Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 01.11.2024
DOI : 10.15672/hujms.1472753

Research Article

A classification of generalized skew-derivations on multilinear polynomials in prime rings

Basudeb Dhara*[1], Gurninder S. Sandhu[2], Nripendu Bera[3]

[1] *Department of Mathematics, Belda College, Belda, Paschim Medinipur, 721424, India*
[2] *Department of Mathematics, Patel Memorial National College, Rajpura, 140401, India*
[3] *Department of Mathematics, Jadavpur University, Kolkata, 700032, India*

Abstract

In this article, we are intended to examine generalized skew-derivations that act as Jordan homoderivations on multilinear polynomials in prime rings. More specifically, we show that if F is generalized skew-derivation of a prime ring R with associated automorphism α such that the relation

$$F(X^2) = F(X)^2 + F(X)X + XF(X)$$

holds for all $X \in f(R)$, where $f(x_1, \dots, x_n)$ is a noncentral valued multilinear polynomial over extended centroid C, then either $F = 0$ or $F = -id_R$ or $F = -id_R + \alpha$ (where id_R denotes the identity map of R).

Mathematics Subject Classification (2020). 16W25, 16N60, 16R50

Keywords. Derivation, generalized derivation, generalized skew derivation, prime ring

1. Introduction

In this paper we consider that R always be a prime ring with center $Z(R)$, Q_r its right Martindale ring of quotients and $C = Z(Q_r)$, called the *extended centroid* of the prime ring R. We refer the reader to [4] for the related all properties and the definitions of these objects.

An additive mapping $d : R \mapsto R$ on R is said to be a derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. The additive mapping $d : R \mapsto R$ is said to be a Jordan derivation if $d(a^2) = d(a)a + ad(a)$ for all $a \in R$. Thus it is clear that every derivation is a Jordan derivation, but the converse is not true in general. An additive mapping G on R is said to be a generalized derivation if $G(ab) = G(a)b + ad(b)$ for all $a, b \in R$, where d is a derivation of R. It is clear that any derivation on R is a generalized derivation on R. Moreover any map f of R with form $f(x) = a'x + xb'$, where $a', b' \in R$, is a generalized derivation, known as *inner generalized derivation*.

An additive mapping $d : R \longrightarrow R$ is said to be a *skew derivation* of R if

$$d(ab) = d(a)b + \alpha(a)d(b)$$

*Corresponding Author.
Email addresses: basu_dhara@yahoo.com (B. Dhara), gurninder_rs@pbi.ac.in (G. S. Sandhu), nripendub@gmail.com (N. Bera)
Received: 24.04.2024; Accepted: 05.10.2024

for all $a, b \in R$, where α is *associated automorphism* of d. Also an additive mapping G on R is said to be a *generalized skew derivation* of R if there exists a skew derivation d of R and associated automorphism α such that

$$G(ab) = G(a)b + \alpha(a)d(b)$$

for all $a, b \in R$, where d is *associated skew derivation* and α is *associated automorphism* of G. The definition of generalized skew derivations is a unified notion of generalized derivations and skew derivations, which are considered as classical linear mappings of non-associative algebras, and have been investigated by many researchers from various views [7–10, 22, 25].
We all known that automorphisms, derivations and skew derivations of R can be extended both to Q_r. In [9], Chang extends the definition of generalized skew derivations to the right Martindale ring of quotient Q_r of R as follows:
By a generalized skew derivation we mean an additive mapping $G : Q_r \longrightarrow Q_r$ such that $G(ab) = G(a)b + \alpha(a)d(b)$ for all $a, b \in Q_r$, where d is skew derivation of R and also α is automorphism of R. Even more, there exists $G(1) = a' \in Q_r$ such that $G(x) = a'x + d(x)$ for all $x \in R$. In another language, any generalized skew derivation of R can be extended to the right Martindale ring of quotient Q_r.

An additive mapping $F : R \to R$ is called a homomorphism or an anti-homomorphism on R if $F(ab) = F(a)F(b)$ or $F(ab) = F(b)F(a)$ holds for all $a, b \in R$ respectively. The additive mapping F is called a Jordan homomorphism, if $F(a^2) = F(a)^2$ holds for all $a \in R$. A unified concept of a derivation and homomorphism has been introduced by El Sofy Aly [15] as: an additive mappings $\delta : R \to R$ is said to be a homoderivation if it satisfies

$$\delta(xy) = \delta(x)\delta(y) + \delta(x)y + x\delta(y)$$

for all $x, y \in R$.

For example, let $R = \mathbb{Z}(1, \sqrt{5}) = \{a + b\sqrt{5} : \ a, b \in \mathbb{Z}\}$, be a ring. Then a mapping $\delta : R \to R$ such that $a + b\sqrt{5} \mapsto -2b\sqrt{5}$ is a homoderivation.

Let us consider the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$ and a mapping $\delta : R \to R$ such that $\delta \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. It is easy to verify that δ is an example of homoderivation.

Moreover, like Jordan derivation, we can define the notion of Jordan homoderivation. An additive mapping $\delta : R \to R$ satisfying $\delta(x^2) = \delta(x)^2 + \delta(x)x + x\delta(x)$ for all $x \in R$ is called a Jordan homoderivation. Thus every homoderivation is a Jordan homoderivation, but the converse is not true in general.

Let us consider the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ and a mapping $\delta : R \to R$ such that $\delta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. It is easy to verify that $\delta(x^2) = \delta(x)^2 + \delta(x)x + x\delta(x)$ for all $x \in R$, that is, δ is an example of Jordan homoderivation.

Many papers in the literature determine the structure of prime rings R as well as structure of additive mappings which acts as a (anti)homomorphism, homomorphism, or Jordan homomorphism on some appropriate subsets of the prime ring R.

At the beginning of this line of investigation Bell and Kappe [5, Theorem 3] proved that there are no non-zero derivations of a prime ring R which acts as a homomorphism or anti-homomorphism on a non-zero right ideal of the prime ring R. Later, this result was extended to a non-central Lie ideal of a prime ring of characteristic not 2 (see [29]).

In [1–3, 16, 27, 28, 31], generalized derivations have also been discussed when it acts as homomorphisms or anti-homomorphisms or Lie homomorphisms or Jordan homomorphisms in prime rings.

From the above cited results, the reader can notice that generalized derivations can act as Jordan homomorphisms. When this is the case, the complete description of the additive mappings can be obtained (see [16]). In the light of this discussion, one can think of the question that what could be the form of a generalized derivation that acts as Jordan homoderivation? A complete answer to this question is given by Bera and Dhara [6] as the following:

Theorem A. *Let R be a prime ring with characteristic is not 2 and $f(x_1,\ldots,x_n)$ be a noncentral multilinear polynomial over $C(=Z(U))$ where U be the Utumi ring of quotients of R. If F,G and H are three generalized derivations on R satisfying*

$$F(x^2)=G(x)^2+H(x)x+xH(x)$$

for all $x\in f(R)$, then one of the following holds:

1. *there exist a derivation d on R and $\lambda_1,\lambda_2,\lambda_3\in C$ such that $F(x)=\lambda_1x+d(x)$, $G(x)=\lambda_2x$ and $H(x)=\lambda_3x+d(x)$ for all $x\in R$, with $\lambda_1=\lambda_2^2+2\lambda_3$;*
2. *there exist a derivation d on R, $a_1\in U$ and $\lambda_1,\lambda_2,\lambda_3\in C$ such that $F(x)=\lambda_1x+d(x)$, $G(x)=\lambda_2x$ and $H(x)=\lambda_3x+[a_1,x]+d(x)$ for all $x\in R$, with $f(R)^2\in C$ and $\lambda_1=\lambda_2^2+2\lambda_3$;*
3. *there exist $a_1,a_2\in U$ and $\lambda_1,\lambda_2,\lambda_3\in C$ such that $F(x)=\lambda_1x+[a_1,x]$, $G(x)=\lambda_2x$ and $H(x)=\lambda_3x+[a_2,x]$ for all $x\in R$, with $f(R)^2\in C$ and $\lambda_1=\lambda_2^2+2\lambda_3$.*

As a reduction of above theorem, we have

Theorem B. *Let R be a prime ring with characteristic is not 2 and $f(x_1,\ldots,x_n)$ be a noncentral multilinear polynomial over $C(=Z(U))$ where U be the Utumi ring of quotients of R. If F is a generalized derivation on R satisfying*

$$F(u^2)=F(u)^2+F(u)u+uF(u)$$

for all $u\in f(R)$, then either $F=0$ or $F(x)=-x$ for all $x\in R$.

Therefore, it is natural to ask that whether the above theorem is true in the settings of generalized skew-derivations or not. In this note, we give an affirmative answer to this question and obtained the common description of a generalized skew derivation and a Jordan homoderivation. More precisely, we shall prove the following theorem:

Theorem 1.1. *Let R be a noncommutative prime ring of characteristic is not 2, Q_r its right Martindale ring of quotients, $C=Z(Q_r)$ the extended centroid of the prime ring R and $f(x_1,\ldots,x_n)$ a noncentral multilinear polynomial over C. Assume that F is a generalized skew-derivation of R and α is the associated automorphism of F. If*

$$F(u^2)=F(u)^2+F(u)u+uF(u)$$

for all $u\in f(R)$, then one of the following holds:

1. *$F=0$;*
2. *$F(x)=-x$ for all $x\in R$;*
3. *$F(x)=-x+\alpha(x)$ for all $x\in R$.*

It is also important to observe that the assumption of primeness in the hypothesis of our theorem is not redundant, the following example illustrates this:

Example 1.2. Let $R=\left\{\begin{pmatrix}0&a_1&b_1\\0&0&c_1\\0&0&0\end{pmatrix}:a_1,b_1,c_1\in\mathbb{Z}\right\}$, which is a ring over the set of integers. It can be easily seen that R is not a prime ring. Define

$$F\begin{pmatrix}0&a_1&b_1\\0&0&c_1\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&a_1\\0&0&0\\0&0&0\end{pmatrix},\ d\begin{pmatrix}0&a_1&b_1\\0&0&c_1\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&c_1\\0&0&0\\0&0&0\end{pmatrix},$$

and

$$\alpha\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a_1 & b_1 \\ 0 & 0 & -c_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, F is a generalized skew-derivation of R with associated skew derivation d and automorphism α and satisfies $F(X^2) = F(X)^2 + F(X)X + XF(X)$ on R. But F takes none of the forms given in the conclusion of Theorem 1.1.

In all what follows, for any multilinear polynomial over the extended centroid C of the prime ring R we will adopt the following notation:

$$f(x_1,\dots,x_n) = x_1x_2\dots x_n + \sum_{\sigma\in S_n,\sigma\neq id} \alpha_\sigma x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$. We always consider that characteristic of the prime ring R is not 2 and $f(x_1,\dots,x_n)$ is a non-central valued in the prime ring R.

We also recall that, if we consider R is a prime ring then C must be a field. Even more, R is a subring of the right Martindale ring of quotients Q_r and Q_r is a prime ring with identity. It should be remarked that Q_r is centrally closed prime C-algebra.

Also, it is known to all that I, R and Q_r satisfy the same generalized polynomial identities (GPI) with coefficients in Q_r, the right Martindale ring of quotients (see [13]). Moreover, I, R, and Q_r satisfy the same generalized polynomial identities (GPI) with automorphisms (see [12, Theorem 1]).

2. The mapping is inner generalized skew derivation

In this section we consider that F is an inner generalized skew derivation associated with an automorphism. We denote $f(R) = \{f(x_1,\dots,x_n)|x_1,\dots,x_n \in R\}$.

Proposition 2.1. *Let R be a noncommutative prime ring of characteristic not 2, Q_r be its right Martindale ring of quotient and C be its extended centroid. Suppose that $f(x_1,\dots,x_n)$ be a noncentral multilinear polynomial over C. If $F(x) = ax + pxp^{-1}b$ for all $x \in R$ and for some $a, b, p \in Q_r$ such that*

$$F(x^2) = F(x)^2 + F(x)x + xF(x)$$

for all $x \in f(R)$, then one of the following holds:

(1) *$F = 0$;*
(2) *$F(x) = -x$ for all $x \in R$;*
(3) *$F(x) = -x + pxp^{-1}$ for all $x \in R$.*

Since $F(x) = ax + pxp^{-1}b$, our hypothesis

$$F(x^2) = F(x)^2 + F(x)x + xF(x)$$

for all $x \in f(R)$ yields

$$\begin{aligned} px^2(p^{-1}b) &= axax + axpx(p^{-1}b) + px(p^{-1}ba)x \\ &+px(p^{-1}bp)x(p^{-1}b) + px(p^{-1}b)x + xax + xpx(p^{-1}b) \end{aligned} \tag{2.1}$$

for all $x \in f(R)$.
This can be written as

$$px^2a_1 = axax + axpxa_1 + pxa_2x + pxa_3xa_1 + pxa_1x + xax + xpxa_1 \tag{2.2}$$

for all $x \in f(R)$, where $a_1 = p^{-1}b, a_2 = p^{-1}ba, a_3 = p^{-1}bp$. Thus, we consider the generalized polynomial

$$\begin{aligned}&\Psi(x_1,\ldots,x_n)=\\&pf(x_1,\ldots,x_n)^2a_1-af(x_1,\ldots,x_n)af(x_1,\ldots,x_n)\\&-af(x_1,\ldots,x_n)pf(x_1,\ldots,x_n)a_1-pf(x_1,\ldots,x_n)a_2f(x_1,\ldots,x_n)\\&-pf(x_1,\ldots,x_n)a_3f(x_1,\ldots,x_n)a_1-pf(x_1,\ldots,x_n)a_1f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)af(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)pf(x_1,\ldots,x_n)a_1.\end{aligned}$$

To prove the above proposition we need the following Lemmas.

Lemma 2.2. *If*

$$\Psi(x_1,\ldots,x_n)=0$$

is a trivial generalized polynomial identity for R, then either $p\in C$ or $a_1\in C$ or $a+a_3,p^{-1}(a+1)\in C$.

Proof. Since R and Q_r satisfy the same generalized polynomial identities (see [13]), Q_r satisfies $\Psi(x_1,\ldots,x_n)=0$. Let $T=Q_r*_C C\{x_1,\ldots,x_n\}$ is the free product of Q_r and $C\{x_1,\ldots,x_n\}$, the free C-algebra of noncommutating indeterminates $x_1,\ldots,x_n$. Then $\Psi(x_1,\ldots,x_n)$ is a zero element in the free product T.

Assume that $f(x_1,\ldots,x_n)=X$. Let $a_1\notin C$. Then

$$pX^2a_1-aXaX-aXpXa_1-pXa_2X-pXa_3Xa_1-pXa_1X-XaX-XpXa_1=0\in T.$$

Since $a_1\notin C$, we have from above

$$pX^2a_1-aXpXa_1-pXa_3Xa_1-XpXa_1=0\in T,$$

that is

$$(pX-aXp-pXa_3-Xp)Xa_1=0\in T.$$

If $p\in C$, we are done. So assume that $p\notin C$. Then $\{p,a,1\}$ is linearly C-dependent. There exist $\lambda_1,\lambda_2,\lambda_3\in C$ such that $\lambda_1p+\lambda_2a+\lambda_3=0$. Since $p\notin C$, $\lambda_2\neq 0$. Hence $a=\beta p+\gamma$ for some $\beta,\gamma\in C$. Thus from above

$$(pX-\beta pXp-\gamma Xp-pXa_3-Xp)Xa_1=0\in T.$$

Since $p\notin C$, we have from above that

$$(pX-\beta pXp-pXa_3)Xa_1=0\in T \tag{2.3}$$

and

$$(-\gamma Xp-Xp)Xa_1=0\in T. \tag{2.4}$$

The equation (2.3) implies that $pX(1-\beta p-a_3)Xa_1=0$ implying $1-\beta p-a_3=0$, i.e., $1+\gamma-a-a_3=0$, i.e., $a+a_3\in C$.

The equation (2.4) implies that $-Xp(\gamma+1)Xa_1=0$ implying $\gamma+1=0$, i.e., $a-\beta p+1=0$, i.e., $p^{-1}(a+1)=\beta\in C$. □

Lemma 2.3 ([17], Lemma1)**.** *Suppose that C be an infinite field and $m\geq 2$. If $A_1,\ldots,A_k$ are not scalar matrices in $M_m(C)$ then there exists some invertible matrix $P_1\in M_m(C)$ such that any matrices $P_1A_1P_1^{-1},\ldots,P_1A_kP_1^{-1}$ have all non-zero entries.*

Lemma 2.4. *Let C be an infinite field and $R=M_m(C)$ be the ring of all $m\times m$ matrices over C, $m\geq 2$. If R satisfies*

$$\Psi(x_1,\ldots,x_n)=0,$$

then either $p\in C\cdot I_m$ or $a_1\in C\cdot I_m$ or $a+a_3,p^{-1}(a+1)\in C\cdot I_m$.

Proof. We assume first that $a_1 \notin C \cdot I_m$, $p \notin C \cdot I_m$ and $a + a_3 \notin C \cdot I_m$. Then by Lemma 2.3, there exists an invertible matrix B such that Ba_1B^{-1}, BpB^{-1} and $B(a+a_3)B^{-1}$ have all non-zero entries. Let $\phi(x) = BxB^{-1}$ for all $x \in R$. Assume that $\phi(a) = \sum a_{ij}e_{ij}$, $\phi(a_1) = \sum a'_{ij}e_{ij}$, $\phi(p) = \sum p_{ij}e_{ij}$ and $\phi(a_3) = \sum a''_{ij}e_{ij}$. Then for any $i \neq j$, all the entries $a'_{ij}, p_{ij}, (a + a'')_{ij}$ are non zeros.

Since ϕ is an inner automorphism, evidently

$$\begin{aligned} \phi(p)x^2\phi(a_1) = \phi(a)x\phi(a)x + \phi(a)x\phi(p)x\phi(a_1) + \phi(p)x\phi(a_2)x \\ +\phi(p)x\phi(a_3)x\phi(a_1) + \phi(p)x\phi(a_1)x + x\phi(a)x + x\phi(p)x\phi(a_1) \end{aligned} \tag{2.5}$$

for all $x \in f(R)$. Let e_{hk} be the matrix unit, that is, the matrix whose only (h,k)th-entry is 1 and all other entries are zero. Since $f(x_1,\ldots,x_n)$ is not central valued, by [23] (see also [24]), there exist sequence of matrices $r_1,\ldots,r_n \in M_m(C)$ such that $f(r_1,\ldots,r_n) = \gamma e_{hk}$, where $0 \neq \gamma \in C$, $h \neq k$. In (2.5), replacing the particular value of $f(x_1,\ldots,x_n)$ we have

$$\begin{aligned} 0 = \phi(a)e_{hk}\phi(a)e_{hk} + \phi(a)e_{hk}\phi(p)e_{hk}\phi(a_1) + \phi(p)e_{hk}\phi(a_2)e_{hk} \\ +\phi(p)e_{hk}\phi(a_3)e_{hk}\phi(a_1) + \phi(p)e_{hk}\phi(a_1)e_{hk} + e_{ij}\phi(a)e_{hk} + e_{hk}\phi(p)e_{hk}\phi(a_1). \end{aligned} \tag{2.6}$$

Left and right multiplying by e_{hk}, it yields

$$e_{hk}\phi(a)e_{hk}\phi(p)e_{hk}\phi(a_1)e_{hk} + e_{hk}\phi(p)e_{hk}\phi(a_3)e_{hk}\phi(a_1)e_{hk} = 0,$$

that is

$$e_{hk}(\sum a_{ij}e_{ij})e_{hk}(\sum p_{ij}e_{ij})e_{hk}(\sum a'_{ij}e_{ij})e_{hk}$$

$$+e_{hk}(\sum p_{ij}e_{ij})e_{hk}(\sum a''_{ij}e_{ij})e_{hk}(\sum a'_{ij}e_{ij})e_{hk} = 0.$$

This implies

$$a_{kh}p_{kh}a'_{kh} + p_{kh}a''_{kh}a'_{kh} = 0$$

that is

$$p_{kh}a'_{kh}(a + a'')_{kh} = 0.$$

This is a contradiction, since $a'_{ij}, p_{ij}, (a + a'')_{ij}$ are all non zeros for any $i \neq j$.

Thus we can conclude that either $a_1 \in C \cdot I_m$ or $p \in C \cdot I_m$ or $a + a_3 \in C \cdot I_m$.

Again, left multiplying by $e_{hk}\phi(p)^{-1}$ and right by e_{hk}, (2.6) yields

$$e_{hk}\phi(p^{-1}a)e_{hk}\phi(p)e_{hk}\phi(a_1)e_{hk} + e_{hk}\phi(p)^{-1}e_{hk}\phi(p)e_{hk}\phi(a_1)e_{hk} = 0.$$

Assuming $\phi(p^{-1}a) = \sum p''_{ij}e_{ij}$ and $\phi(p)^{-1} = \sum p'_{ij}e_{ij}$, we have

$$p''_{kh}p_{kh}a'_{kh} + p'_{kh}p_{kh}a'_{kh} = 0$$

that is

$$p_{kh}a'_{kh}(p'' + p')_{kh} = 0.$$

Thus as above by same argument, either $p \in C \cdot I_m$ or $a_1 \in C \cdot I_m$ or $p^{-1}a + p^{-1} \in C \cdot I_m$.

Hence if $p \notin C \cdot I_m$ and $a_1 \notin C \cdot I_m$, then $a + a_3 \in C \cdot I_m$ and $p^{-1}a + p^{-1} \in C \cdot I_m$. Therefore conclusion follows.

□

Lemma 2.5. *Let C be a finite field and let $R = M_m(C)$ be the ring of all $m \times m$ matrices over C, $m \geq 2$. If char $(R) \neq 2$ and R satisfies*

$$\Psi(x_1,\ldots,x_n) = 0,$$

then either $p \in C \cdot I_m$ or $a_1 \in C \cdot I_m$ or $a + a_3, p^{-1}(a+1) \in C \cdot I_m$.

Proof. Let K be an extension field of C such that K is infinite. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Note that $f(x_1, \ldots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Now the generalized polynomial (GP) $\Psi(x_1, \ldots, x_n)$ is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $x_1, \ldots, x_n$. Now linearizing the identity with respect to first argument, that is, replacing x_1 with $x_1 + y_1$ we have

$$\Psi(x_1 + y_1, \ldots, x_n) = 0.$$

This can be written as

$$\Psi(x_1, \ldots, x_n) + \Psi(y_1, \ldots, x_n) + \Theta_1(x_1, \ldots, x_n, y_1) = 0,$$

where

$$\begin{aligned}
&\Theta_1(x_1, \ldots, x_n, y_1) = \\
&pf(x_1, \ldots, x_n)f(y_1, \ldots, x_n)a_1 + pf(y_1, \ldots, x_n)f(x_1, \ldots, x_n)a_1 \\
&- af(x_1, \ldots, x_n)af(y_1, \ldots, x_n) - af(y_1, \ldots, x_n)af(x_1, \ldots, x_n) \\
&- af(x_1, \ldots, x_n)pf(y_1, \ldots, x_n)a_1 - af(y_1, \ldots, x_n)pf(x_1, \ldots, x_n)a_1 \\
&- pf(x_1, \ldots, x_n)a_2f(y_1, \ldots, x_n) - pf(y_1, \ldots, x_n)a_2f(x_1, \ldots, x_n) \\
&- pf(x_1, \ldots, x_n)a_3f(y_1, \ldots, x_n)a_1 - pf(y_1, \ldots, x_n)a_3f(x_1, \ldots, x_n)a_1 \\
&- pf(x_1, \ldots, x_n)a_1f(y_1, \ldots, x_n) - pf(y_1, \ldots, x_n)a_1f(x_1, \ldots, x_n) \\
&- f(x_1, \ldots, x_n)af(y_1, \ldots, x_n) - f(y_1, \ldots, x_n)af(x_1, \ldots, x_n) \\
&- f(x_1, \ldots, x_n)pf(y_1, \ldots, x_n)a_1 - f(y_1, \ldots, x_n)pf(x_1, \ldots, x_n)a_1.
\end{aligned}$$

Note that $\Theta_1(x_1, \ldots, x_n, y_1)$ is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $x_2, \ldots, x_n$ such that $\Theta_1(x_1, \ldots, x_n, x_1) = 2\Psi(x_1, \ldots, x_n)$. Since $\Psi(x_1, \ldots, x_n) = 0 = \Psi(y_1, \ldots, x_n)$, we have from above that

$$\Theta_1(x_1, \ldots, x_n, y_1) = 0.$$

Again linearizing the above identity with respect to x_2 we shall get

$$\Theta_2(x_1, \ldots, x_n, y_1, y_2) = 0$$

such that $\Theta_2(x_1, \ldots, x_n, x_1, x_2) = 2^2\Psi(x_1, \ldots, x_n)$. Continuing this process of linearization, finally we shall get

$$\Theta_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$$

such that $\Theta_n(x_1, \ldots, x_n, x_1, \ldots, x_2) = 2^n\Psi(x_1, \ldots, x_n)$.

Note that $\Theta_n(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a multilinear generalized polynomial in $2n$ indeterminates. Clearly $\Theta_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$ is a generalized polynomial identity (GPI) for R and $\overline{R}$ too. Since $char(C) \neq 2$, we have $\Psi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{R}$ and hence conclusion follows by Lemma 2.4. □

Following corollary is straightforward.

Corollary 2.6. *Let C be a field and let $R = M_m(C)$ be the ring of all $m \times m$ matrices over C, $m \geq 2$. If char $(R) \neq 2$ and $a, p, a_1, a_2, a_3 \in R$ such that*

$$px^2a_1 = axax + axpxa_1 + pxa_2x + pxa_3xa_1 + pxa_1x + xax + xpxa_1$$

for all $x \in R$, then either $p \in C \cdot I_m$ or $a_1 \in C \cdot I_m$ or $a + a_3, p^{-1}(a + 1) \in C \cdot I_m$.

Lemma 2.7. *Let R be a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over C, such that $dim_C V = \infty$. If char $(R) \neq 2$ and R satisfies*

$$\Psi(x_1, \ldots, x_n) = 0,$$

then either $p \in C$ or $a_1 \in C$ or $a + a_3, p^{-1}(a + 1) \in C$.

Proof. Since $\dim_C V = \infty$, by [30, Lemma 2], the set $f(R)$ is dense on R. Thus by hypothesis, R satisfies

$$px^2a_1 = axax + axpxa_1 + pxa_2x + pxa_3xa_1 + pxa_1x + xax + xpxa_1.$$

It is well known that for any $v \in R$, $[v, soc(R)] = (0)$ implies $v \in C$. In this case also we want to prove that either $p \in C$ or $a_1 \in C$ or $a + a_3, p^{-1}(a + 1) \in C$. On contrary, we assume that $p \notin C$ and $a_1 \notin C$ and either $a + a_3 \notin C$ or $p^{-1}(a+1) \notin C$. Then there exist $v_1, v_2, v_3, v_4 \in soc(R)$ such that

(1) $[p, v_1] \neq 0$;
(2) $[a_1, v_2] \neq 0$
(3) either $[a + a_3, v_3] \neq 0$ or $[p^{-1}(a + 1), v_4] \neq 0$.

By Litoff's theorem [21, p. 280], there exists idempotent $e \in soc(R)$ such that

(1) $v_1, v_2, v_3, v_4 \in eRe$;
(2) $pv_i, v_ip, a_1v_i, v_ia_1, a_3v_i, v_ia_3, av_i, v_ia, p^{-1}v_i, v_ip^{-1} \in eRe$ for all $i = 1, \ldots, 4$.

Moreover, $eRe \cong M_{k'}(C)$, the ring of all $k' \times k'$ matrices over C.

Since R satisfies

$$px^2a_1 = axax + axpxa_1 + pxa_2x + pxa_3xa_1 + pxa_1x + xax + xpxa_1, \tag{2.7}$$

eRe satisfies

$$(epe)x^2(ea_1e) = (eae)x(eae)x + (eae)x(epe)x(ea_1e) + (epe)x(ea_2e)x$$
$$+(epe)x(ea_3e)x(ea_1e) + (epe)x(ea_1e)x + x(eae)x + x(epe)x(ea_1e).$$

Then by Corollary 2.6, either $epe \in Ce$ or $ea_1e \in Ce$ or $e(a + a_3)e, ep^{-1}(a + 1)e \in Ce$. This leads to a contradiction with the choices of v_1, v_2, v_3, v_4 in $soc(R)$. □

Proof of Proposition 2.1.

Since R and Q_r satisfy the same generalized polynomial identities (see [13]), Q_r satisfies $\Psi(x_1, \ldots, x_n) = 0$.

If $\Psi(x_1, \ldots, x_n) = 0$ is trivial GPI for Q_r, then by Lemma 2.2 either $p \in C$ or $p^{-1}b \in C$ or $a + a_3, p^{-1}(a + 1) \in C$.

If $\Psi(x_1, \ldots, x_n) = 0$ is a non-trivial GPI for Q_r, by Martindale's theorem [26], Q_r is a primitive ring with a nonzero socle and with C as its associated division ring. By Jacobson's theorem [19, p.75], Q_r is isomorphic to a dense ring of linear transformations of any vector space V over the field C. At first we assume that V is a finite dimensional vector space over a field C, i.e, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is noncentral valued of R, R must be noncommutative and so $m \geq 2$. In this case by applying Lemma 2.4 and Lemma 2.5, we get either $p \in C$ or $p^{-1}b \in C$ or $a + p^{-1}bp, p^{-1}(a + 1) \in C$.

On the other hand, if V is an infinite dimensional vector space over the field C, then by Lemma 2.7 we conclude either $p \in C$ or $p^{-1}b \in C$ or $a + p^{-1}bp, p^{-1}(a + 1) \in C$.

Thus we divide the rest part of the proof in the following two cases:

Case-i. When $p \in C$ or $p^{-1}b \in C$.

If p or $p^{-1}b$ are central, then F becomes a generalized derivation. Then by Theorem B, we have our conclusions.

Case-ii. When $a + p^{-1}bp, p^{-1}(a + 1) \in C$.

Let $p^{-1}(a + 1) = \gamma \in C$. Thus $a = \gamma p - 1$.

Next let $a + p^{-1}bp = \lambda \in C$. This implies $\gamma p - 1 + p^{-1}bp \in C$, that is, $\gamma p + p^{-1}bp \in C$. Assume that $\gamma p + p^{-1}bp = \lambda' \in C$. This implies $p^{-1}b = \lambda' p^{-1} - \gamma$. Thus $F(x) = ax + pxp^{-1}b = (\gamma p - 1)x + px(\lambda' p^{-1} - \gamma) = -x + \lambda' pxp^{-1}$ for all $x \in R$.

Then by hypothesis, we have $\lambda'(\lambda' - 1)pf(R)^2p^{-1} = 0$ which implies $\lambda'(\lambda' - 1) = 0$. This implies $\lambda' = 0$ or $\lambda' = 1$. Therefore, either $F(x) = -x$ for all $x \in R$ or $F(x) = -x + pxp^{-1}$ for all $x \in R$.

We now consider that F is an inner generalized skew derivation having an associated automorphism α. More precisely:

Proposition 2.8. *Let R be a non-commutative prime ring of characteristic is not 2, Q_r be its right Martindale quotient ring and C be its extended centroid. Suppose that $f(x_1, \dots, x_n)$ be a non-central multilinear polynomial over C, $F(x) = ax + \alpha(x)c$ for some $a, c \in Q_r$ and α is an automorphism of R, such that*

$$F(x^2) = F(x)^2 + F(x)x + xF(x)$$

for all $x \in f(R)$. Then one of the following holds:

(1) $F = 0$;
(2) $F(x) = -x$ *for all* $x \in R$;
(3) $F(x) = -x + \alpha(x)$ *for all* $x \in R$.

Proof. Firstly we recall that, in case α is an inner automorphism on the prime ring R, then in light of Proposition 2.1, we get our conclusions.
Therefore in what follows we may assume that α is not inner.
In view of [11] we know that R and Q_r satisfy the same generalized polynomial identities (GPI) with automorphisms. Therefore

$$\begin{aligned} \Phi(x_1, \dots, x_n) = af(x_1, \dots, x_n)^2 + \alpha(f(x_1, \dots, x_n)^2)c - (af(x_1, \dots, x_n) \\ +\alpha(f(x_1, \dots, x_n))c)^2 - (af(x_1, \dots, x_n) + \alpha(f(x_1, \dots, x_n))c)f(x_1, \dots, x_n) \\ -f(x_1, \dots, x_n)(af(x_1, \dots, x_n) + \alpha(f(x_1, \dots, x_n))c) = 0 \end{aligned} \quad (2.8)$$

is also satisfied by the right Martindale quotient ring Q_r. Moreover, Q_r is a centrally closed prime C-algebra. Also if $c = 0$, then F is a generalized derivation of R again we are done by Theorem B.
Thus assume that $c \neq 0$. In this case, by [12, Main Theorem] we assume that $\Phi(x_1, \dots, x_n)$ is a non-trivial generalized identity for R and for Q_r. By [20, Theorem 1], we have RC has non-zero socle and Q_r is primitive. Since α is an outer automorphism and any $(x_i)^\alpha$-word degree in $\Phi(x_1, \dots, x_n)$ is equal to 2 and $char(R) = 0$ or $char(R) = p > 2$, then by [12, Theorem 3], Q_r satisfies the generalized polynomial identity (GPI)

$$\begin{aligned} &af(x_1, \dots, x_n)^2 + f^\alpha(y_1, \dots, y_n)^2c - (af(x_1, \dots, x_n) + f^\alpha(y_1, \dots, y_n)c)^2 \\ &- (af(x_1, \dots, x_n) + f^\alpha(y_1, \dots, y_n)c)f(x_1, \dots, x_n) \\ &- f(x_1, \dots, x_n)(af(x_1, \dots, x_n) + f^\alpha(y_1, \dots, y_n)c) = 0 \end{aligned} \quad (2.9)$$

where we denote by $f^\alpha(x_1, \dots, x_n)$ the polynomial obtained from $f(x_1, \dots, x_n)$ by replacing each coefficient γ_σ with $\alpha(\gamma_\sigma)$. Also notice that $f^\alpha(x_1, \dots, x_n)$ is not central valued on R. By (2.9), Q_r satisfies both

$$-\Big(af(x_1, \dots, x_n)\Big)^2 - f(x_1, \dots, x_n)af(x_1, \dots, x_n) = 0 \quad (2.10)$$

and

$$f^\alpha(y_1, \dots, y_n)^2c - \Big(f^\alpha(y_1, \dots, y_n)c\Big)^2 = 0. \quad (2.11)$$

By (2.10), we have $(a+1)f(x_1,\dots,x_n)af(x_1,\dots,x_n)=0$. This implies $a=0$ or $a=-1$. On the other hand the relation (2.11) reduces to

$$f^{\alpha}(y_1,\dots,y_n)\Big(f^{\alpha}(y_1,\dots,y_n)c-cf^{\alpha}(y_1,\dots,y_n)c\Big)=0. \tag{2.12}$$

By [18, Lemma 2.4], this relation implies $c\in C$. Since $c\neq 0$, we have from (2.11) above that $f^{\alpha}(y_1,\dots,y_n)\Big(f^{\alpha}(y_1,\dots,y_n)-f^{\alpha}(y_1,\dots,y_n)c\Big)=0$, that is $f^{\alpha}(y_1,\dots,y_n)^2(1-c)=0$. This implies $c=1$. If $a=-1$ and $c=1$, we have $F(x)=-x+\alpha(x)$ for all $x\in R$, as desired. Thus we are to consider the case when $a=0$ and $c=1$. In this case by (2.9), Q_r satisfies the generalized polynomial identity

$$f^{\alpha}(y_1,\dots,y_n)f(x_1,\dots,x_n)+f(x_1,\dots,x_n)f^{\alpha}(y_1,\dots,y_n)=0. \tag{2.13}$$

Assuming $p=f^{\alpha}(y_1,\dots,y_n)$, we have $pf(X)+f(X)p=0$ for all $X=(x_1,\dots,x_n)\in Q_r^n$. This implies $p\in C$, that is, $f(x_1,\dots,x_n)$ is central valued, a contradiction. □

3. The proof of Theorem 1.1

In light of the results contained in the previous Section, Theorem 1.1 is proved if one of the following holds:

- $d=0$, that is, F is centralizer on R;
- α is an identity mapping on R, that is, F is a generalized derivation on R;
- d is inner skew derivation of R, that is, F is an inner generalized skew derivation on R.

Therefore in all that follows, we may assume that

- d is nonzero;
- α is not an identity mapping on R;
- d is not an inner skew derivation on R.

Let us also recall the following:

Fact 3.1. Let R be a prime ring, α be an X-outer automorphism of R and D be an X-outer skew derivation of R. If $\Phi(x_i,D(x_i),\alpha(x_i))$ is a generalized polynomial identity (GPI) for R, then R also satisfies the generalized polynomial identity $\Phi(x_i,y_i,z_i)$, where x_i, y_i and z_i are distinct indeterminates ([14, Theorem 1]).

The Proof of Theorem 1.1:

Here we can write $F(x)=ax+d(x)$ for all $x\in R$, where $a\in Q_r$ and d is a skew derivation of R (see [9]). By [14, Theorem 2] we know that R and Q_r satisfy the same generalized polynomial identities with a single skew derivation. Thus Q_r satisfies

$$\begin{aligned}\Psi(x_1,\dots,x_n,d(x_1),\dots,d(x_n))=&\\ af(x_1,\dots,x_n)^2+d(f(x_1,\dots,x_n)^2)&\\ -\Big(af(x_1,\dots,x_n)+d(f(x_1,\dots,x_n)\Big)^2&\\ -(af(x_1,\dots,x_n)+d(f(x_1,\dots,x_n))f(x_1,\dots,x_n)&\\ -f(x_1,\dots,x_n)(af(x_1,\dots,x_n)+d(f(x_1,\dots,x_n))=0,&\end{aligned} \tag{3.1}$$

that is

$$\begin{aligned}af(x_1,\ldots,x_n)^2+d(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n)\\+\alpha(f(x_1,\ldots,x_n))d(f(x_1,\ldots,x_n))-\Big(af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n)\Big)^2\\-(af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n)\\-f(x_1,\ldots,x_n)(af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n))=0.\end{aligned}\tag{3.2}$$

The action of any skew derivation d on a monomial of $f(x_1,\ldots,x_n)$ can be described as follows:

$$\begin{aligned}d\Big(\gamma_\sigma\cdot x_{\sigma(1)}\cdots x_{\sigma(n)}\Big)=d(\gamma_\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)}\\+\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}.\end{aligned}$$

So from above we have

$$\begin{aligned}d(f(x_1,\ldots,x_n))=f^d(x_1,\ldots,x_n)+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\\\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}.\end{aligned}$$

Using the above value of $d(f(x_1,\ldots,x_n))$ in (3.2), we get

$$\begin{aligned}af(x_1,\ldots,x_n)^2+f^d(x_1,\ldots,x_n)f(x_1,\ldots,x_n)\\+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}f(x_1,\ldots,x_n)\\+\alpha(f(x_1,\ldots,x_n))f^d(x_1,\ldots,x_n)+\alpha(f(x_1,\ldots,x_n))\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)}).\\d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}-\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)^2-af(x_1,\ldots,x_n)^2\\-f^d(x_1,\ldots,x_n)f(x_1,\ldots,x_n)-\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)}).\\x_{\sigma(j+2)}\cdots x_{\sigma(n)}f(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)af(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)f^d(x_1,\ldots,x_n)\\-f(x_1,\ldots,x_n)\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}=0.\end{aligned}\tag{3.3}$$

By our assumption of this section, d is not inner. As d is outer, using [14] Q_r satisfies the following:

$$
\begin{aligned}
&af(x_1,\dots,x_n)^2+f^d(x_1,\dots,x_n)f(x_1,\dots,x_n)\\
&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}f(x_1,\dots,x_n)\\
&+\alpha(f(x_1,\dots,x_n))f^d(x_1,\dots,x_n)\\
&+\alpha(f(x_1,\dots,x_n))\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\\
&-\Big(af(x_1,\dots,x_n)+f^d(x_1,\dots,x_n)+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)}).\\
&y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)^2-af(x_1,\dots,x_n)^2-f^d(x_1,\dots,x_n)f(x_1,\dots,x_n)\\
&-\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}f(x_1,\dots,x_n)\\
&-f(x_1,\dots,x_n)af(x_1,\dots,x_n)-f(x_1,\dots,x_n)f^d(x_1,\dots,x_n)\\
&-f(x_1,\dots,x_n)\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}=0.
\end{aligned}
\tag{3.4}
$$

After splitting above expression, left hand side of the equation can be written as sum of the monomials having no y_i and sum of the monomials having y_i, that is, $\Psi_1(x_1,\dots,x_n)+\Psi_2(x_1,\dots,x_n,y_1,\dots,y_n)=0$. Assuming $y_1=\cdots=y_n=0$, we can write that

$$\Psi_1(x_1,\dots,x_n)=\Psi_2(x_1,\dots,x_n,y_1,\dots,y_n)=0.$$

Note that $\psi_2(x_1,\cdots,x_n,y_1,\cdots,y_n)$ contains sum of monomials having $y_i's$ of degree one (which is denoted by Ω_1) and sum of monomials having $y_i's$ of degree two (which is denoted by Ω_2). Thus

$$\Omega_1(x_1,\cdots,x_n,y_1,\cdots,y_n)+\Omega_2(x_1,\cdots,x_n,y_1,\cdots,y_n)=0.$$

Replacing y_i with λy_i in $\Psi_2(x_1,\dots,x_n,y_1,\dots,y_n)=0$, where λ is any positive integer, this can be written as

$$\lambda\Omega_1(x_1,\dots,x_n,y_1,\dots,y_n)+\lambda^2\Omega_2(x_1,\dots,x_n,y_1,\dots,y_n)=0.$$

Now assuming $\lambda=1$ and $\lambda=2$ respectively in above equation, we shall obtain a system of 2 homogeneous equations, the coefficient matrix of the system is a vander Monde matrix

$$\begin{pmatrix}1&1\\2&2^2\end{pmatrix}.$$

Since the determinant of the matrix is nonzero, it follows immediately that

$$\Omega_2(x_1,\dots,x_n,y_1,\dots,y_n)=0$$

which implies

$$\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)^2=0. \tag{3.5}$$

If α is an inner automorphism, that is $\alpha(x) = qxq^{-1}$, for all $x \in R$ and an invertible $q \in Q_r$, then we write (3.5) as follows

$$\Big(\sum_{\sigma\in S_n} \gamma_\sigma \sum_{j=0}^{n-1} qx_{\sigma(1)} \cdots x_{\sigma(j)}q^{-1}y_{\sigma(j+1)}x_{\sigma(j+2)} \cdots x_{\sigma(n)}\Big)^2 = 0. \tag{3.6}$$

Replacing each y_i by qy_i, for $i = 1, 2, \dots, n$ we get $qf(x_1, \dots, x_n)qf(x_1, \dots, x_n) = 0$, implies that $f(x_1, \dots, x_n)qf(x_1, \dots, x_n) = 0$. Then by [18, Lemma 2.6] we get $q \in C$. Therefore α is an identity mapping on R, contradiction. Hence we may assume that α is not inner. Thus, by (3.5) we have that Q_r satisfies

$$\Big(\sum_{\sigma\in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} z_{\sigma(1)} \cdots z_{\sigma(j)}y_{\sigma(j+1)}x_{\sigma(j+2)} \cdots x_{\sigma(n)}\Big)^2 = 0. \tag{3.7}$$

that is $f^\alpha(x_1, \dots, x_n)^2 = 0$ is an identity for Q_r, which is again a contradiction. Thus the proof is completed.

Acknowledgements

The authors would like to thank referee for his/her valuable comments and suggestions which have helped the authors to improve the manuscript.

Author contributions. All the co-authors have contributed equally in all aspects of the preparation of this submission.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. This work is supported by a grant from Science and Engineering Research Board (SERB), DST, New Delhi, India. Grant No. is MTR/2022/000568. This work is done when Dr. G. S. Sandhu visited Belda College for a period of one week from 11 Dec., 2023 to 17 Dec., 2023 under the support of this project grant.

Data availability. No data was used for the research described in the article.

References

[1] E. Albaş and N. Argaç, *Generalized derivations of prime rings*, Algebra Colloq. **11**, 399-410, 2004.

[2] A. Ali and D. Kumar, *Generalized derivations as homomorphisms or as anti-homomorphisms in a prime ring*, Hacet. J. Math. Stat. **38**, 17-20, 2009.

[3] A. Ali, N. Rehman and S. Ali, *On Lie ideals with derivations as homomorphisms and anti-homomorphisms*, Acta Math. Hungar. **101**, 79-82, 2003.

[4] K. I. Beidar, W. S. Martindale III and A. V. Mikhalev, *Rings with generalized identities*, Pure and Applied Math. 196, Marcel Dekker, New York, 1996.

[5] H. E. Bell and L. C. Kappe, *Rings in which derivations satisfy certain algebraic conditions*, Acta Math. Hungar. **53** (3-4), 339-346, 1989.

[6] N. Bera and B. Dhara, *Jordan Homoderivation behavior of generalized derivations in prime rings*, Ukrainian Math. J. **75** (9), 1178-1194, 2023.

[7] J. C. Chang, *Generalized skew derivations with annihilating Engel conditions*, Taiwanese J. Math. **12**, 1641-1650, 2008.

[8] J. C. Chang, *Generalized skew derivations with nilpotent values on Lie ideals*, Monatsh. Math. **161**, 155-160, 2010.

[9] J. C. Chang, *On the identity* $h(x) = af(x)+g(x)b$, Taiwanese J. Math. **7** (1), 103-113, 2003.

[10] H. W. Cheng and F. Wei, *Generalized skew derivations of rings*, Adv. Math.(China) **35**, 237-243, 2006.

[11] C. L. Chuang, *Differential identities with automorphisms and antiautomorphisms I*, J. Algebra **149**, 371-404, 1992.

[12] C. L. Chuang, *Differential identities with automorphisms and antiautomorphisms II*, J. Algebra **160** (1), 130-171, 1993.

[13] C. L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103** (3), 723-728, 1988.

[14] C. L. Chuang and T. K. Lee, *Identities with a single skew derivation*, J. Algebra **288** (1), 59-77, 2005.

[15] M. M. El Sofy Aly, *Rings with some kinds of mappings*, M.Sc. Thesis, Cairo University, Branch of Fayoum, 2000.

[16] V. De Filippis, *Generalized Derivations as Jordan Homomorphisms on Lie Ideals and Right Ideals*, Acta Mathematica Sinica, **25** (12), 1965-1974, 2009.

[17] V. De Filippis and O. M. Di Vincenzo, *Vanishing derivations and centralizers of generalized derivations on multilinear polynomials*, *Comm. Algebra* **40**, 1918-1932, 2012.

[18] V. De Filippis, B. Dhara and N. Bera, *Generalized skew derivations and generalization of commuting maps on prime rings*, Beitr. Algebra Geom. **63**, 599-620, 2022.

[19] N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Pub. 37, Amer. Math. Soc., Providence, RI, 1964.

[20] V. K. Kharchenko, *Generalized identities with automorphisms*, Algebra and Logic, **14**, 132-148, 1975.

[21] C. Lanski, *Differential identities, Lie ideals, and Posner's theorem*, Pacific J. Math. **134**, 275-297, 1988.

[22] T. K. Lee, *Generalized skew derivations characterized by acting on zero products*, Pacific J. Math. **216**, 293-301, 2004.

[23] T. K. Lee, *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sinica **20** (1), 27-38, 1992.

[24] U. Leron, *Nil and power central polynomials in rings*, *Trans. Amer. Math. Soc.* **202**, 97-103, 1975.

[25] K. S. Liu, *Differential identities and constants of algebraic automorphisms in prime rings*, Ph.D. Thesis, National Taiwan University 2006.

[26] W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12**, 576-584, 1969.

[27] N. Rehman, *On generalized derivations as homomorphisms and anti-homomorphisms*, Glas. Mat. **39** (1), 27-30, 2004.

[28] G. Scudo, *Generalized derivations acting as Lie homomorphisms on polynomials in prime rings*, Southeast Asian Bull. Math. **38**, 563-572, 2014.

[29] Y. Wang and H. You, *Derivations as homomorphisms or anti-homomorphisms on Lie ideals*, Acta Math. Sinica **32** (6), 1149-1152, 2007.

[30] T. L. Wong, *Derivations with power central values on multilinear polynomials*, Algebra Colloq. **3**, 369-478, 1996.

[31] X. Xu, J. Ma and F. Niu, *Compositions, derivations and polynomials*, Indian J. Pure Appl. Math. **44** (4), 543-556, 2013.