Çankaya University Journal of Science and Engineering
Volume 8 (2011), No. 2, 189–200

On Coatomic Semimodules over Commutative Semirings

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Özet. Bu makale, değişmeli halkalar üzerindeki koatomik modüller ve yarıbasit modüller hakkında iyi bilinen bazı sonuçları, değişmeli yarıhalkalar üzerindeki koatomik ve yarıbasit yarımodüllere genelleştirmiştir. Benzer sonuçlara ulaşmadaki temel zorluk, altyarımodüllerin sağlaması gereken ekstra özellikleri ortaya çıkarmaktır. Yarı modüllerin çalışılmasında yarımodüllerin k-altyarımodüllerinin önemli oldukları ispatlanmıştır.[†]

Anahtar Kelimeler. Yarıhalka, koatomik yarımodüller, yarıbasit yarımodüller, k-tümlenmiş yarımodüller.

Abstract. This paper generalizes some well known results on coatomic and semisimple modules in commutative rings to coatomic and semisimple semimodules over commutative semirings. The main difficulty is figuring out what additional hypotheses the subsemimodules must satisfy to get similar results. It is proved that k-subsemimodules of semimodules are important in the study of semimodules.

Keywords. Semiring, coatomic semimodules, semisimple semimodules, k-supplemented semimodules.

1. Introduction

Study of semirings has been carried out by several authors since there are numerous applications of semirings in various branches of mathematics and computer sciences (see [6],[7] and [9]). It is well known that for a finitely generated module M, every proper submodule of M is contained in a maximal submodule. As an attempt to generalize this property of finitely generated modules we have coatomic modules. In [12], Zöschinger calls a module M coatomic if every proper submodule of M is contained in a maximal submodule of M. The main part of this paper is devoted to

Received March 15, 2011; accepted July 30, 2011.

[†] Türkçe özet ve anahtar kelimeler, orijinal İngilizce metindeki ilgili kısmın doğrudan tercümesi olup *Çankaya University Journal of Science and Engineering* editörlüğü tarafından yazılmıştır. | Turkish abstract and the keywords are written by the editorial staff of *Çankaya University Journal of Science and Engineering* which are the direct translations of the related original English text.

ISSN 1309 - 6788

extending some basic results of the notion semisimple and coatomic modules from the theory of modules to the theory of semimodules.

For the sake of completeness, we state some definitions and notations used throughout. By a commutative semiring we mean an algebraic system $R = (R, +, \cdot)$ such that $(R, +)$ and $(R, \cdot)$ are commutative semigroups, connected by $a(b+c) = ab+ac$ for all $a, b, c \in R$, and there exists $0 \in R$ such that $r + 0 = r$ and $r \cdot 0 = 0 \cdot r = 0$ for all $r \in R$. Throughout this paper let R be a commutative semiring. A (left) semimodule M over a semiring R is a commutative additive semigroup which has a zero element, together with a mapping from $R \times M$ into M (sending (r, m) to rm) such that $(r + s)m = rm + sm$, $r(m + p) = rm + rp$, $r(sm) = (rs)m$ and $0m = r0_M = 0_M$ for all $m, p \in M$ and $r, s \in R$). Let M be a semimodule over a semiring R, and let N be a subset of M. We say that N is a subsemimodule of M, or an R-subsemimodule of M, precisely when N is itself an R-semimodule with respect to the operations for M (so $0_M \in N$). It is easy to see that if $r \in R$, then $rM = \{rm : \ m \in M\}$ is a subsemimodule of M. The semiring R is considered to be also a semimodule over itself. In this case, the subsemimodules of R are called ideals of R.

Let M be a semimodule over a semiring R. A k-subsemimodule (subtractive subsemimodule) N is a subsemimodule of M such that if x, $x + y \in N$, then $y \in N$ (so $\{0_M\}$ is a k-subsemimodule of M). A semimodule M is called a simple semimodule if M has no non-zero k-subsemimodule. A subsemimodule (k-subsemimodule) N of a semimodule M is called a maximal subsemimodule (maximal k-subsemimodule) if L is a subsemimodule (k-subsemimodule) of M such that $N \subsetneqq L$, then $L = M$. A subsemimodule N of a semimodule M over a semiring R is called a partitioning subsemimodule (= Q_M-subsemimodule) if there exists a non-empty subset Q_M of M such that $M = \bigcup\{q + N : q \in Q_M\}$ and if $q_1, q_2 \in Q_M$ then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$. It is easy to see that if $M = Q_M$, then $\{0\}$ is a Q_M-subsemimodule of M.

Let M be a semimodule over a semiring R, and let N be a Q_M-subsemimodule of M. We put $M/N = \bigcup\{q + N : q \in Q_M\}$. Then M/N forms a commutative additive semigroup which has zero element under the binary operation $\oplus$ defined as follows: $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q_M$ is the unique element such that $q_1 + q_2 + N \subseteq q_3 + N$. Then M/N is a semimodule over semiring R by mapping $R \times M/N$ into M/N (sending $(r, q+N)$ to $rq+N$) and zero element q_0+N with $q_0 = 0_M$ (see [3]). We call this R-semimodule the residue class semimodule or factor semimodule of M modulo N. A semimodule M over a semiring R is called an M-cancellative semimodule if whenever $rm = rn$ for elements $m, n \in M$ and $r \in R$, then $m = n$.

2. k-Supplement Subsemimodules

In this section we define k-supplement subsemimodules and the k-radical of a semimodule. We extend some definitions and results of Wisbauer [11] to semimodules over semirings.

Definition 2.1. Let M be a semimodule over a commutative semiring R.

(a) A subsemimodule N of M is essential (k-essential) in M, abbreviated $N \trianglelefteq M$ ($N \trianglelefteq_k M$), if for every subsemimodule (k-subsemimodule) L of M, $N \cap L = 0$ implies $L = 0$.

(b) A subsemimodule K of M is k-superfluous (k-small) in M, abbreviated $K \ll_k M$, in case for every k-subsemimodule L of M, $K + L = M$ implies $L = M$.

(c) Let $\{M_\lambda\}_\Lambda$ be a non-empty family of k-subsemimodules of M. If $M = \sum_{\lambda\in\Lambda} M_\lambda$ and $M_\lambda \cap (\sum_{\mu\neq\lambda} M_\mu) = 0$ for each $\lambda \in \Lambda$. Then M is called the (internal) direct sum of the k-subsemimodules $\{M_\lambda\}_\Lambda$. This is written as $M = \bigoplus_\Lambda M_\lambda$ and M_λ are called direct summands of M. If only $M_\lambda \cap (\sum_{\mu\neq\lambda} M_\mu) = 0$ for each $\lambda \in \Lambda$ is satisfied, then $\{M_\lambda\}_\Lambda$ is called an independent family of k-subsemimodules. It is easy to see that, if M is a direct sum of the k-subsemimodules $\{M_\lambda\}_\Lambda$, then 0_M has a unique representation. A semimodule M is called indecomposable if $M \neq 0$ and it cannot be written as a direct sum of non-zero k-subsemimodules.

(d) Let N be a subsemimodule of M. If N' is a subsemimodule of M maximal with respect to $N \cap N' = 0$, then we say that N' is an M-complement of N. By Zorn's Lemma, we can show that every subsemimodule of M has an M-complement which is a k-subsemimodule of M.

Remark 2.2. For any infinite family $\{N_i\}_{i\in\Lambda}$ of subsemimodules of M, a sum is defined by $\sum_{\lambda\in\Lambda} N_\lambda = \{\sum_{k=1}^{r} n_{\lambda_k} | r \in \boldsymbol{N}, \lambda_k \in \Lambda, n_{\lambda_k} \in N_{\lambda_k}\}$. This is a subsemimodule in M. It is easy to see that every sum of k-subsemimodules is also a k-subsemimodule of M (see [3, Lemma 2]).

Lemma 2.3 (Semimodularity Law)**.** *Let M be a semimodule over semiring R and let N and K be subsemimodules of M. Let L be a k-subsemimodule of M with $N \subseteq L$. Then $L \cap (N + K) = N + (L \cap K)$.*

Proof. Let $x \in N + (L \cap K)$. Then $x = n + a$ for some $n \in N \subseteq L$ and $a \in L \cap K$. Therefore $x = n + a \in L \cap (N + K)$. Now, let $y \in L \cap (N + K)$. Then $y = n' + k$ for some $n' \in N$ and $k \in K$. Hence $k \in L$, since L is a k-subsemimodule of M. Therefore $y = n' + k \in N + (L \cap K)$. □

Proposition 2.4. *Let M be a semimodule over a semiring R and N and N' be k-subsemimodules of M with N' is a M-complement of N. Then,*

(i) $N \oplus N' \trianglelefteq M$.
(ii) *If N' be a Q_M-subsemimodule of M, then $(N \oplus N')/N' \trianglelefteq_k M/N'$.*

Proof. (i) Let $0 \neq L$ be a subsemimodule of M with $(N \oplus N') \cap L = 0$. Now, we show that $N \cap (N' \oplus L) = 0$. Let $n = n' + l$ for some $n \in N$, $n' \in N'$ and $l \in L$. Therefore $l \in N \oplus N'$, since $n, n' \in N \oplus N'$ and $N \oplus N'$ is a k-subsemimodule of M by [4, Lemma 2]. Hence $l \in (N \oplus N') \cap L = 0$ and $n = n' \in N \cap N' = 0$. Then $N \cap (N' \oplus L) = 0$ that is a contradiction with the maximality of N'. Hence $L = 0$ and $N \oplus N' \trianglelefteq M$.

(ii) Let N' be a Q_M-subsemimodule of M. Then $(N \oplus N')/N'$ is a k-subsemimodule of M/N' by [4, Theorem 3]. Let $0 \neq L/N'$ be a k-subsemimodule of M/N'. If $L \cap (N \oplus N') = N'$, then by semimodularity law, $N' \oplus (L \cap N) = N'$. Therefore $L \cap N = 0$ and by maximality of N', $L = N'$, that is a contradiction. So assume that $x \in L \cap (N \oplus N')$ and $x \notin N'$. Since N' is a $Q_M \cap L$-subsemimodule of L and a $Q_M \cap (N \oplus N')$-subsemimodule of $N \oplus N'$ by [2, Lemma 3.4], hence $x \in (q_1 + N) \cap (q_2 + N)$ for some $q_1 \in Q_M \cap L$ and $q_2 \in Q_M \cap (N \oplus N')$. Since $x \notin N'$, then $q_1, q_2 \notin N'$. Therefore $q_1 = q_2$, since N' is a Q_M-subsemimodule of M. Then $q_1 + N = q_2 + N \in L/N' \cap (N \oplus N')/N' \neq 0$. □

Proposition 2.5. *Let M be a semimodule over a semiring R and U be a k-subsemimodule of M and let $M = U + V$ for some subsemimodule V of M. If V' is a maximal k-subsemimodule of V, then $U + V' = M$ or $U + V'$ is a maximal k-subsemimodule of M containing U.*

Proof. Let $U + V' \neq M$ and L be a k-subsemimodule of M with $U + V' \subseteq L \subsetneq M$. Set $L' = L \cap V$. Then $L' = V'$ or $L' = V$ since V' is a maximal subsemimodule of

V. But $L' \neq V$ since $L \neq M$. Therefore $L' = V'$. Let $m \in L$ Therefore $m = u + v$ for some $u \in U$ and $v \in V$. Then $v \in V \cap L = L' = V'$ since L is k-subsemimodule. Thus $m \in U + V'$ and $L = U + V'$. Then $U + V'$ is a maximal k-subsemimodule of M by [4, Lemma 2]. □

Proposition 2.6. *Let M be a semimodule and V be a k-subsemimodule of M. If P is a maximal k-subsemimodule of M, then $V \subseteq P$ or $P \cap V$ is a maximal k-subsemimodule of V.*

Proof. Let $V \nsubseteq P$. Then $P \cap V$ is a proper k-subsemimodule of V. Now, let $P \cap V \subseteq K$ for some proper k-subsemimodule K of V. Then $P + K$ is a k-subsemimodule of M containing P. If $P + K = P$, then $K \subseteq P$ this implies that $K = P \cap V$. If $P + K = M$, then by semimodularity law $V = K + (P \cap V) = K$ that is a contradiction. Hence $P \cap V$ is a maximal k-subsemimodule of V. □

Definition 2.7. Let M be a semimodule over semiring R and let U be a subsemimodule of M. A proper k-subsemimodule V of M is called a k-supplement of U in M if V is minimal element in the set of proper k-subsemimodules L of M with $U + L = M$.

Lemma 2.8. *Let M be a semimodule over a semiring R and U be a k-subsemimodule of M. If V is a k-subsemimodule of M, then V is a k-supplement of U if and only if $M = U + V$ and $U \cap V \ll_k V$.*

Proof. Let $(U \cap V) + K = V$ for some k-subsemimodule K of V. Then $M = U + V = U + (U \cap V) + K = U + K$. Then $V = K$ by the minimality of V. Now let $M = U + V$ and $U \cap V \ll V$. Let $M = U + K$ for some k-subsemimodule K of V. Then by semimodularity law, $V = K + (U \cap V)$ and since $U \cap V \ll_k V$, then $V = K$. Thus V is a supplement of U. □

Definition 2.9. Let M be a semimodule over semiring R. The k-subsemimodule $\bigcap_{P \subseteq M}\{P \mid P \text{ is a maximal } k\text{-subsemimodule of } M\}$ of M, is called the k-radical of M, written as $\mathrm{Rad}_k(M)$.

Proposition 2.10. *Let M be a semimodule over a semiring R and U be a proper k-subsemimodule of M. If V is a k-supplement of U in M, then*

(i) *If M is a finitely generated semimodule, then V is also finitely generated.*
(ii) *If $\mathrm{Rad}_k(V) = V$, then $V \subseteq \mathrm{Rad}_k(M)$.*

Proof. (i) Let $\{x_1, x_2, ..., x_n\}$ be a generating set of M. Then for each $i = 1, 2, ..., n$, there exists $u_i \in U$ and $v_i \in V$ such that $x_i = u_i + v_i$. Set V' the subsemimodule of V generated by $\{v_1, v_2, ..., v_n\}$. Hence $M = V' + U$ and by the minimality of V, this means $V = V'$.

(ii) Let P be a maximal k-subsemimodule of M. If $V \nsubseteq P$, then $P \cap V$ is a maximal k-subsemimodule of V by Proposition 2.6. Since $\mathrm{Rad}_k(V) = V$, then $P \cap V = V$ and hence $V \subseteq P$. Therefore $V \subseteq \mathrm{Rad}_k(M)$. □

3. Coatomic Semimodules

In this section we define coatomic semimodules and we introduce their relations with some other semimodules.

Definition 3.1. Let R be a semiring. The R-semimodule M is called coatomic if every proper k-subsemimodule of M is contained in a maximal k-subsemimodule.

Theorem 3.2. *Every finitely generated semimodule is coatomic.*

Proof. Let M be a finitely generated semimodule and N be a k-subsemimodule of M. Let Δ be the set of all proper k-subsemimodules M' of M with $N \subseteq M'$. Since N is a k-subsemimodule, then Δ is not empty. Of course, the relation of inclusion, $\subseteq$ is a partial order on Δ. Let $\{L_i\}_{i\in I}$ be a chain of elements of Δ for some index set I. Set $L = \bigcup_{i\in I} L_i$. It is clear that L is a subsemimodule of M and $N \subseteq L$. Let x, $x + y \in L$. Therefore $x \in L_s$ and $x + y \in L_k$ for some element $s, k \in I$. Without loss of generality, we can assume that $L_s \subseteq L_k$. Therefore $y \in L_k \subseteq L$, since L_k is a k-subsemimodule of M. Hence L is a k-subsemimodule of M. Now, we show that $L \neq M$. Let $M = \sum_{i=1}^{n} Rm_i$ and $L = M$. Therefore $m_i \in L_{k_i}$ for some $k_i \in I$ for each $i = 1, 2, ..., n$. Choose $j \in \{k_1, ..., k_s\}$ such that L_j is the biggest element in $\{L_{k_1}, ..., L_{k_s}\}$. Therefore $m_i \in L_j$ for each $i = 1, 2, .., n$. Hence $L_j = M$, that is a contradiction. Then by using Zorn's Lemma Δ has a maximal element, which is a maximal k-subsemimodule of M containing N. □

Definition 3.3. Let M be a semimodule over semiring R. Then M is called a k-supplemented semimodule, if every k-subsemimodule of M has a k-supplement in M.

Proposition 3.4. *Let M be a k-supplemented semimodule and* $\mathrm{Rad}_k(M) \ll M$. *Then M is coatomic.*

Proof. Let U be a proper k-subsemimodule of M and let V be a k-supplement of U. If $\mathrm{Rad}_k(V) = V$, then $V \subseteq \mathrm{Rad}_k(M)$ by Proposition 2.10. Then $V \ll M$ by [10, Proposition 3]. This implies that $M = U + V = U$ that is a contradiction. So V has a maximal k-subsemimodule, say V'. Then $V' + U$ is a maximal k-subsemimodule of M containing U by Proposition 2.5, since $V' \neq V$ and V is a k-supplement of U. So M is coatomic. □

Proposition 3.5. *Let M be a coatomic semimodule over a semiring R. Then $\mathrm{Rad}_k(M) \ll_k M$.*

Proof. Let $M = \mathrm{Rad}_k(M) + L$ for some k-subsemimodule L of M. If $L \neq M$, then there is a maximal k-subsemimodule P of M containing L. Since $\mathrm{Rad}_k(M) \subseteq P$, so $M = P$ is a contradiction. Thus $L = M$. □

Proposition 3.6. *Let M be a semimodule over a semiring R and let N be a Q_M-subsemimodule of M. Then the following assertions hold:*

(i) *If M is a coatomic semimodule, then M/N is also coatomic.*
(ii) *If N and M/N are coatomic, then M is coatomic.*

Proof. (i) Let K/N be a proper k-subsemimodule of M/N. Therefore K is a proper k-subsemimodule of M by [2, Theorem 3.6]. Since M is coatomic, there exists a maximal k-subsemimodule P of M with $N \subseteq K \subseteq P$. It is easy to see that P/N is a maximal k-subsemimodule of M/N by [2, Theorem 3.5].

(ii) Let N and M/N be coatomic and let X be a proper k-subsemimodule of M. If $X + N \neq M$, then $(X + N)/N$ is a proper k-subsemimodule of M/N by [4, Theorem 3] and by assumption, there is a maximal k-subsemimodule L/N of M/N containing $(X + N)/N$. Thus X and $X + N$ is contained in maximal k-subsemimodule of M. If $X + N = M$, then $X \cap N$ is a proper k-subsemimodule of N since $X \neq M$. So $X \cap N$ is contained in a maximal k-subsemimodule N' of N. So $X + N'$ is a maximal k-subsemimodule of M by Proposition 2.5. Then M is coatomic. □

Theorem 3.7. *Let M be a semimodule over a semiring R and let N be a Q_M-subsemimodule of M. If N is a small subsemimodule of M, then M is a coatomic semimodule if and only if M/N is a coatomic semimodule.*

Proof. If M is a coatomic semimodule, then M/N is also coatomic by Proposition 3.6. Suppose that M/N is a coatomic semimodule and N is a small subsemimodule of M. Let L be a proper k-subsemimodule of M. If $N \subseteq L$, then L/N is a k-subsemimodule of M/N. By assumption L/N is contained in a maximal k-subsemimodule P/N of M/N. Then L is contained in maximal k-subsemimodule P of M. If not, Then $(L+N)/N$ is a k-subsemimodule of M/N by [4, Theorem 3]. If $(L+N)/N = M/N$ then $L+N = M$ by [4, Theorem 4] and since N is a small subsemimodule of M, then $M = L$ that is a contradiction. So $(L+N)/N$ is a proper k-subsemimodule of M/N. Thus $(L+N)/N$ is contained in a maximal k-subsemimodule of M/N. Thus L and $L+N$ is contained in a maximal k-subsemimodule of M. Then M is coatomic. □

4. Semisimple Semimodules

In this section we define semisimple semimodules and we show that every semisimple semimodule is a coatomic semimodule.

Definition 4.1. Let $\{N_\alpha\}_{\alpha\in\Lambda}$ be an indexed set of simple k-subsemimodules of semimodule M over a semiring R. If M is a direct sum of this set, then $M = \bigoplus_{\alpha\in\Lambda} N_\alpha$ is a semisimple decomposition of M. A semimodule M is said to be semisimple in case it has a semisimple decomposition.

Proposition 4.2. *Let $\{N_\lambda\}_\Lambda$ be a family of simple k-subsemimodules of the R-semimodule M with $M = \sum_\Lambda N_\lambda$. Then for every proper k-subsemimodule K of M, there is an index set $\Lambda_K \subset \Lambda$ such that $M = K \oplus (\bigoplus_{\Lambda_K} N_\lambda)$.*

Proof. Let K be a proper subtractive subsemimodule of M. If $K \cap N_\lambda \neq 0$ for each $\lambda \in \Lambda$. Then $K \cap N_\lambda = N_\lambda$ for each $\lambda \in \Lambda$, since N_λ is simple and $K \cap N_\lambda$ is a subtractive subsemimodule of N_λ by [4, Lemma 2]. This implies that $K = M$. Now, choose a subset $\Lambda_K \subset \Lambda$ maximal with respect to the property that $\{N_\lambda\}_{\Lambda_K}$ is an independent family of simple k-subsemimodules with $K \cap \sum_{\Lambda_K} N_\lambda = 0$. Then $L = K + \sum_{\Lambda_K} N_\lambda$ is a direct summand, that is, $L = K \oplus (\bigoplus_{\Lambda_K} N_\lambda)$. By Remark 2.2, L is a k-subsemimodule of M. It suffices to show that $L = M$. Since $L \cap N_\lambda$ is a k-subsemimodule of N_λ and N_λ is simple for each $\lambda \in \Lambda$, then $L \cap N_\lambda = 0$ or $L \cap N_\lambda = N_\lambda$. If for some $\lambda \in \Lambda \setminus \Lambda_K$, $L \cap N_\lambda = 0$, this is a contradiction to the maximality of Λ_K. Hence we get $N_\lambda \subseteq L$ for all $\lambda \in \Lambda$ and $L = M$. □

Example 4.3. In a semimodule every cyclic subsemimodule need not be a k-subsemimodule. Let $R = \{0, 1, u\}$ be the idempotent semiring in which $1 + u = u + 1 = u$ and $M = R$ as an R-semimodule. Then cyclic subsemimodule $N = \{0, u\}$ is not k-subsemimodule, since $1 + u = u \in N$ but $1 \notin N$ [7, Example 6.4].

Definition 4.4. Let M be a semimodule over a semiring R. Then M is called a subtractive semimodule, if every cyclic subsemimodule of M is a k-subsemimodule.

Example 4.5. Let R be a partitioning semiring [5, Definition 2.2] and let $M = R$ as an R-semimodule. Then M is a k-semimodule, since every partitioning semimodule is a k-subsemimodule.

Theorem 4.6. *Let M be a subtractive semimodule. Then the following properties are equivalent:*

(i) *M is semisimple.*
(ii) *M is a sum of simple k-subsemimodules.*
(iii) *M has no proper essential k-subsemimodule.*
(iv) *Every k-subsemimodule of M is a direct summand.*

Proof. (i) $\Leftrightarrow$ (ii) by Lemma 4.2.

(i) $\Rightarrow$ (iv) by Lemma 4.2.

(iii) $\Rightarrow$ (iv): Let N be a k-subsemimodule of M and N' be a k-subsemimodule of M which is an M-complement of N, then by Proposition 2.4, $N \oplus N' \trianglelefteq M$. Hence $N \oplus N' = M$.

(iv) $\Rightarrow$ (iii) is clear.

(iv) $\Rightarrow$ (ii): Let $0 \neq m \in M$. Then Rm is a k-subsemimodule of M by assumption. So Rm has a maximal k-subsemimodule U, by Proposition 3.2. It is easy to see that U is a k-subsemimodule of M. Then $M = U \oplus V'$ for some k-subsemimodule V' of M. Set $V = V' \cap Rm$, then V is a k-subsemimodule of M by [4, Lemma 2]. We show that V is a simple k-subsemimodule (minimal k-subsemimodule) of M. Let K be a k-subsemimodule of V. Therefore $U \subseteq U + K \subseteq Rm$. If $U + K = U$, then $K \subseteq U \cap V \subseteq U \cap V' = 0$. If $U + K = Rm$, then for every $v \in V$, $v = u + k$ for some $k \in K$ and $u \in U$. Since V is a k-subsemimodule and $K \subseteq V$, then $u \in V \cap U = 0$. Then $v = k \in K$, hence $V = K$. So V is a simple k-subsemimodule of M. Therefore every non-zero subsemimodule of M contains a simple k-subsemimodule. Let L be the sum of all simple k-subsemimodules of M. Then L is a k-subsemimodule of M and there is a k-subsemimodule L' of M with $M = L \oplus L'$. Since L' cannot have any simple k-subsemimodule, it must be zero. □

Lemma 4.7. *Let M be a semisimple semimodule over semiring R. Then every k-subsemimodule of M is semisimple.*

Proof. Let $M = \bigoplus_{\lambda\in\Lambda} N_\lambda$ such that N_λ is a simple k-subsemimodule of M for every $\lambda \in \Lambda$ and let K be a k-subsemimodule of M. Then there is an index set $\Lambda_K \subset \Lambda$ such that $M = K \oplus (\bigoplus_{\Lambda_K} N_\lambda)$ by Proposition 4.2. Set $I = \Lambda \setminus \Lambda_K$. We show that $K = \bigoplus_{\lambda\in I} N_\lambda$. Let $\lambda \in I$. Then $K \cap N_\lambda \neq 0$ and since N_λ is simple we have $K \cap N_\lambda = N_\lambda$. Therefore $\bigoplus_{\lambda\in I} N_\lambda \subseteq K$. Now, let $x \in K$. Then $x = a + b$ for some $a \in \bigoplus_{\Lambda_K} N_\lambda$ and $b \in \bigoplus_I N_\lambda$. Then $b \in K$ and since K is a k-subsemimodule of M, we have $a \in K$. Thus $a \in K \cap (\bigoplus_{\Lambda_K} N_\lambda) = 0$. Therefore $x = b \in \bigoplus_I N_\lambda$. Then $K = \bigoplus_I N_\lambda$. □

Definition 4.8. Let M be a semimodule over a semiring R. As socle of M, we denote the sum of all simple (minimal) k-subsemimodules of M. If there are no simple k-subsemimodules in M, we put $\mathrm{Soc}(M) = 0$

Proposition 4.9. *Let M be a subtractive semimodule. Then*

$$\begin{aligned}\mathrm{Soc}(M) &= \sum\{K \mid K \textit{ is a simple } k\textit{-subsemimodule of } M\}\\ &= \bigcap\{L \mid L \textit{ is an essential } k\textit{-subsemimodule of } M\}.\end{aligned}$$

Proof. Let L be an essential k-subsemimodule of M. Then for every simple k-subsemimodule K of M, we have $0 \neq L \cap K$. So $L \cap K = K$, since $L \cap K$ is a k-subsemimodule and K is simple. Then $K \subseteq L$. This implies that $\mathrm{Soc}(M)$ is contained in every essential k-subsemimodule. Put $L_0 = \bigcap\{L \mid L$ is a subtractive essential subsemimodule of $M\}$, so L_0 is a k-subsemimodule of M. We show that L_0 is semisimple. Let K be a k-subsemimodule of L_0 and K' be a M-complement of K which is a k-subsemimodule of M. Then $K \oplus K' \trianglelefteq M$ by Proposition 2.4 and consequently $L_0 \subseteq K \oplus K'$. By semimodularity, this yields $L_0 = L_0 \cap (K \oplus K') = K \oplus (L_0 \cap K')$. Therefore K is a direct summand of L_0 and L_0 is semisimple by Theorem 4.6. Hence $L_0 \subseteq \mathrm{Soc}(M)$. □

Theorem 4.10. *Let M be a semisimple semimodule. Then M is a coatomic semimodule.*

Proof. Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ such that M_λ is a simple k-subsemimodule of M for every $\lambda \in \Lambda$ and let N be a k-subsemimodule of M. Then there is an index set $\Lambda_N \subset \Lambda$ such that $M = N \oplus (\bigoplus_{\lambda\in\Lambda_N} M_\lambda)$ by Proposition 4.2. Since $N \neq M$, then $\Lambda_N \neq \emptyset$. Let $\beta \in \Lambda_N$. Then $L = N \oplus (\bigoplus_{\lambda\in\Lambda_N\setminus\{\beta\}} M_\lambda)$ is a k-subsemimodule of M and $M = L \oplus M_\beta$. We show that L is a maximal k-subsemimodule of M. Let L' be a k-subsemimodule of M with $L \subseteq L' \subsetneqq M$. Then $L' \cap M_\beta = 0$. Let $x \in L'$. Then

$x = y + z$ for some $y \in L$ and $z \in M_\beta$. Therefore $z \in L' \cap M_\beta = 0$ since L' is a k-subsemimodule. Thus $L = L'$. Therefore L is a maximal k-subsemimodule of M and $N \subseteq L$. □

Theorem 4.11. *Let M be a semimodule. Then M is semisimple if and only if M is coatomic and every maximal k-subsemimodule of M is a direct summand.*

Proof. (⇒) Let M be a semisimple semimodule. Then M is coatomic and every maximal k-subsemimodule of M is a direct summand by Theorem 4.6 and Proposition 4.10.

(⇐) Let $\mathrm{Soc}(M) \neq M$. Since $\mathrm{Soc}(M)$ is a k-subsemimodule of M and M is coatomic, then $\mathrm{Soc}(M)$ is contained in a maximal k-subsemimodule P of M. By assumption there is a k-subsemimodule K of M such that $M = P \oplus K$. Now we show that K is a simple k-subsemimodule of M. Let K_0 be a non-zero k-subsemimodule of K. If $P + K_0 = P$, then $K_0 \subseteq P \cap K = 0$ that is a contradiction. So $P + K_0 = M$. Therefore by semimodularity $K = K_0 + (P \cap K) = K_0$. Thus K is a simple k-subsemimodule of M and $K \subseteq \mathrm{Soc}(M) \subseteq P$. Then $P = M$ that is a contradiction. So, $\mathrm{Soc}(M) = M$ and M is semisimple. □

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