field ψ is not unitary. So, for three-dimensional C_{12}-manifold we put $V = \mathrm{e}^{-\rho}\psi$ where $\mathrm{e}^\rho = \|\psi\|$, we get immediately that $\{\xi, V, \varphi V\}$ is an orthonormal frame. We refer to this basis as "fundamental basis".

From ([2], Corollary 5.1) with $\beta = 0$ we obtain the components of the Levi-Civita connection corresponds to the fundamental base as follows:

Proposition 2.1. *For any 3-dimensional corner manifold, we have*

$$\begin{array}{lll} \nabla_\xi \xi = \mathrm{e}^\rho V, & \nabla_V \xi = 0, & \nabla_{\varphi V} \xi = 0, \\ \nabla_\xi V = -\mathrm{e}^\rho \xi + \kappa \mathrm{e}^{-2\rho}\varphi V, & \nabla_V V = \varphi V(\rho)\varphi V, & \nabla_{\varphi V} V = a\varphi V, \\ \nabla_\xi \varphi V = -\kappa \mathrm{e}^{-2\rho} V, & \nabla_V \varphi V = -\varphi V(\rho) V, & \nabla_{\varphi V}\varphi V = -aV. \end{array}$$

where $\kappa = g(\nabla_\xi \psi, \varphi\psi)$ and $a = \mathrm{e}^{-\rho}\mathrm{div}\psi + \mathrm{e}^\rho - V(\rho)$.

Using Theorem this Proposition, standard calculations yield

$$\begin{cases} R(\xi, V)\xi = \big(\mathrm{e}^{2\rho} - V(\mathrm{e}^\rho)\big)V - \varphi V(\mathrm{e}^\rho)\varphi V, \\ R(\xi, \varphi V)\xi = -\varphi V(\mathrm{e}^\rho)V - a\mathrm{e}^\rho \varphi V, \\ R(V, \varphi V)\xi = 0, \\ R(V, \varphi V)V = b\varphi V, \end{cases} \tag{2.13}$$

where $b = a^2 + V(a) - \varphi V\big(\varphi V(\rho)\big) + \big(\varphi V(\rho)\big)^2$.

The Ricci tensor S, which is defined for all $X, Y \in \mathfrak{X}(M)$ by

$$S(X,Y) = g\big(R(X,\xi)\xi, Y\big) + g\big(R(X,\psi)\psi, Y\big) + g\big(R(X,\varphi\psi)\varphi\psi, Y\big),$$

is completely described by

$$\begin{cases} S(\xi,\xi) = \mathrm{div}\psi, \\ S(V,V) = -b - \mathrm{e}^{2\rho} + V(\mathrm{e}^\rho), \\ S(\varphi V, \varphi V) = a\mathrm{e}^\rho - b, \\ S(V, \varphi V) = \varphi V(\mathrm{e}^\rho), \\ S(\xi, V) = S(\xi, \varphi V) = 0. \end{cases} \tag{2.14}$$

From (2.14), the scalar curvature r is given by

$$r = \mathrm{div}\psi - \mathrm{e}^{2\rho} + V(\mathrm{e}^\rho) - 2b + a\mathrm{e}^\rho. \tag{2.15}$$

2.2. Ricci soliton

A Ricci soliton is a natural generalization of an Einstein metric. A pseudo-Riemannian manifold (M, g) is called a Ricci soliton if it admits a smooth vector field V (potential vector field) on M such that

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{2.16}$$

where $\mathcal{L}_V g$ is the Lie-derivative of g along V given by:

$$(\mathcal{L}_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X), \tag{2.17}$$

S is the Ricci curvatur tensor, λ is a constant called soliton constant and X, Y are arbitrary vector fields on M. A Ricci soliton is said to be expanding, steady or shrinking according to λ being positive, zero or negative, respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold.

A Ricci soliton becomes a trivial Ricci soliton, if the potential vector field V is a Killing vector field. Recently, Pigola et al. [14], introduced the notion of almost Ricci soliton by allowing the soliton constant λ in Ricci soliton to be smooth function.

3. Construction of 3-dimensional almost contact metric structures

In this section we will present an explicit way to construct an almost contact metric structure. It is simple and practical, based on linear algebra and basic definitions of structural elements.

Let (M, g) be a 3-dimensional oriented Riemmannian manifold. For an arbitrary fixed point $p \in M$, by $\{e_i\}_{1\leq i\leq 3}$ we denote an adapted frame in the tangent space T_pM. So, we define a unit vector field ξ by

$$\xi = \sum_{i=1}^{3} \xi^i e_i, \tag{3.1}$$

where $\xi^i \in \mathcal{C}^\infty(M)$ and $\sum_{i=1}^{3}(\xi^i)^2 = 1$. Consequently, the g-dual of ξ is the differential 1-form η given by

$$\eta = \sum_{i=1}^{3} \xi^i \theta^i, \tag{3.2}$$

where $\{\theta^i\}_{1\leq i\leq 3}$ is the dual coframe.
Note: we will use the convention of Einstein. (Whenever an index is repeated, it is a dummy index). i.e.,

$$\xi = \xi^i e_i \qquad and \qquad \eta = \xi^i \theta^i.$$

Now let us start looking for the φ. We put

$$\varphi e_i = \sum_j \varphi_i^j e_j, \tag{3.3}$$

where φ_i^j are functions on M. Using relationships (2.1) i.e.,

$$g(\varphi e_i, e_j) = -g(e_i, \varphi e_j) \quad and \quad g(\varphi e_i, \varphi e_j) = g(e_i, e_j) - \eta(e_i)\eta(e_j),$$

we get the following system

$$\begin{cases} \varphi_i^j = -\varphi_j^i \\ \varphi_i^a \varphi_j^a = \delta_{ij} - \xi^i \xi^j. \end{cases} \tag{3.4}$$

By noting that i and j are fixed, from this system one easily obtains

$$\varphi_i^i = 0, \qquad \varphi_i^a \varphi_i^a = 1 - (\xi^i)^2 \qquad and \qquad \varphi_i^a \varphi_j^a = -\xi^i \xi^j \quad for \quad i \neq j.$$

For every $i, j, k \in \{1,2,3\}$ with $i \neq j$, $i \neq k$ and $j \neq k$, the first two equations above gives the following

$$\begin{cases} (\varphi_i^j)^2 + (\varphi_i^k)^2 = 1 - (\xi^i)^2 \\ (\varphi_j^i)^2 + (\varphi_j^k)^2 = 1 - (\xi^j)^2 \\ (\varphi_k^i)^2 + (\varphi_k^j)^2 = 1 - (\xi^k)^2. \end{cases}$$

Subtracting the second equation from the first taking it into account $\varphi_i^j = -\varphi_j^i$, we find

$$(\varphi_i^k)^2 - (\varphi_j^k)^2 = (\xi^j)^2 - (\xi^i)^2. \tag{3.5}$$

From the third equation in the above system, we have

$$(\varphi_k^j)^2 = (\varphi_j^k)^2 = 1 - (\xi^k)^2 - (\varphi_k^i)^2.$$

So, (3.5) becomes

$$\begin{aligned} 2(\varphi_i^k)^2 &= 1 + (\xi^j)^2 - (\xi^k)^2 - (\xi^i)^2 \\ &= 1 + (\xi^j)^2 - \big(1 - (\xi^j)^2\big) \\ &= 2(\xi^j)^2, \end{aligned}$$

which give

$$\varphi_i^k = \epsilon\xi^j \qquad where \qquad \epsilon = \pm 1. \tag{3.6}$$

Note that φ is completely defined with ξ.

Based on these facts, we give the following Theorem:

Theorem 3.1. *Let (M^3, g) be a 3-dimensional oriented Riemannian manifold. If ξ is a global unit vector field written in the form $\xi = \xi^i e_i$ where $\{e_i\}_{1\leq i\leq 3}$ is a local orthonormal basis on M then there exist an infnite number of almost contact metric structures (φ, ξ, η, g) where*

$$\varphi = \epsilon \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix},$$

with $\epsilon = \pm 1$ and η is the g-dual of ξ.

Proof. For the proof, it is easy to check the conditions (2.1). □

4. From Ricci soliton to trans-Sasakian structures

We assume that (g, V, λ) is a Ricci soliton on M. That is,

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{4.1}$$

We know that the potential vector field V is global on M but is not necessarily unitary, we put

$$\xi = \frac{V}{f} \qquad where \qquad f = \|V\|. \tag{4.2}$$

Following Theorem 3.1, we have a family of almost contact metric structures (φ, ξ, η, g).

Assuming that $(M, \varphi, \xi, \eta, g)$ is a trans-Sasakian manifold and don't forget that the trans-Sasakian structure (φ, ξ, η, g) is normal which implies that $\nabla_\xi \xi = 0$.

So, with the help of (4.1), (2.17) and (2.6) we have

$$(\mathcal{L}_V g)(X, \xi) + 2S(X, \xi) + 2\lambda g(X, \xi) = 0, \tag{4.3}$$

implies

$$\big(2\lambda - 2\xi(\beta) + \xi(f) + 4(\alpha^2 - \beta^2)\big)\eta(X) + X(f) - 2X(\beta) - 2\varphi X(\alpha) = 0,$$

putting $X = \xi$ in (4.1) yields

$$\lambda - 2\xi(\beta) + \xi(f) + 2(\alpha^2 - \beta^2) = 0. \tag{4.4}$$

Again, using (4.1), (2.17) and (2.6), for all vector fields X and Y on M one can get

$$X(f)\eta(Y) + Y(f)\eta(X) - 2\beta f g(\varphi^2 X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{4.5}$$

Since $S(X, Y) = g(QX, Y)$ where Q is the Ricci operator then, the equation (4.5) gives

$$2QX = -2\lambda X + 2\beta f \varphi^2 X - X(f)\xi - \eta(X)\mathrm{grad} f. \tag{4.6}$$

On the other hand, from (2.7) one can get

$$QX = \Big(\frac{r}{2}+\xi(\beta)-(\alpha^2-\beta^2)\Big)X-\Big(\frac{r}{2}+\xi(\beta)-3(\alpha^2-\beta^2)\Big)\eta(X)\xi$$
$$+\eta(X)\big(\varphi\mathrm{grad}\alpha-\mathrm{grad}\beta\big)-\big(X(\beta)+\varphi X(\alpha)\big)\xi. \quad (4.7)$$

Now, using (4.6) and (4.7) we obtain

$$-\Big(2\lambda+2\beta f+r+2\xi(\beta)-2(\alpha^2-\beta^2)\Big)X-\big(X(f)-2X(\beta)-2\varphi X(\alpha)\big)\xi$$
$$+\eta(X)\Big(\big(2\beta f+r+2\xi(\beta)-6(\alpha^2-\beta^2)\big)\xi-2\varphi\mathrm{grad}\alpha+2\mathrm{grad}\beta-\mathrm{grad}f\Big)=0. \quad (4.8)$$

In (4.8) replacing X by ξ it follows that

$$\mathrm{grad}f=-\Big(2\lambda+4(\alpha^2-\beta^2)+\xi(f)-2\xi(\beta)\Big)\xi-2\varphi\mathrm{grad}\alpha+2\mathrm{grad}\beta, \quad (4.9)$$

which gives

$$X(f)=-\Big(2\lambda+4(\alpha^2-\beta^2)+\xi(f)-2\xi(\beta)\Big)\eta(X)+2\varphi X(\alpha)+2X(\beta). \quad (4.10)$$

Putting these values in (4.8), we get

$$0 = \Big(\lambda+\beta f+\frac{r}{2}+\xi(\beta)-(\alpha^2-\beta^2)\Big)X$$
$$- \Big(2\lambda+\beta f+\frac{r}{2}-\xi(\beta)+(\alpha^2-\beta^2)+\xi(f)\Big)\eta(X)\xi. \quad (4.11)$$

Now, taking $X=\xi$ and then replacing X with φX in (4.11), we get the following system

$$\begin{cases} \lambda+\beta f+\frac{r}{2}+\xi(\beta)-(\alpha^2-\beta^2)=0, \\ 2\lambda+\beta f+\frac{r}{2}-\xi(\beta)+(\alpha^2-\beta^2)+\xi(f)=0, \end{cases} \quad (4.12)$$

finally, using (4.4) and (4.12) we get

$$\begin{cases} \beta=\frac{-1}{2f}\big(3\lambda+r+\xi(f)\big), \\ \beta^2-\alpha^2=\frac{1}{2}\big(\lambda+\xi(f)-2\xi(\beta)\big). \end{cases} \quad (4.13)$$

This leads to the following:

Theorem 4.1. *Let $(g, V=f\xi, \lambda)$ be an almost Ricci soliton on a 3-dimensional oriented Riemannian manifold where $f=\|V\|$. If the system*

$$\begin{cases} \beta=\frac{-1}{2f}\big(3\lambda+r+\xi(f)\big), \\ \beta^2-\alpha^2=\frac{1}{2}\big(\lambda+\xi(f)\big), \end{cases} \quad (4.14)$$

hold, then (M,φ,ξ,η,g) is a trans-Sasakian manifold of type (α,β) where φ is defined in theorem 3.1 and $\eta=(\xi)^\flat$.

For the sake of illustration we give the following examples:

Example 4.1. Let $M=\mathbb{S}^2\times\mathbb{R}=\{(x,y,t)\in\mathbb{R}^3\}$ and $\{e_1,e_2,e_3\}$ be the frame of vector fields on M given by

$$e_1=(1+x^2+y^2)\frac{\partial}{\partial x}, \qquad e_2=(1+x^2+y^2)\frac{\partial}{\partial y}, \qquad e_3=\frac{1}{\sqrt{2}}\frac{\partial}{\partial t}.$$

We define a Riemannian metric g by

$$g=\frac{1}{(1+x^2+y^2)^2}(dx^2+dy^2)+2\,dt^2.$$

Let ∇ be the Riemannian connection of g, then we have

$$[e_1,e_2]=2xe_2-2ye_1, \qquad [e_1,e_3]=[e_2,e_3]=0.$$

By using the Koszul formula for the Riemannian metric g, the non zero components of the Levi-Civita connection corresponding to g are given by:

$$\nabla_{e_1}e_1 = 2ye_2, \qquad \nabla_{e_1}e_2 = -2ye_1, \qquad \nabla_{e_2}e_1 = -2xe_2, \qquad \nabla_{e_2}e_2 = 2xe_1.$$

The non-vanishing curvature tensor R components are computed as

$$R(e_1, e_2)e_1 = -4e_2 \qquad R(e_1, e_2)e_2 = 4e_1.$$

The Ricci curvature S components and the scalar curvature r are computed as

$$S(e_1, e_1) = S(e_2, e_2) = 4, \qquad S(e_3, e_3) = 0 \qquad r = 8.$$

For $V = 4te_3$, the only component that is not zero is

$$(\mathcal{L}_V g)(e_3, e_3) = 8.$$

Now, we can easily see that $(g, V, -4)$ is a Ricci soliton. So, taking $\xi = e_3$ and note that $\nabla_\xi\xi = 0$. From (3.1) we get

$$\varphi = \epsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then, (φ, ξ, η, g) is an almost contact metric structure on M with $\eta = dt$. One easily see that the system (4.13) is satisfied with

$$f = 4t, \qquad \alpha = -\sum_{i=1}^{3} g(\nabla_{e_i}\xi, \varphi e_i) = 0 \qquad and \qquad \beta = \sum_{i=1}^{3} g(\nabla_{e_i}\xi, e_i) = 0,$$

which allows us to conclude that (φ, ξ, η, g) is a trans-Sasakian structure of type $(0, 0)$ i.e., cosymplectic structure.

Example 4.2. Let $M = \mathbb{R}^3 = \{(x, y, z) \in \mathbb{R}^3\}$ and $\{e_1, e_2, e_3\}$ be the frame of vector fields on M given by

$$e_1 = \mathrm{e}^{-z}\frac{\partial}{\partial x}, \qquad e_2 = \mathrm{e}^{-z}\frac{\partial}{\partial y}, \qquad e_3 = \mathrm{e}^{-z}\frac{\partial}{\partial z}.$$

We define a Riemannian metric g by

$$g = \mathrm{e}^{2z}(dx^2 + dy^2 + dz^2).$$

By using the Koszul formula for the Riemannian metric g, the non zero components of the Levi-Civita connection corresponding to g are given by:

$$\nabla_{e_1}e_1 = -\mathrm{e}^{-z}e_3, \qquad \nabla_{e_1}e_3 = \mathrm{e}^{-z}e_1, \qquad \nabla_{e_2}e_2 = -\mathrm{e}^{-z}e_3, \qquad \nabla_{e_2}e_3 = \mathrm{e}^{-z}e_2.$$

The Ricci curvature S components and the scalar curvature r are computed as

$$S(e_1, e_1) = S(e_2, e_2) = -\mathrm{e}^{-2z}, \qquad S(e_3, e_3) = 0 \qquad r = -2\mathrm{e}^{-2z}.$$

For $V = \frac{1}{2}\mathrm{e}^{-z}e_3$, the only component that is not zero is

$$(\mathcal{L}_V g)(e_1, e_1) = (\mathcal{L}_V g)(e_2, e_2) = -(\mathcal{L}_V g)(e_3, e_3) = \mathrm{e}^{-2z}.$$

Now, we can easily see that $(g, V, \frac{1}{2}\mathrm{e}^{-2z})$ is a Ricci soliton. So, taking $\xi = e_3$ and note that $\nabla_\xi\xi = 0$. From (3.1) we get

$$\varphi = \epsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then, (φ, ξ, η, g) is an almost contact metric structure on M with $\eta = dt$. One easily see that the system (4.13) is satisfied with

$$f = \frac{1}{2}\mathrm{e}^{-z}, \qquad \alpha = -\sum_{i=1}^{3} g(\nabla_{e_i}\xi, \varphi e_i) = 0 \qquad and \qquad \beta = \sum_{i=1}^{3} g(\nabla_{e_i}\xi, e_i) = \mathrm{e}^{-z},$$

which allows us to conclude that (φ, ξ, η, g) is a trans-Sasakian structure of type $(0, 1)$ i.e., Kenmotsu structure.

Example 4.3. (3D cigar soliton)
Let $M = \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R} = \{(x, r, t) \in \mathbb{R}^3 / x > 0\}$ and let $\{e_1, e_2, e_3\}$ be linearly independent vector fields given by

$$e_1 = (1+x^2)\frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial r}, \qquad e_3 = \frac{1}{x}\frac{\partial}{\partial t}.$$

and $\{\theta^1, \theta^2, \theta^3\}$ be the dual frame of differential 1-forms such that

$$\theta^1 = \frac{1}{1+x^2}dx, \qquad \theta^2 = dr, \qquad \theta^3 = xdt.$$

We define a Riemannian metric g by $g = \sum_{i=1}^{3} \theta^i \otimes \theta^i$, That is the form

$$g = \frac{1}{(1+x^2)^2}dx^2 + dy^2 + x^2dz^2.$$

The potential vector field is given by $V = \text{grad}f = 2x(1+x^2)e_1$ where the potential function is $f = \ln(1+x^2)$. With simple computations we find

$$S = -2(1+x^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad and \quad \mathcal{L}_V g = 4(1+x^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can easily notice that $\mathcal{L}_V g + 2S = 0$ which implies that (M, g, V) is a steady Ricci soliton.

So, taking $\xi = e_1$ and note that $\nabla_\xi \xi = 0$. From (3.1) we get

$$\varphi = \epsilon\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

then, (φ, ξ, η, g) is an almost contact metric structure on M with $\eta = \frac{1}{1+x^2}dx$. One easily can get

$$f = 2x(1+x^2), \qquad \alpha = 0 \qquad and \qquad \beta = \frac{1+x^2}{2x}.$$

Unfortunately, system (4.13) doesn't hold with these data, and the structure (φ, ξ, η, g) isn't trans-Sasakian.

5. From Ricci soliton to corner structures

In the remaining part of the paper, we focus on the case of $\nabla_\xi \xi \neq 0$. More precisely, we will construct a corner structure starting from a Ricci soliton on a 3-dimensional oriented Riemannian manifold (M, g).

Assume that (g, U, λ) is a Ricci soliton on M. Similarly as in the above section, we put

$$\xi = \frac{U}{f} \qquad where \qquad f = \|U\|. \tag{5.1}$$

From Theorem 3.1, we have a family of almost contact metric structures (φ, ξ, η, g).

Suppose that $(M, \varphi, \xi, \eta, g)$ is a corner manifold and using (4.1), (2.17) and (2.11), for all vector fields X and Y on M one can get

$$X(f)\eta(Y) + f\eta(X)\omega(Y) + Y(f)\eta(X) + f\eta(Y)\omega(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0,$$

or equivalently,

$$2QX + 2\lambda X + X(f)\xi + fe^\rho\eta(X)V + fe^\rho\theta(X)\xi + \eta(X)\text{grad}f = 0. \tag{5.2}$$

where $\psi = e^{\rho} V$ and $\omega = e^{\rho}\theta$. On the other hand, we have

$$\begin{aligned} QX &= \sum_{i=1}^{3} g(QX, e_i)e_i = \sum_{i=1}^{3} S(X, e_i)e_i \\ &= \sum_{i,j=1}^{3} g(X, e_j)S(e_j, e_i)e_i \\ &= \eta(X)\big(S(\xi,\xi)\xi + S(\xi, V)V + S(\xi, \varphi V)\varphi V\big) \\ &\quad +\theta(X)\big(S(V,\xi)\xi + S(V, V)V + S(V, \varphi V)\varphi V\big) \\ &\quad -\theta(\varphi X)\big(S(\varphi V,\xi)\xi + S(\varphi V, V)V + S(\varphi V, \varphi V)\varphi V\big). \end{aligned}$$

By using (2.14), we obtain

$$QX = S_{11}\eta(X)\xi + \theta(X)\big(S_{22}V + S_{23}\varphi V\big) - \theta(\varphi X)\big(S_{23}V + S_{33}\varphi V\big), \tag{5.3}$$

where $S_{22} = S(V, V)$, $S_{23} = S(V, \varphi V)$ and $S_{33} = S(\varphi V, \varphi V)$.

Now, from (5.2) and (5.3), one can get

$$\begin{aligned} 0 &= 2\lambda X + \Big(2S_{11}\eta(X) + X(f) + fe^{\rho}\theta(X)\Big)\xi \\ &\quad +\Big(2\theta(X)S_{22} - 2\theta(\varphi X)S_{23} + fe^{\rho}\eta(X)\Big)V \\ &\quad +\Big(2\theta(X)S_{23} - 2\theta(\varphi X)S_{33}\Big)\varphi V + \eta(X)\mathrm{grad} f. \end{aligned} \tag{5.4}$$

Takink $X = \xi$ we obtain

$$\mathrm{grad} f = -\big(2\lambda + 2S_{11} + \xi(f)\big)\xi - fe^{\rho}V, \tag{5.5}$$

and

$$X(f) = -\big(2\lambda + 2S_{11} + \xi(f)\big)\eta(X) - fe^{\rho}\theta(X). \tag{5.6}$$

Replace (5.5) and (5.6) in (5.4), with $X = \eta(X)\xi + \theta(X)V - \theta(\varphi X)\varphi V$, we get

$$\begin{aligned} 0 &= -\big(\lambda + (S_{11} + \xi(f)\big)\eta(X)\xi + \big((\lambda + S_{22})\theta(X) - S_{23}\theta(\varphi X)\big)V \\ &\quad -\big((\lambda + S_{33})\theta(\varphi X) - S_{23}\theta(X)\big)\varphi V = 0. \end{aligned}$$

Hence the system

$$\begin{cases} \lambda = -S_{22} = -S_{33}, \\ \xi(f) = S_{22} - S_{11}, \\ S_{23} = 0. \end{cases}$$

By examining the above system we can discover that the manifold achieves the relationship

$$S = -\lambda g - \xi(f)\eta \otimes \eta.$$

That is, (M, g) is an η-Einstein manifold and this is consistent with Theorems 3.1 and 3.2 in [1].

Therefore, we present the following theorem

Theorem 5.1. *Let $(g, U = f\xi, \lambda)$ be an almost Ricci soliton on a 3-dimensional oriented Riemannian manifold where $f = \|U\|$. If the system*

$$\begin{cases} \lambda = -S(V, V) = -S(\varphi V, \varphi V), \\ \xi(f) = S(V, V) - S(\xi, \xi), \\ S(V, \varphi V) = 0. \end{cases} \tag{5.7}$$

hold, then $(M, \varphi, \xi, \eta, g)$ is a corner η-Einstein manifold where φ is defined in theorem 3.1 and $\eta = (\xi)^{\flat}$.

Example 5.1. Let $M = \mathbb{R}^3 = \{(x, y, z) \in \mathbb{R}^3\}$ and $\{e_1, e_2, e_3\}$ be the frame of vector fields on M given by

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \qquad e_2 = e^{-z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}.$$

We define a Riemannian metric g by

$$g = e^{2z}(dx^2 + dy^2) + dz^2.$$

By using the Koszul formula for the Riemannian metric g, the non zero components of the Levi-Civita connection corresponding to g are given by:

$$\nabla_{e_1} e_1 = -e_3, \qquad \nabla_{e_1} e_3 = e_1, \qquad \nabla_{e_2} e_2 = -e_3, \qquad \nabla_{e_2} e_3 = e_2.$$

The Ricci curvature S is given by

$$S = -2g.$$

So, (M, g) is an Einstein manifold. For $V \in \{e_1, e_2\}$, one can get

$$\mathcal{L}_V g = 0,$$

i.e., $(g, V, 2)$ is a trivial Ricci soliton. So, taking $\xi = e_1$ (resp., $\xi = e_2$), from (3.1) we get

$$\varphi = \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

resp.,

$$\varphi = \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Easily we can show that we have constructed two corner manifolds from a trivial Ricci soliton.

6. Conclusion and outlook

Knowing that the existence of a Ricci soliton (g, V, λ) on an odd-dimensional oriented Riemannian manifold ensures the existence of a global vector field called "potential vector field", and since this vector field is not unitary in general, we have extracted a unitary vector ξ by the formula $\xi = \frac{V}{\|V\|}$.

In three-dimension, based on ξ, which is called the characteristic vector, we gave a method to construct an almost contact metric structure (φ, ξ, η, g). Then, we gave the necessary and sufficient conditions to know the nature of this structure.

It is possible to take V as a characteristic vector field with a corresponding deformation of the metric in such a way that V is a unit vector field.

Finally, it would be interesting to extend theorems 4.1 and 5.1, or a similar result, to higher dimensions.

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