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The Concept of Synchronization from the Observer's Viewpoint

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Özet. Bu makalede, sürekli zaman dinamik sistemleri için eşzamanlama kavramı bir gözlemci bakış açısından incelenmiştir. Bu kavramın eşzamanlama kavramının bir genellemesi olduğu ispatlanmıştır. İçinde iki dinamik sistemin bağıl olasılık eşzamanlandığı kümenin noktalarının geleceğinin bir bağıl olasılık senkronizasyonu tarafından belirlenen homeomorfizme göre aynı olduğu ispatlanmıştır. Bağıl olasılık eşzamanlamasının topolojik eşlenim bağıntısı altında kararlı olduğu sonucuna varılmıştır.†

Anahtar Kelimeler. Eşzamanlama, bağıl olasılık ölçümü, gözlemci, bağıl eşlenim.

Abstract. In this paper the concept of synchronization for continuous time dynamical systems from the viewpoint of an observer is considered. It is proved that: this concept is a generalization of the notion of synchronization. It is proved that the future of the points of the set in which two dynamical systems are relative probability synchronized is the same up to the homeomorphism determined by a relative probability synchronization. The persistence of relative probability synchronization under a topological conjugate relation is deduced.

Keywords. Synchronization, relative probability measure, observer, relative conjugacy.

1. Introduction

The intersection between dynamical systems [1], differential equations [2, 3] and measure theory [4, 5] has been one of the interesting research topics raised in the last decade. The concept of synchronization is placed in this intersection. Synchronization is one of the main tools for considering chaotic systems. Recently this topic has been considered via fuzzy theory [6]. In this paper we present a new approach to this concept. One of the main objects in physical phenomena is the "observer".

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A modeling for an observer of a set X is a fuzzy set $\mu : X \to [0,1]$ [7, 8]. In fact these kinds of fuzzy sets are called "one dimensional observers". Although mathematically in the definition of fuzzy sets we can replace the interval $[0,1]$ with the other kinds of latices, physically these are not the same. One must pay attention to the point that an observer is not a stochastic observer, because we do not have any probability measure on the space.

In this paper we would like to use of the notion of observer to define *the relative probability synchronization* for topological dynamical systems.

A similarity for the omega limit sets of the points of a set in which two dynamical systems are relative probability synchronized is deduced (see Theorem 2.3). We will prove that: relative probability synchronization is an equivalence relation up to the suitable observers and the semi-dynamics of the spaces. We will also show that topological conjugacy preserves this notion.

2. Relative Probability Synchronization

We assume that X and Y are two metric spaces, and μ is a one dimensional observer of X [7, 8], that is, $\mu : X \to [0,1]$ is a fuzzy set [9]. We also assume that $\{\varphi^t : X \to X \mid t \in \mathbb{R}\}$ is a topological dynamical system on X, with continuous time, that is,

(i) φ^0 is the identity map;
(ii) $\varphi^t \circ \varphi^s = \varphi^{t+s}$, for all $t, s \in \mathbb{R}$;
(iii) $\varphi^t : X \to X$ is a continuous map, for all $t \in \mathbb{R}$.

Moreover we assume that $f : X \to X$ is a continuous map. This map creates the new observers for X. In fact if $E \subseteq X$, then $m_\mu^f(E) : X \longrightarrow [0,1]$ is a fuzzy set defined by

$$m_\mu^f(E)(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(f^i(x))\mu(f^i(x)),$$

where χ_E is the characteristic function of E. The observer $m_\mu^f(E)$ is called the relative probability measure of E with respect to an observer μ and the topological semi-dynamics f [5].

If we restrict ourself to a probability space $(X, \mathcal{B}, m)$ and we take the characteristic function χ_X as an observer, and if we assume that $f : X \longrightarrow X$ is a measure preserving map, then for given $x \in X$, and $E \in \mathcal{B}$ the Birkhoff ergodic theorem [10] implies that

$$m_\mu^f(E)(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(f^i(x)) = m(E) \quad \text{almost every where.}$$

So relative probability measure is an extension of the notion of probability measure.

Definition 2.1. A topological dynamical system $\{\varphi^t : X \to X \mid t \in \mathbb{R}\}$ is relative probability synchronized to a topological dynamical system $\{\psi^t : Y \to Y \mid t \in \mathbb{R}\}$ up to a topological semi-dynamical system $f : X \to X$ and an observer μ if there exists a homeomorphism $r : X \to Y$ such that $m_\mu^f(Z) = m_\mu^f(X)$, where $Z = \{x \in X \mid \lim_{t\to\infty} d_X(\varphi^t(x), (r^{-1} \circ \psi^t \circ r)(x)) = 0\}$. In this case we use of the notation $(X, \varphi)_{f,r}^\mu \simeq (Y, \psi)$.

Now we show that: the notion of relative probability synchronization is a generalization of the notion of synchronization. For this purpose we assume that (X, β, m) and (Y, η, n) are two probability spaces with the probability measures m and n and sigma algebras β and η respectively. Moreover we assume that X and Y are two metric spaces.

Theorem 2.1. *Suppose $(X, \varphi)_{f,r}^{\chi_X} \simeq (Y, \psi)$ and $f : X \to X$ is a measure preserving homeomorphism. Moreover assume that $Z \in \beta$ where*

$$Z = \left\{ x \in X \;\middle|\; \lim_{t\to\infty} d_X(\varphi^t(x), (r^{-1} \circ \psi^t \circ r)(x)) = 0 \right\},$$

then $m(Z) = m(X)$ for almost all points of X.

Proof. The Birkhoff ergodic theorem [11] implies that $m_{\chi_X}^f(Z)(x) = m(Z)$ almost everywhere and $m_{\chi_X}^f(Z)(x) = m(X)$ almost everywhere, where $x \in X$. Since $m_{\chi_X}^f(Z)(x) = m_{\chi_X}^f(X)(x)$ for all $x \in X$, then $m(Z) = m(X)$ almost everywhere. □

Theorem 2.2. *If A and B are two disjoint subsets of X, then*

$$m_\mu^f(A \cup B) = m_\mu^f(A) + m_\mu^f(B).$$

Proof. Let $x \in X$ be given. Then

$$\begin{aligned} m_\mu^f(A \cup B)(x) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{A\cup B}(f^i(x))\mu(f^i(x)) \\ &= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (\chi_A + \chi_B)(f^i(x))\mu(f^i(x)) \\ &= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(f^i(x))\mu(f^i(x)) + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_B(f^i(x))\mu(f^i(x)) \\ &= m_\mu^f(A)(x) + m_\mu^f(B)(x). \end{aligned}$$

□

We denote the omega limit set of a point $x \in X$ for a dynamical system $\{\varphi^t : X \to X \mid t \in \mathbb{R}\}$ by $\omega^\varphi(x)$.

Theorem 2.3. *Let $(X,\varphi)^\mu_{f,r} \simeq (Y,\psi)$. Moreover let $x \in Z$. Then $\omega^\varphi(x) = r^{-1}(\omega^\psi(r(x)))$.*

Proof. If $q \in \omega^\varphi(x)$, then there is a sequence $\{t_n\}$ with $t_n \to \infty$ such that $\lim_{t_n\to\infty} \varphi^{t_n}(x) = q$. Since $x \in Z$, then

$$\begin{aligned} &\lim_{t_n\to\infty} d_X(q, (r^{-1} \circ \psi^{t_n} \circ r)(x)) \\ &\qquad \le \lim_{t_n\to\infty} d_X(\varphi^{t_n}(x), (r^{-1} \circ \psi^{t_n} \circ r)(x)) + \lim_{t_n\to\infty} d_X(\varphi^{t_n}(x), q) = 0. \end{aligned}$$

So

$$d_X(q, (r^{-1} \circ \psi^{t_n} \circ r)(x)) = 0.$$

Since r is a homeomorphism, then

$$d_X(r(q), (\psi^{t_n} \circ r)(x)) = 0.$$

Thus $r(q) \in \omega^\psi(r(x))$. So $\omega^\varphi(x) \subseteq r^{-1}(\omega^\psi(r(x)))$. By the similar calculations we can deduce $r^{-1}(\omega^\psi(r(x))) \subseteq \omega^\varphi(x)$. Thus $\omega^\varphi(x) = r^{-1}(\omega^\psi(r(x)))$. □

Theorem 2.4. *Let $(X,\varphi)^\mu_{f,r} \simeq (Y,\psi)$, and $\omega^\varphi(p) \cap Z = (r^{-1}(\omega^\psi(r(p)))) \cap Z$, for a given $p \in X$. Then $m_\mu^f(\omega^\varphi(p)) = m_\mu^f(r^{-1}(\omega^\psi(r(p))))$.*

Proof. Since $(X,\varphi)^\mu_{f,r} \simeq (Y,\psi)$, then

$$m_\mu^f(Z) = m_\mu^f(X) = m_\mu^f(X \cap Z) + m_\mu^f(X \cap Z^c) = m_\mu^f(Z) + m_\mu^f(Z^c).$$

So $m_\mu^f(Z^c) = 0$. We have

$$\begin{aligned} m_\mu^f(\omega^\varphi(p)) &= m_\mu^f(\omega^\varphi(p) \cap Z) + m_\mu^f(\omega^\varphi(p) \cap Z^c) \\ &= m_\mu^f(\omega^\varphi(p) \cap Z) + 0 \\ &= m_\mu^f(r^{-1}(\omega^\psi(r(p))) \cap Z) + m_\mu^f(r^{-1}(\omega^\psi(r(p))) \cap Z^c) \\ &= m_\mu^f(r^{-1}(\omega^\psi(r(p)))). \end{aligned}$$

□

3. Equivalence Relation and Conjugacy

Now we prove that the relative probability synchronization is an equivalence relation.

Theorem 3.1. (i) *Let* $(X,\varphi)_{f,r}^\mu \simeq (Y,\psi)$. *Then the dynamical system* $(Y, \{\psi^t \mid t \in \mathbb{R}\})$ *is relative probability synchronized to* $(X, \{\varphi^t \mid t \in \mathbb{R}\})$ *up to the topological semi-dynamical system* $r \circ f \circ r^{-1}$ *and the observer* $\mu \circ r^{-1}$.

(ii) *Let* $(X,\varphi)_{f,r}^\mu \simeq (Y,\psi)$ *and* $(Y,\psi)_{r\circ f \circ r^{-1},s}^{\mu\circ r^{-1}} \simeq (T,\rho)$. *Then*

$$(X,\varphi)_{s\circ r\circ f\circ r^{-1}\circ s^{-1}, s\circ r}^{\mu\circ r^{-1}\circ s^{-1}} \simeq (T,\rho).$$

Proof. (i) The set $\{y \in Y \mid \lim_{t\to\infty} d_Y(\psi^t(y), (r\circ\varphi^t\circ r^{-1})(y)) = 0\}$ is equal to $r(Z)$. Let $y \in r(Z)$ and $y = r(x)$. Then

$$\begin{aligned} m_{\mu\circ r^{-1}}^{r\circ f\circ r^{-1}}(r(Z))(y) &= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_{r(Z)}(r\circ f^i\circ r^{-1}(y))\mu\circ r^{-1}(r\circ f^i \circ r^{-1}(y)) \\ &= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_{r(Z)}(r\circ f^i)(x))\mu(r\circ f^i(x)) \\ &= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_Z(f^i(x))\mu(f^i(x)) = m_\mu^f(Z)(x). \end{aligned}$$

Similarly we deduce

$$m_{\mu\circ r^{-1}}^{r\circ f\circ r^{-1}}(r(X))(y) = m_\mu^f(X)(x).$$

Thus

$$m_{\mu\circ r^{-1}}^{r\circ f\circ r^{-1}}(r(Z))(y) = m_{\mu\circ r^{-1}}^{r\circ f\circ r^{-1}}(Y)(y).$$

(ii) If $Z = \{x \in X \mid \lim_{t\to\infty} d_X(\varphi^t(x), r^{-1}\circ\psi^t\circ r(x)) = 0\}$ then

$$s\circ r(Z) = \left\{t \in T \;\middle|\; \lim_{t\to\infty} d_T(\rho^t(u), r^{-1}\circ s^{-1}\circ\rho^t\circ s\circ r(u)) = 0\right\}.$$

For $u \in (s \circ r)(Z)$ and $x = r^{-1} \circ s^{-1}(u)$ we have

$$
\begin{aligned}
& m_{\mu \circ r^{-1} \circ s^{-1}}^{s \circ r \circ f \circ r^{-1} \circ s^{-1}}(s \circ r)(Z)(u) \\
&= \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{(s \circ r)(Z)}(s \circ r \circ f^i \circ r^{-1} \circ s^{-1}(u)) \mu \circ r^{-1} \circ s^{-1}(s \circ r \circ f^i \circ r^{-1} \circ s^{-1})(u)) \\
&= \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_Z(f^i(x)) \mu(f^i(x)) = m_\mu^f(Z)(x).
\end{aligned}
$$

We also have

$$m_{\mu \circ r^{-1} \circ s^{-1}}^{s \circ r \circ f \circ r^{-1} \circ s^{-1}}(s \circ r)(X)(u) = m_\mu^f(Z)(x).$$

Hence the equality

$$m_\mu^f(Z)(x) = m_\mu^f(X)(x)$$

implies that

$$m_{\mu \circ r^{-1} \circ s^{-1}}^{s \circ r \circ f \circ r^{-1} \circ s^{-1}}(T)(u) = m_{\mu \circ r^{-1} \circ s^{-1}}^{s \circ r \circ f \circ r^{-1} \circ s^{-1}}(s \circ r)(Z)(u).$$

Thus

$$(X, \varphi)_{s \circ r \circ f \circ r^{-1} \circ s^{-1}, s \circ r}^{\mu \circ r^{-1} \circ s^{-1}} \simeq (T, \rho).$$

$\square$

Two topological dynamical systems $(X, \{\varphi^t \mid t \in \mathbb{R}\})$ and $(X, \{\psi^t \mid t \in \mathbb{R}\})$ are called topologically conjugate if there exists a homeomorphism $u : X \to X$ such that $u \circ \varphi^t = \psi^t \circ u$. The next theorem implies that if two dynamical systems are topologically conjugate then they are relative probability synchronized.

Theorem 3.2. *Let X be a metric space and $(X, \{\varphi^t \mid t \in \mathbb{R}\})$ and $(X, \{\psi^t \mid t \in \mathbb{R}\})$ be two conjugate dynamical systems on X with the conjugacy $u : X \to X$. Then $(X, \varphi)_{f,u}^\mu \simeq (X, \psi)$.*

Proof. Let

$$Z = \left\{ x \in X \mid \lim_{t \to \infty} d_X(\varphi^t(x), u^{-1} \circ \psi^t \circ u(x)) = 0 \right\}.$$

Since $\varphi^t = u^{-1} \circ \psi^t \circ u$ then $Z = X$. Thus $m_\mu^f(Z) = m_\mu^f(X)$. $\square$

Example 3.1. Two topological dynamical systems on $\mathbb{S}^2$ with the center $(0,0,0)$ where their orbits are illustrated in Figure 1, are not synchronized, and they are not topologically conjugate but they are relative probability synchronized with the identity map $i : \mathbb{S}^2 \to \mathbb{S}^2$ and the observer $\mu : \mathbb{S}^2 \to [0,1]$ defined by

$$\mu(x, y, z) = \begin{cases} z \text{ if } z \geq 0, \\ 0 \text{ otherwise.} \end{cases}$$

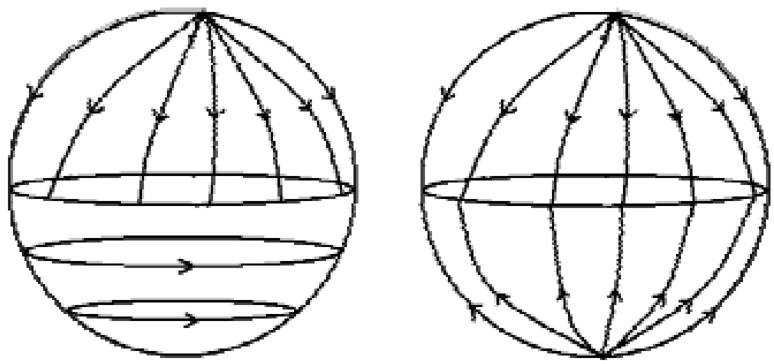

FIGURE 1. This is an example of two relative probability synchronized dynamical systems which are not topologically conjugate.

4. Conclusion

Constant observers appear in many sensitive systems for example in the unified chaotic system [1, 3], which is the following system:

$$\begin{cases} \dot{x} = (25\,\theta + 10)(y - x), \\ \dot{y} = (28 - 35\,\theta)x - xz + (29\,\theta - 1)y, \\ \dot{z} = xy - \left(\dfrac{8+\theta}{3}\right) z. \end{cases}$$

The parameter θ has an essential role in this system for example $\theta = 0$ implies Lorenz system [1, 12], $\theta = 28/35$ implies Lü and Chen system [13] and $\theta = 1$ implies Chen's system [14]. In fact θ is a constant observer on $\mathbb{R}^3$ [15], that is, it is a constant function from $\mathbb{R}^3$ to $[0, 1]$. The more complicated case will happen when we assume that θ is a non-constant observer. The consideration of this system from the viewpoint of relative probability synchronization when $\theta : \mathbb{R}^3 \to [0, 1]$ is an observer is a topic for further research.

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