http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 4, Pages 929–940 (2024)
DOI:10.31801/cfsuasmas.1473166
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: April 24, 2024; Accepted: October 23, 2024

ON A CLASS OF FOURTH-ORDER NEUTRAL DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENTS

M. Emre KAVGACI

Ankara University, Department of Mathematics, 06100 Ankara, TÜRKİYE

Abstract. In this paper, we investigate a fourth-order neutral differential equation characterized by piecewise constant arguments. Our study focuses on establishing both the existence and uniqueness of solutions to this equation, incorporating a prescribed initial condition. In addition, we investigate the stability analysis of the above-mentioned equation and show that the zero solution of this equation cannot be asymptotically stable and indicate under what conditions it is unstable. Through rigorous mathematical analysis and theoretical exploration, this research contributes to the deeper understanding of fourth-order neutral differential equations with piecewise constant arguments, offering insights into their solution behavior and stability properties.

1. Introduction

In this work, we investigate the fourth-order neutral differential equation with piecewise constant arguments (NDEPCA)

$$\frac{d^4}{dt^4}\Big(x(t)+px(t-1)\Big)=qx([t-1]),\quad t\geq 0, \tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\quad -1\leq t\leq 0, \tag{2}$$

where p and q are real constants, $[.]$ denotes the greatest integer function and $\varphi\in C([-1,0],\mathbb{R})$ is an initial function.

Our aim is to show the existence and uniqueness of solutions for the initial value problem (1)–(2) and to demonstrate that its zero solution cannot be asymptotically stable. Obtaining the solutions of neutral differential equations with piecewise constant arguments using difference equations offers numerous advantages. In this study, we show that equation (1) exhibits the same asymptotic properties as the

2020 *Mathematics Subject Classification.* 34K40, 34K20, 39A30.
Keywords. Piecewise arguments, stability, neutral differential equations.
ekavgaci@ankara.edu.tr; 0000-0002-8605-4346.

corresponding fifth-order difference equation. The paper is structured as follows: In the first section, previous studies are presented to provide the necessary motivation for this paper. Additionally, important definitions and theorems related to neutral differential equations and difference equations are provided. In the second section, the existence and uniqueness of solutions for the initial value problem (1)–(2) are demonstrated, and it is shown that the zero solution of a fourth-order equation of type (1) cannot be asymptotically stable. Section 3 consists of a numerical example.

Differential equations with piecewise constant arguments (DEPCA) were pioneered by Busenberg, Cooke, Shah, and Wiener in their seminal works [12, 36]. These equations bridge the realms of difference and differential equations, incorporating both discrete and continuous dynamics at integer points. This connection is particularly evident in epidemic models, where the interplay between discrete events and continuous processes naturally emerges. Following this research, numerous significant problems spanning the vibration of spring-mass systems, biomedicine, electronic processes, epidemic diseases, isolated mechanisms and some significant properties of the solutions have been investigated through the utilization of DEPCA [1]- [3], [5]-[11], [13]-[20], [22, 23, 26, 29, 30, 35, 37].

However there are only a few articles that issued on neutral differential equations with piecewise constant arguments (NDEPCA) (see [4, 24, 27, 28], [31]–[34],[38]). Some stability and oscillation results for NDEPCA have been discussed in [34], where the oscillatory behavior and stability of the trivial solution of first- and second-order NDEPCA were analyzed:

$$\frac{d}{dt}\Big(y(t) + py(t-1)\Big) = -qy([t-1]),$$

and

$$\frac{d^2}{dt^2}\Big(y(t) + py(t-1)\Big) = -qy([t-1]). \tag{3}$$

It was proved that the trivial solution of Eq. (3) is not asymptotically stable. Later, in [33], Papaschinopoulos obtained a unique solution for the third-order NDEPCA

$$\frac{d^3}{dt^3}\Big(y(t) + py(t-1)\Big) = -qy([t-1]), \tag{4}$$

and demonstrated that the zero solution of Eq. (4) is not asymptotically stable.

The gap in the literature, along with these earlier studies, motivates us to explore the asymptotic behavior of Eq. (1).

Now, let us give definition:

Definition 1. *A function $x : [-1,\infty) \to \mathbb{R}$ is a solution of the initial value problem (1)-(2) if the following conditions are satisfied:*

(i) x *and* $\dfrac{dx}{dt} \in C([-1,\infty),\mathbb{R})$,

(ii) $\frac{d^2}{dt^2}\Big(x(t)+px(t-1)\Big)=\beta(t)$ *and* $\frac{d^3}{dt^3}\Big(x(t)+px(t-1)\Big)=\alpha(t)$ *exist on* $[0,\infty)$ *and* $\beta(t)$ *and* $\alpha(t)$ *are continuous on* $[0,\infty)$,

(iii) $\frac{d^4}{dt^4}\Big(x(t)+px(t-1)\Big)$ *exist on* $[0,\infty)$ *with the possible exception at the point* $[t]\in[0,\infty)$ *where one-sided derivatives exists;*

(iv) x *satisfies Eq. (1) on each interval* $[n,n+1)$ *with* $n=0,1,2,...$ *and initial condition (2) on the interval* $[-1,0]$.

Before giving the main theorems, consider the $k-th$ order difference equation

$$x_{n+k}+p_1x_{n+k-1}+p_2x_{n+k-2}+...+p_kx_n=0, \qquad (5)$$

where $p_i, i=1,2,...,k$ are real numbers. Also, we can write the corresponding characteristic equation of (5) as follows:

$$p(\lambda)=\lambda^k+p_1\lambda^{k-1}+...+p_k. \qquad (6)$$

Now, we should remember the following well-known some theorems for difference equations:

Theorem 1. *([21], p246.) The zero solution of Eq. (5) is asymptotically stable if and only if* $|\lambda|<1$ *for all roots* λ *of Eq. (6).*

Theorem 2. *(Schur-Cohn Criterion or Jury Conditions, [25]) The roots of the Eq. (6) lie inside the unit disk if and only if the following hold:*
(i) $p(1)>0$,
(ii) $(-1)^kp(-1)>0$,
(iii) consider the matrix $A_1^{\pm},A_2^{\pm},...$ *for* $i=1,2,...k$,

$$A_i^{\pm}=\begin{pmatrix}1 & p_{k-1} & p_{k-2} & \dots & p_{k-i+1}\\ 0 & 1 & p_{k-1} & \dots & p_{k-i+2}\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}\pm\begin{pmatrix}p_{i-1} & p_{i-2} & \dots & p_1 & p_0\\ p_{i-2} & p_{i-3} & \dots & p_0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ p_0 & 0 & \dots & 0 & 0\end{pmatrix}$$

and determinants $|A_1^{\pm}|>0,|A_3^{\pm}|>0,...,|A_{k-1}^{\pm}|>0$ *(for k is even) or* $|A_2^{\pm}|>0,|A_4^{\pm}|>0,...,|A_{k-1}^{\pm}|>0$ *(for k is odd).*

Theorem 3. *([21], Theorem 5.12) The zero solution of Eq. (5) is unstable if*

$$|p_1|-\sum_{i=2}^{k}|p_i|>1.$$

2. Main Results

Theorem 4. *The initial value problem (1)-(2) has a unique solution* $x(t)$ *with* $x(-1)=c_{-1}$ *and* $x(0)=c_0$.

Proof. Let us consider,

$$\frac{d}{dt}\Big(x(t)+px(t-1)\Big)\Big|_{t=0}=\gamma_0,$$
$$\frac{d^2}{dt^2}\Big(x(t)+px(t-1)\Big)\Big|_{t=0}=\beta_0,$$
$$\frac{d^3}{dt^3}\Big(x(t)+px(t-1)\Big)\Big|_{t=0}=\alpha_0,$$

and $x(-1)=\varphi(-1)=c_{-1}, x(0)=\varphi(0)=c_0$. We apply the method of steps to show the existence and uniqueness of the solution of (1)-(2). Let $x_0(t)\equiv x(t)$ on the interval $0\le t<1$,

$$\frac{d^4}{dt^4}\Big(x(t)+px(t-1)\Big)=qx(-1)=q\varphi(-1)=qc_{-1}.$$

Integrating this equation from 0 to t, we obtain

$$\frac{d^3}{dt^3}\Big(x(t)+px(t-1)\Big)=\alpha_0+qc_{-1}t,$$

and again, integrating from 0 to t, we get

$$\frac{d^2}{dt^2}\Big(x(t)+px(t-1)\Big)=\beta_0+\alpha_0 t+qc_{-1}\frac{t^2}{2},$$

and also integrating this equation from 0 to t, we obtain

$$\frac{d}{dt}\Big(x(t)+px(t-1)\Big)=\gamma_0+\beta_0 t+\alpha_0\frac{t^2}{2}+qc_{-1}\frac{t^3}{6},$$

and finally, if we integrate this equation from 0 to t, we obtain

$$x(t)+px(t-1)=x(0)+px(-1)+\gamma_0 t+\beta_0\frac{t^2}{2}+\alpha_0\frac{t^3}{6}+qc_{-1}\frac{t^4}{24}.$$

On the interval $0\le t<1$, we can rewrite this equation as follows:

$$x_0(t)\equiv x(t)=-p\varphi(t-1)+c_0+pc_{-1}+\gamma_0 t+\beta_0\frac{t^2}{2}+\alpha_0\frac{t^3}{6}+qc_{-1}\frac{t^4}{24},$$

where $x(0)=c_0, x(-1)=c_{-1}$. Let $x_1(t)\equiv x(t)$ be a solution of (1)-(2) for $t\in[1,2)$. Let us consider,

$$\frac{d}{dt}\Big(x(t)+px(t-1)\Big)\Big|_{t=1}=\gamma_1,$$
$$\frac{d^2}{dt^2}\Big(x(t)+px(t-1)\Big)\Big|_{t=1}=\beta_1,$$
$$\frac{d^3}{dt^3}\Big(x(t)+px(t-1)\Big)\Big|_{t=1}=\alpha_1,$$

with the path followed in the previous step, we obtain

$$x_1(t)\equiv x(t)=-px_0(t-1)+c_1+pc_0+\gamma_1(t-1)+\beta_1\frac{(t-1)^2}{2}+\alpha_1\frac{(t-1)^3}{6}+qc_0\frac{(t-1)^4}{24}. \tag{7}$$

By the continuity of $x(t)$ at $t = 1$, one can write clearly

$$c_1 = (1-p)c_0 + (p + \frac{q}{24})c_{-1} + \gamma_0 + \frac{\beta_0}{2} + \frac{\alpha_0}{6}, \tag{8}$$

and

$$\begin{cases} \alpha_1 = \alpha_0 + qc_{-1}, \\ \beta_1 = \beta_0 + \alpha_0 + \frac{q}{2}c_{-1}, \\ \gamma_1 = \gamma_0 + \beta_0 + \frac{\alpha_0}{2} + \frac{q}{6}c_{-1}. \end{cases} \tag{9}$$

If we put (9) and (8) into Eq. (7), we obtain for $1 \le t < 2$,

$$x_1(t) \equiv x(t) = -p\Big[-p\varphi(t-2) + c_0 + pc_{-1} + \gamma_0(t-1) + \beta_0\frac{(t-1)^2}{2} + \alpha_0\frac{(t-1)^3}{6} + qc_{-1}\frac{(t-1)^4}{24}\Big] + c_0 + (p+\frac{q}{24})c_{-1} + \gamma_0 + \frac{\beta_0}{2} + \frac{\alpha_0}{6}.$$

We will do it for the general case. Now, let us assume that, respectively, $x_n(t) \equiv x(t)$ be a solution of (1)-(2) on the interval $n \le t < n+1$ and $x_{n+1}(t) \equiv x(t)$ be a solution of (1)-(2) on the interval $n+1 \le t < n+2$, let us consider

$$\frac{d}{dt}\Big(x(t) + px(t-1)\Big)\Big|_{t=n} = \gamma_n \text{ and } \frac{d}{dt}\Big(x(t) + px(t-1)\Big)\Big|_{t=n+1} = \gamma_{n+1}, \tag{10}$$

$$\frac{d^2}{dt^2}\Big(x(t) + px(t-1)\Big)\Big|_{t=n} = \beta_n \text{ and } \frac{d^2}{dt^2}\Big(x(t) + px(t-1)\Big)\Big|_{t=n+1} = \beta_{n+1}, \tag{11}$$

$$\frac{d^3}{dt^3}\Big(x(t) + px(t-1)\Big)\Big|_{t=n} = \alpha_n \text{ and } \frac{d^3}{dt^3}\Big(x(t) + px(t-1)\Big)\Big|_{t=n+1} = \alpha_{n+1}, \tag{12}$$

in the same way, $x_n(t) = x(t)$ can be written as

$$x_n(t) \equiv x(t) = -px_{n-1}(t-1) + c_n + pc_{n-1} + \gamma_n(t-n) + \beta_n\frac{(t-n)^2}{2} + \alpha_n\frac{(t-n)^3}{6} + qc_{n-1}\frac{(t-n)^4}{24}, \tag{13}$$

for $t \in [n, n+1)$, where $c_n = x(n)$ and $c_{n-1} = x(n-1)$. Finally, on the interval $n+1 \le t < n+2$, we derive

$$x_{n+1}(t) \equiv x(t) = -px_{n-1}(t-1) + c_{n+1} + pc_n + \gamma_{n+1}(t-n-1) + \beta_{n+1}\frac{(t-n-1)^2}{2} + \alpha_{n+1}\frac{(t-n-1)^3}{6} + \frac{q}{24}c_n(t-n-1)^4. \tag{14}$$

Because of the continuity of $x(t)$ at $t = n+1$, it must be the case that

$$\lim_{t\to n+1} x_n(t) = \lim_{t\to n+1} x_{n+1}(t) \text{ for } n = 0, 1, 2, ...$$

Therefore from (13) and (14), we get

$$c_{n+1} + (p-1)c_n + (-p - \frac{q}{24})c_{n-1} = \gamma_n + \frac{\beta_n}{2} + \frac{\alpha_n}{6}, \quad n = 0, 1, 2, ... \tag{15}$$

By the continuity at $t = n + 1$ and for $n = 0, 1, 2, ...$, from (10), (11) and (12), we can write following equations:

$$\begin{cases} \gamma_{n+1} = \gamma_n + \beta_n + \frac{\alpha_n}{2} + \frac{q}{6}c_{n-1}, \\ \beta_{n+1} = \beta_n + \alpha_n + \frac{q}{2}c_{n-1}, \\ \alpha_{n+1} = \alpha_n + qc_{n-1}. \end{cases}$$

From these equations, we can write α_n, β_n, and γ_n as follows:

$$\begin{cases} \alpha_n = \alpha_{n+1} - qc_{n-1}, \\ \beta_n = \beta_{n+1} - \alpha_{n+1} + \frac{q}{2}c_{n-1}, \\ \gamma_n = \gamma_{n+1} - \beta_{n+1} + \frac{1}{2}\alpha_{n+1} - \frac{q}{6}c_{n-1}. \end{cases} \tag{16}$$

Therefore, from (16) and (15), we obtain

$$c_{n+1} + (p-1)c_n + (-p + \frac{q}{24})c_{n-1} = \gamma_{n+1} - \frac{\beta_{n+1}}{2} + \frac{\alpha_{n+1}}{6}, \quad n = 0, 1, 2, ... \tag{17}$$

If we replace n with $n + 1$ in Eq. (15), we get

$$c_{n+2} + (p-1)c_{n+1} + (-p - \frac{q}{24})c_n = \gamma_{n+1} + \frac{\beta_{n+1}}{2} + \frac{\alpha_{n+1}}{6} \tag{18}$$

From Eq. (17), Eq. (18) and by using (16), we can write

$$\begin{aligned} c_{n+2} + (p-4)c_{n+1} + (6 - 4p - \frac{q}{24})c_n + (-4p + 6p - \frac{11q}{24})c_{n-1} \\ +(1 - 4p - \frac{11q}{24})c_{n-2} + (p - \frac{q}{24})c_{n-3} = 0, \quad n = 2, 3, ... \end{aligned}$$

Therefore, we obtain the fifth-order difference equation for $n = -1, 0, 1, ...$

$$\begin{aligned} c_{n+5} + (p-4)c_{n+4} + (6 - 4p - \frac{q}{24})c_{n+3} + (-4 + 6p - \frac{11q}{24})c_{n+2} \\ +(1 - 4p - \frac{11q}{24})c_{n+1} + (p - \frac{q}{24})c_n = 0, \end{aligned} \tag{19}$$

with the initial conditions

$$\begin{aligned} c_{-1} = \varphi(-1), \quad c_0 = \varphi(0), \quad c_1 = (1-p)c_0 + (p + \frac{q}{24})c_{-1} + \gamma_0 + \frac{\beta_0}{2} + \frac{\alpha_0}{6}, \\ c_2 = (p^2 - p + 1 + \frac{q}{24})c_0 + (-p^2 + p + \frac{q(15-p)}{24})c_{-1} + (2-p)\gamma_0 + (\frac{4-p}{2})\beta_0 \\ +(\frac{8-p}{6})\alpha_0 \\ c_3 = (1-p)c_2 + (p + \frac{q}{24})c_1 + \frac{7q}{12}c_0 + \frac{25q}{12}c_{-1} + \gamma_0 + \frac{5\beta_0}{2} + \frac{19\alpha_0}{6}, \end{aligned} \tag{20}$$

This initial value problem has a unique solution. Then the solution $x(t)$ of (1)-(2) defined by (13) is unique on the interval $n \leq t < n + 1$. Thus, the proof is completed. □

Now, the solution methodology for (1)-(2) can be succinctly described by referring to Lemma 3 in [34], which offers a comprehensive approach.

$$x(t) + px(t-1) = v(t), \quad t \geq 0,$$

with the initial function

$$x(t) = \varphi(t), \quad -1 \leq t \leq 0,$$

is the continuous function given by

$$x(t) = (-p)^{n+1}\varphi(\theta - 1) + \sum_{k=0}^{n}(-p)^{n-k}v(k+\theta), \quad t \geq 0,$$

where $v(k+\theta)$ can be obtain from (13) as follows:

$$v(t) = c_n + pc_{n-1} + \gamma_n(t-n) + \frac{\beta_n}{2}(t-n)^2 + \frac{\gamma_n}{6}(t-n)^3 + \frac{q}{24}c_{n-1}(t-n)^4,$$

we get the solution of (1)-(2) as in the form

$$x(t) = (-p)^{n+1}\Big[\varphi(\theta-1) + \sum_{k=0}^{n}(-p)^{-k-1}[c_k + (p+\frac{q}{24}\theta^4)c_{k-1} + \gamma_k\theta + \frac{1}{2}\beta_k\theta^2 + \frac{1}{6}\alpha_k\theta^3]\Big], \tag{21}$$

where $\varphi \in C([-1,0],\mathbb{R})$, $t = n + \theta$ with $0 \leq \theta \leq 1$ and $n = -1, 0, 1, ...$

Now, we investigate the stability nature behaviour of solutions of the general fifth order linear difference equation with constant coefficients of the form

$$c_{n+5} + a_4c_{n+4} + a_3c_{n+3} + a_2c_{n+2} + a_1c_{n+1} + a_0c_n = 0, \quad n = -1, 0, 1, ... \tag{22}$$

where $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$. The characteristic equation of Eq. (22) is

$$p(\lambda) = \lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \tag{23}$$

The following lemma gives necessary and sufficient conditions for the asymptotic stability of the zero solution of Eq. (22).

Lemma 1. *The zero solution of Eq.* (22) *is asymptotically stable if and only if the following conditions hold:*

(I) $1 + a_3 + a_1 > |a_4 + a_2 + a_0|$

(II) $1 - a_0^2 > |a_1 - a_4a_0|,$

(III) $a_0^4 + a_0^3a_2 + a_0^3a_4 - a_0^2a_1a_3 - a_0^2a_1 - a_0^2a_3^2 - a_0^2a_3 - a_0^2a_4^2 - 2a_0^2 + a_0a_1^2a_4 + a_0a_1a_2 + a_0a_1a_3a_4 + 2a_0a_1a_4 + 2a_0a_2a_3 - a_0a_2a_4^2 - a_0a_2 + a_0a_3a_4 - a_0a_4^3 - a_0a_4 - a_1^3 - a_1^2a_3 - a_1^2 + a_1a_2a_4 + a_1a_4^2 + a_1 - a_2^2 - a_2a_4 + a_3 + 1 > 0,$

(IV) $a_0^4 - a_0^3a_2 - a_0^3a_4 + a_0^2a_1a_3 + a_0^2a - 1 + 2a_0^2a_2a_4 - a - 0^2a_3^2 + a_0^2a_3 - a_0^2a_4^2 - 2a_0^2 - a_0a_1^2a_4 - 3a_0a_1a_2 + a_0a_1a_3a_4 + 2a_0a_1a_4 + 2a_0a_2a_3 - a_0a_2a_4^2 + a_0a_2 - 3a_0a_3a_4 + a_0a_4^3 + a_0a_4 + a_1^3 - a_1^2a_3 - a_1^2 + a_1a_2a_4 + 2a_1a_3 - a_1a_4^2 - a_1 - a_2^2 + a_2a_4 - a_3 + 1 > 0.$

Proof. By Theorem 1, the zero solution of Eq. (22) is asymptotically stable if and only if each root of λ of Eq. (23) satisfies $|\lambda| < 1$. Using the condition (i) and (ii) in Theorem 2, we can easily obtain

$$1 + a_4 + a_3 + a_2 + a_1 + a_0 > 0 \quad \text{and} \quad 1 - a_4 + a_3 - a_2 + a_1 - a_0 > 0, \qquad (24)$$

it can be easily seen that the conditions (24) are equivalent to condition (I). Using the condition (iii) in Theorem 2, we can write

$A_2^+ = \begin{pmatrix} 1+a_1 & a_4+a_0 \\ a_0 & 1 \end{pmatrix}$, $A_2^- = \begin{pmatrix} 1-a_1 & a_4-a_0 \\ -a_0 & 1 \end{pmatrix}$ and

$$A_4^+ = \begin{pmatrix} 1+a_3 & a_4+a_2 & a_3+a_1 & a_2+a_0 \\ a_2 & 1+a_1 & a_4+a_0 & a_3 \\ a_1 & a_0 & 1 & a_4 \\ a_0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_4^- = \begin{pmatrix} 1-a_3 & a_4-a_2 & a_3-a_1 & a_2-a_0 \\ -a_2 & 1-a_1 & a_4-a_0 & a_3 \\ -a_1 & -a_0 & 1 & a_4 \\ -a_0 & 0 & 0 & 1 \end{pmatrix}.$$

We can say that the determinants of $A_2^{\pm}$ and $A_4^{\pm}$ must be positive. If numerical calculations are performed, the conditions $(II), (III)$, and (IV) are obtained. □

Theorem 5. *The zero solution of Eq.* (1) *is not asymptotically stable.*

Proof. Applying Lemma 1 to Eq. (19), we obtain that the zero solution of Eq. (19) is asymptotically stable if and only if

(a) $p < 1$ and $q < 0$,

(b) $2 - 4p - (p-4)(p - \frac{q}{24}) - (p - \frac{q}{24})^2 - \frac{11q}{24} > 0$ and $4p + (p-4)(p - \frac{q}{24}) + (p - \frac{q}{24})^2 + \frac{11q}{24} > 0$,

(c) $-\frac{q}{864}(-3456p^3 - 24p^2(7q+432) + p(13q^2 - 3504q + 3456) - 121q^2 + 3672q + 10368) > 0$ and $\frac{q^2}{96}(-24p^2 + p(q-144) - 9(q+24)) > 0$.

However, these conditions are inconsistent. If we solve these inequalities, we can approximately obtain $p > 87.332$ and $-12(\sqrt{p^2 - 90p + 233} - 3p + 15) < q < 12(\sqrt{p^2 - 90p + 233} + 3p - 15)$. This, however, contradicts the condition that $p < 1$. As a result, the zero solution of Eq. (19) is not asymptotically stable. It is clear that from the Eq. (21), the zero solution of the Eq. (19) is not asymptotically stable then the zero solution of (1) is not asymptotically stable. □

Theorem 6. *The zero solution of* (1) *is unstable if the condition*

$$|p-4| - |p - \frac{q}{24}| - |1 - 4p - \frac{11q}{24}| - |-4 + 6p - \frac{11q}{24}| - |6 - 4p - \frac{q}{24}| > 1 \qquad (25)$$

is hold.

Proof. We will apply Theorem 3 to prove this result. In difference equation (19), it's clear that $p_1 = p-4, p_2 = 6-4p-\frac{q}{24}, p_3 = -4+6p-\frac{11q}{24}, p_4 = 1-4p-\frac{11q}{24}$ and $p_5 = p - \frac{q}{24}$. So, under the condition of (25), the inequality (3) is satisfied and the solution c_n of the Eq. (19) is unstable. When the solution of Eq. (19) is unstable it is observed that solution $x(t)$ of (1) is unstable. □

3. Example

Example 1. *Let us consider fourth-order neutral differential equation with piecewise argument*

$$\frac{d^4}{dt^4}\Big(x(t) - x(t-1)\Big) = -x([t-1]), \quad t \geq 0, \tag{26}$$

and initial function

$$\varphi(t) = t, \quad -1 \leq t \leq 0. \tag{27}$$

This initial value problem is a special case of (1)-(2) with $p = -1, q = -1$ and $\varphi(t) = t$. We can obtain corresponding difference equation of Eq. (26) from (19) *as follows:*

$$c_{n+5} - 5c_{n+4} + \frac{241}{24}c_{n+3} - \frac{229}{24}c_{n+2} + \frac{131}{24}c_{n+1} - \frac{23}{24}c_n = 0, \quad n = -1, 0, 1, ... \tag{28}$$

and also, if $\alpha_0 = \beta_0 = \gamma_0 = 0$ is taken in the equations (20), we can write the initial conditions: $c_{-1} = -1, c_0 = 0, c_1 = \frac{25}{24}, c_2 = \frac{8}{3}, c_3 = \frac{3647}{576}$. Thus, the difference equation (28) *has a unique solution c_n. It can be clearly seen that the solution c_n of Eq. (28) is not asymptotically stable. Finally, if the c_n solution is substituted into equation* (21) *for $n = -1, 0, 1, ...$ and the equations* (16) *are used, the $x(t)$ solution of equation* (26) *is found. This solution is not asymptotically stable (See Figure 1).*

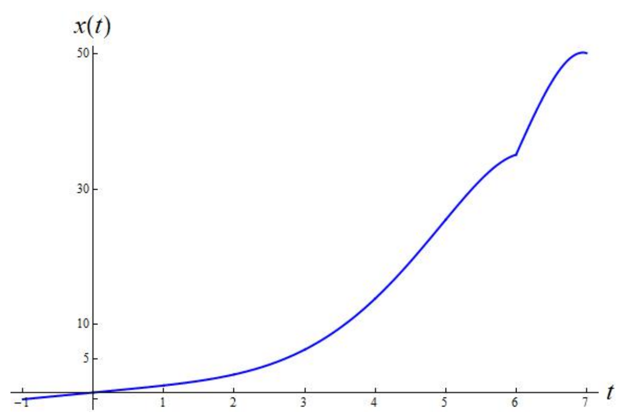

Figure 1. Solution $x(t)$ of initial value problem (26)-(27).

4. Conclusion

In this study, we have investigated a fourth-order neutral differential equation with piecewise constant arguments. Our analysis has focused on demonstrating the existence and uniqueness of solutions for the equation, along with a specified initial condition. Through rigorous mathematical analysis, we have established the conditions necessary for stability in the considered equation. Our findings contribute to the understanding of differential equations with piecewise constant arguments and provide valuable insights into their behavior and stability properties. This work not only enhances theoretical understanding but also offers practical implications for various applications where such equations arise. In this study, we have demonstrated that the zero solution of a fourth-order neutral differential equation with piecewise constant arguments of type (1) is not asymptotically stable. Further research could explore extensions of these results to more complex systems or investigate additional properties of similar equations. Also, these analyses can be made more generalized. Moreover, the oscillation state of the solutions of the equations (1) and (4) can be investigated. This is an open problem.

Declaration of Competing Interests The author declares that they have no competing interest.

Acknowledgement The author is very grateful to the referees whose thoughtful comments and valuable suggestions significantly contributed to the quality of the paper.

References

[1] Abdel-Aty, M., Kavgaci, M. E., Stavroulakis, I.P., Zidan, N., A Survey on sharp oscillation conditions of differential equations with several delays, *Mathematics*, 8(9) 1492, (2020). https://doi.org/10.3390/math8091492

[2] Aftabizadeh, A. R., Wiener, J., Oscillatory and periodic solutions of an equation alternately of retarded and advanced type, *Applicable Analysis*, 23 (1986), 219-231. https://doi.org/10.1080/00036818608839642

[3] Aftabizadeh, A. R., Wiener, J., Xu, J., Oscillatory and periodic solutions of delay differential equations with piecewise constant argument, *Proc. Amer. Math. Soc.*, 99(4) (1987), 673–679. https://doi.org/10.2307/2046474

[4] Agarwal, R .P., Grace, S. R., Asymptotic stability of certain neutral differential equations, *Math. Comput. Model*, 31 (2000), 9–15. https://doi.org/10.1016/S0895-7177(00)00056-X

[5] Akhmet, M. U., Integral manifolds of differential equations with piecewise constant argument of generalized type, *Nonlinear Anal.*, 66 (2) (2007), 367–383. https://doi.org/10.1016/j.na.2005.11.032

[6] Akhmet, M. U., On the reduction principle for differential equations with piecewise constant argument of generalized type, *J. Math. Anal. Appl.*, 336(1) (2007), 646–663. https://doi.org/10.1016/j.jmaa.2007.03.010

[7] Bereketoglu, H., Seyhan, G., Ogun, A., Advanced impulsive differential equations with piecewise constant arguments, *Math. Model. Anal.*, 15(2) (2010), 175–187. https://doi.org/10.3846/1392-6292.2010.15.175-187

[8] Bereketoglu, H., Seyhan, G., Karakoc, F., On a second order differential equation with piecewise constant mixed arguments, *Carpathian Journal of Mathematics,* 27(1) (2011), 1–12. http://www.jstor.org/stable/43997668

[9] Bereketoglu, H., Oztepe, G. S., Convergence of the solution of an impulsive differential equation with piecewise constant arguments, *Miskolc Math. Notes*, 14(3) (2013), 801-815. https://doi.org/10.18514/MMN.2013.595

[10] Bereketoglu, H., Oztepe, G. S., Asymptotic constancy for impulsive differential equations with piecewise constant argument, *Bull. Math. Soc. Sci. Math.*, 57 (2014), 181–192. https://www.jstor.org/stable/43678896

[11] Berezansky, L., Braverman, E., Pinelas, S., Exponentially decaying solutions for models with delayed and advanced arguments: Nonlinear effects in linear differential equations, *Proc. Amer. Math. Soc.*, 151 (2023), 4261-4277. https://doi.org/10.1090/proc/16383

[12] Busenberg, S., Cooke, K. L., Models of Vertically Transmitted Diseases with Sequential-Continuous Dynamics, Nonlinear Phenomena in Mathematical Sciences, Academic Press, New York, 1982, 179–187. https://doi.org/10.1016/B978-0-12-434170-8.50028-5

[13] Chiu, K. S., Periodic solutions of impulsive differential equations with piecewise alternately advanced and retarded argument of generalized type, *Rocky Mountain J. Math.*, 52(1) (2022), 87 - 103. https://doi.org/10.1216/rmj.2022.52.87

[14] Chiu, K. S., Sepulveda, I. B., Nonautonomous impulsive differential equations of alternately advanced and retarded type, *Filomat* 37(23) (2023), 7813–7829. https://doi.org/10.2298/FIL2323813C

[15] Chiu, K. S., Pinto, M., Periodic solutions of differential equations with a general piecewise constant argument and applications, *Electron. J. Qual. Theory Differ. Equ.*, 2010(46) (2010), 1–19.

[16] Chiu, K. S., Li, T., Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments, *Math. Nachr.*, 292 (2019), 2153–2164. https://doi.org/10.1002/mana.201800053

[17] Cooke, K. L., Wiener, J., Retarded differential equations with piecewise constant delays, *J. Math. Anal. Appl.*, 99 (1984), 265–297.

[18] Cooke, K. L., Wiener, J., An equation alternately of retarded and advanced type, *Proc. Amer. Math. Soc.*, 99 (1987), 726-732. https://doi.org/10.1090/S0002-9939-1987-0877047-8

[19] Dai, L., Singh, M. C., On oscillatory motion of spring-mass systems subjected to piecewise constant forces, *J. Sound Vib.*, 173 (1994), 217–233. https://doi.org/10.1006/jsvi.1994.1227

[20] Dai, L., Singh, M.C., An analytical and numerical method for solving linear and nonlinear vibration problems, *Int. J. Solids Struct.*, 34 (1997), 2709–2731. https://doi.org/10.1016/S0020-7683(96)00169-2

[21] Elaydi, S.; An Introduction to Difference Equation, Springer, New York, 2015.

[22] Gyori, I., Ladas, G., Linearized oscillations for equations with piecewise constant argument, *Differential and Integral Equations*, 2 (1989), 123-131.

[23] Gyori, I., On approximation of the solutions of delay differential equations by using piecewise constant arguments, *Int. J. Math. Math. Sci.*, 14 (1991), 111–126.

[24] Huang, Y. K., Oscillations and asymptotic stability of solutions of first-order neutral differential equations with piecewise constant argument, *J. Math. Anal. Appl.*, 149(1) (1990), 70-85. https://doi.org/10.1016/0022-247X(90)90286-O

[25] Jury, E. I., Theory and Application of the Z-Transform Method Wiley, New York, 1964.

[26] Kavgaci, M.E., Al Obaidi, H., Bereketoglu, H., Some results on a first-order neutral differential equation with piecewise constant mixed arguments, *Period Math Hung.,* 87 (2023), 265–277. https://doi.org/10.1007/s10998-022-00512-3

[27] Muminov, M. I., On the method of finding periodic solutions of second-order neutral differential equations with piecewise constant arguments, *Adv. Difference Equ.*, 2017(336) (2017). doi: 10.1186/s13662-017-1396-7

[28] Muminov, M. I., Murid H. M. A., Existence conditions for periodic solutions of second-order neutral delay differential equations with piecewise constant arguments, *Open Math.*, 18(1) (2020), 93-105. https://doi.org/10.1515/math-2020-0010

[29] Muminov, M. I., Radjabov, T. A., Existence conditions for 2-periodic solutions to a non-homogeneous differential equations with piecewise constant argument, *Examples and Counterexamples*, 5 (2024), 100145. https://doi.org/10.1016/j.exco.2024.100145

[30] Muroya, Y., New contractivity condition in a population model with piecewise constant arguments, *J. Math. Anal. Appl.*, 346 (1) (2008), 65-81. https://doi.org/10.1016/j.jmaa.2008.05.025

[31] Papaschinopoulos, G., Schinas, J., Existence stability and oscillation of the solutions of first-order neutral delay differential equations with piecewise constant argument, *Appl. Anal.*, 44(1-2) (1992), 99-111. http://dx.doi.org/10.1080/00036819208840070

[32] Papaschinopoulos, G., Schinas, J., Some results concerning second and third order neutral delay differential equations with piecewise constant argument, *Czechoslovak Math. J.*, 44 (119) (1994), 501–512.

[33] Papaschinopoulos, G., On a class of third order neutral delay differential equations with piecewise constant argument, *Internat. J. Math. Math. Sci.*, 17 (1994), 113-118. https://doi.org/10.1155/S0161171294000153.

[34] Partheniadis, E. C., Stability and oscillation of neutral delay differential equations with piecewise constant argument, *Differential Integral Equations*, 1 (4) (1988), 459 - 472. https://doi.org/10.57262/die/1372451948

[35] Pinto, M., Asymptotic equivalence of nonlinear and quasilinear differential equations with piecewise constant arguments, *Math. Comput. Modelling*, 49(9-10), (2009), 1750-1758. https://doi.org/10.1016/j.mcm.2008.10.001

[36] Shah, S. M., Wiener, J., Advanced differential equations with piecewise constant argument deviations, *Int. J. Math. Math. Sci.*, 6 (1983), 671-703.

[37] Shen, J.H., Stavroulakis, I.P., Oscillatory and nonoscillatory delay equations with piecewise constant argument, *J. Math. Anal. Appl.*, 248 (2000), 385–401. https://doi.org/10.1006/jmaa.2000.6908

[38] Wang, G. Q., Periodic solutions of a neutral differential equation with piecewise constant arguments, *J. Math. Anal. Appl.*, 326 (2007), 736–747. https://doi.org/10.1016/j.jmaa.2006.02.093