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Research Article

# Jackknifed estimators for generalized linear models with multicollinearity

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## Abstract

Generalized linear models applications have become very popular in recent years. However, if there is a high degree of relationship between the independent variables, the problem of multicollinearity arises in these models. In this paper, we introduce a new Jackknifed two-parameter estimator and a new modified Jackknifed two-parameter estimator in the case of Poisson, negative binomial and gamma distributed response variables in generalized linear models. We examine bias vectors, covariance matrices, and matrix mean squared error of the Jackknifed ridge estimator, modified Jackknifed ridge estimator, Jackknifed Liu estimator, modified Jackknifed Liu estimator, Jackknifed Liu-type estimator and modified Jackknifed Liu-type estimator given in the literature. According to bias vectors and covariance matrices, the superiority of the Jackknifed two-parameter estimator has been demonstrated theoretically. The generalization of some estimation methods for ridge and Liu parameters in generalized linear models is provided. Also, the superiority of the Jackknifed two-parameter estimator and the modified Jackknifed two-parameter estimator are assessed by the simulated mean squared error via Monte-Carlo simulation study where the response follows a Poisson, negative binomial, and gamma distribution with the log link function. Finally, we consider real data applications. The proposed estimators are compared and interpreted.

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## 1. Introduction

Let consider the general linear regression (LR) model,

$$y = X\beta + e, \tag{1.1}$$

where $y$ is an $n \times 1$ vector of responses, $X$ is an $n \times p$ matrix of the explanatory variables, $\beta$ is a $p \times 1$ vector of unknown regression coefficients and $e$ is an $n \times 1$ vector of error terms with

$$E(e) = 0, Cov(e) = \sigma^2 I_n.$$

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The ordinary least squares (OLS) estimator is often used to estimate regression coefficients

$$\hat{\beta} = (X^\top X)^{-1} X^\top y. \tag{1.2}$$

Multicollinearity, linear or near linear dependency among the explanatory variables, in the regression model is an important problem in applications. If multicollinearity exists, then small relative changes in the matrix $X^\top X$ will produce large relative changes in the matrix $(X^\top X)^{-1}$. Thus, the OLS estimator results in a large variance, and it will not be a precise estimator. Alternative estimators designed to combat multicollinearity yield biased estimators. The most popular estimator for combating multicollinearity is the ridge estimator (RE), originally proposed by [23], the Liu estimator (LE) proposed by [27], the Liu-type estimator (LTE) proposed by [31], the two-parameter estimator (TPE) proposed by [46], which has become more popular recently.

The authors then came across the question as to whether it is possible to find an estimator with a smaller bias. Quenouille [45] and Tukey [53] introduced the application of the Jackknife procedure to a biased estimator to reduce bias. The Jackknife procedure provides an estimator that has not only small bias but also all the desirable large sample properties. The use of the Jackknife procedure to reduce the bias of the RE and the properties of the Jackknifed ridge estimator (JRE) were studied by [20, 26, 40, 43, 44, 51, 52]. Hinkley [22] stated that with few exceptions, the Jackknife had been applied to balanced models. Then he proposed the weighted Jackknife procedure for unbalanced models. An excellent review is given by [35]. Miller [36] gives the first detailed account of Jackknifing LR model estimates show that the Jackknife produces consistent results in large samples. Kadiyala [26] proposed a class of almost unbiased estimators which include a bias-corrected RE as a special case. But, unfortunately, his estimator is not operational, since the bias-corrected term includes unknown parameters. Then, Ohtani [44] replaced the unknown parameters with their REs. Batah et al. [12] proposed a new estimator, namely the modified Jackknifed ridge estimator (MJRE), which combines the ideas underlying both the RE and JRE. Özkale [47] proposed JRE in the presence of a LR model with heteroscedastic and/or correlated error. Erdugan and Akdeniz [18] introduced a new estimator for the JRE parameter, which efficiently combines a graphical analysis and an analytical method borrowed from the generalized maximum entropy estimator from [19]. Khurana et al. [29] proposed different estimators to reduce LTE bias, one using the Jackknife technique and the other using the technique proposed in [26]. They also investigated the Bootstrap method of bias correction on the LTE as well. Moreover, we refer to the following papers in which the authors discuss the performance of Jackknife estimators in different types of generalized linear model (GLM): [2–4, 8, 11, 24, 32, 41, 42, 54, 55].

In addition, Algamal et al. [6] proposed a Jackknifed variant of the ridge estimator to reduce biasedness for the Bell regression model. Rasheed et al. [48] have developed the Jackknife approach and its modified version is proposed to model count data with the Conway-Maxwell Poisson regression model. Both of these two estimators have been suggested to reduce the effects of multicollinearity and the biasedness by using the LTE. Similarly, Algamal et al. [7] suggested the JRE and the modified version of the JRE for the Conway-Maxwell Poisson regression model. Seifollahi et al. [50] introduced the Jackknifed Liu-type estimator (JLTE) and its modified version in the Beta regression model, which demonstrate improved bias reduction compared to the original LTE. Similarly, Abduljabbar and Algamal [1] proposed a Jackknifed version of the Kibria and Lukman estimator in the Bell regression model, which combines the Jackknife process with the Kibria and Lukman estimator to reduce biasedness. Like this, Hamad and Algamal [21] proposed a Jackknifed version of the Kibria and Lukman estimator in the inverse Gaussian regression model that combines the Jackknife procedure with the Kibria and Lukman estimator to reduce the biasedness. Algamal et al. [5] developed the Jackknife beta ridge and the modified Jackknife beta ridge estimator to estimate the regression coefficient when multicollinearity exists efficiently.

It is well known that the performance of estimators can be negatively affected by multicollinearity. The presence of multicollinearity has serious effects on the regression coefficients. As in LR, in GLM, interrelationships cause difficulty in interpreting the estimated regression coefficients. It is observed that the jackknifing procedure has not been applied to TPE in GLMs so far. Therefore, the aim of this article is to propose the Jackknifed two-parameter estimator (JTPE) and modified Jackknifed Liu-type estimator (MJTPE) of the Poisson, negative binomial, and gamma distributed response variable to reduce bias and overcome the multicollinearity problem and to discuss its theoretical properties along with designing Monte Carlo simulations to compare the performance of the new estimator with the existing ones numerically. In this case, the recommended estimator has performed well in both reducing MSE and producing less bias. The advantages of these alternative estimators are that they reduce the effect of linearity and improve the precision and accuracy of the estimated independent variables.

The remainder of the paper is organized as follows: Section 2 describes the Jackknifed estimators in the LR and GLM. In addition, we give details of the JTPE and MJTPE obtained for GLMs. Theoretical properties of the listed estimators are derived, and some theorems are proved to compare these estimators in Section 3. We suggest "ggplots" to choose the best parameters k and d in Section 4. A Monte Carlo simulation is designed to compare the performances of the estimators using different simulated data sets having the collinearity problem in Section 5. Real data applications are considered to illustrate the methods discussed in this paper in Section 6 using the different data sets regarding the Poisson, negative binomial and gamma distributions. Finally, some conclusive remarks are presented in Section 7.

## 2. Methodology

This section describes the RE, LE and LTE for the LR, the Poisson regression (PR), negative binomial regression (NBR) and gamma regression (GR) models.

### 2.1. The linear regression estimators

Firstly, we transform the models (1.1) with (1.2) into the following model (2.1) with (2.2). Consider the LR given in model (1.1), let $Q = (q_1, q_2, , q_p)$ be $pxp$ matrix whose columns are normalized eigen vectors of $X^\top X$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_p)$, such that $X^\top X = Q^\top Q$. The LR model in Equation (1.1) can be written as

$$y = Z\gamma + u, \tag{2.1}$$

where $Z = XQ$ ve $\gamma = Q^\top \beta$. The OLS estimators of $\beta$ and $\gamma$ in Equations (1.1) and (2.1) are given by, respectively

$$\hat{\gamma}_{\text{OLS}} = \Lambda^{-1} Z^\top y, \tag{2.2}$$

$$\hat{\beta}_{\text{OLS}} = Q\hat{\gamma}_{\text{OLS}}. \tag{2.3}$$

Note that, due to the relation $\gamma = Q^\top \beta$, any estimator of $\gamma$ has a corresponding $\hat{\beta} = Q\hat{\gamma}$. Hence, it is sufficient to consider only the canonical form. The RE is proposed by [23] as

$$\hat{\gamma}_{\text{RE}} = (\Lambda + kI)^{-1} Z^\top y = A^{-1} Z^\top y = A^{-1}\Lambda\hat{\gamma}_{\text{OLS}} = (I - kA^{-1})\hat{\gamma}_{\text{OLS}},$$

where $k > 0$ is the ridge parameter and $A = (\Lambda + kI)$. Here, it is indicated that the Jackknifed procedure is applied on a transformed set of regressors.

Recently, Khurana et al. [29] showed that transformation is not required and that it is easy to obtain the estimator for the original regression parameter explicitly. As they stated that the MJRE was introduced for the transformed parameter, use the transformed model in Equation (2.1).

Singh et al. [51] proposed the JRE as

$$\hat{\gamma}_{\mathrm{JRE}} = (I + kA^{-1})\hat{\gamma}_{\mathrm{RE}} = (I - k^2A^{-2})\hat{\gamma}_{\mathrm{OLS}},$$

and found that the bias of the Jackknifed procedure of $\hat{\gamma}_{\mathrm{JRE}}$ has smaller than the bias of $\hat{\gamma}_{\mathrm{RE}}$. Then, Jadhav and Kashid [25] obtained the MMSE of the JRE as

$$\begin{aligned}\mathrm{MMSE}(\hat{\gamma}_{\mathrm{JRE}}) &= \mathrm{Var}(\hat{\gamma}_{\mathrm{JRE}}) + \mathrm{Bias}(\hat{\gamma}_{\mathrm{JRE}})(\mathrm{Bias}(\hat{\gamma}_{\mathrm{JRE}}))^\top \\ &= \sigma^2(I - k^2A^{-2})\Lambda^{-1}(I - k^2A^{-2}) + k^4A^{-2}\gamma\gamma^\top A^{-2}.\end{aligned}$$

The MJRE is proposed by [12] as

$$\begin{aligned}\hat{\gamma}_{\mathrm{MJRE}} &= (I - k^2A^{-2})\hat{\gamma}_{\mathrm{RE}} \\ &= (I - k^2A^{-2})(I - kA^{-1})\hat{\gamma}_{\mathrm{OLS}}.\end{aligned}$$

The MMSE of the MJRE is given by

$$\mathrm{MMSE}(\hat{\gamma}_{\mathrm{MJRE}}) = \sigma^2(I-k^2A^{-2})(I-kA^{-1})\Lambda^{-1}(I-kA^{-1})(I-k^2A^{-2})+k^2w_rA^{-1}\gamma\gamma^\top A^{-1}w_r,$$

where $w_r = (I + kA^{-1} - k^2A^{-2})$.

The LE proposed by [27], which Jackknifed procedure is applied on a transformed set of regressors is defined as

$$\hat{\gamma}_{\mathrm{LE}} = (I - (1 - d)B^{-1})\hat{\gamma}_{\mathrm{OLS}},$$

where $0 < d < 1$ shrinkage parameter and $B = (\Lambda + I)$.

Akdeniz Duran and Akdeniz [2] proposed the Jackknifed Liu estimator (JLE) as

$$\hat{\gamma}_{\mathrm{JLE}} = (I - (1 - d)^2B^{-2})\hat{\gamma}_{\mathrm{OLS}}.$$

Then, they obtained the MMSE of the JLE as

$$\mathrm{MMSE}(\hat{\gamma}_{\mathrm{JLE}}) = \sigma^2(I - (1 - d)^2B^{-2})\Lambda^{-1}(I - (1 - d)^2B^{-2}) + (1 - d)^4B^{-2}\gamma\gamma^\top B^{-2}.$$

The modified Jackknifed Liu estimator (MJLE) are proposed by [2] as

$$\hat{\gamma}_{\mathrm{MJLE}} = (I - (1 - d)^2B^{-2})(I - (1 - d)B^{-1})\hat{\gamma}_{\mathrm{OLS}}.$$

The MMSE of the MJLE is given by

$$\begin{aligned}\mathrm{MMSE}(\hat{\gamma}_{\mathrm{MJLE}}) =&\sigma^2(I - (1 - d)^2B^{-2})(I - (1 - d)B^{-1})\Lambda^{-1} \\ &\mathrm{x}(I - (1 - d)B^{-1})(I - (1 - d)^2B^{-2}) + (1 - d)^2w_lB^{-1}\gamma\gamma^\top B^{-1}w_l,\end{aligned}$$

where $w_l = (I + (1 - d)B^{-1} - (1 - d)^2B^{-2})$.

The transformation set of regressors for Jackknifed procedure which LTE proposed by [31] is defined as follows

$$\hat{\gamma}_{\mathrm{LTE}} = (I - (k + d)A^{-1})\hat{\gamma}_{\mathrm{OLS}},$$

where $-\infty < d < \infty$ and $k \geq 0$.

Alkhateeb and Algamal [8] proposed JLTE and the MMSE of the JLTE as follows respectively

$$\hat{\gamma}_{\mathrm{JLTE}} = (I - (k + d)^2A^{-2})\hat{\gamma}_{\mathrm{OLS}},$$

$$\mathrm{MMSE}(\hat{\gamma}_{\mathrm{JLTE}}) = \sigma^2(I - (k + d)^2A^{-2})\Lambda^{-1}(I - (k + d)^2A^{-2}) + (k + d)^4A^{-2}\gamma\gamma^\top A^{-2}.$$

Then, they are proposed the modified Jackknifed Liu-type estimator (MJLTE) as

$$\hat{\gamma}_{\mathrm{MJLTE}} = (I - (k + d)^2A^{-2})(I - (k + d)A^{-1})\hat{\gamma}_{\mathrm{OLS}}.$$

The MMSE of the MJLTE is given by

$$\begin{aligned}\mathrm{MMSE}(\hat{\gamma}_{\mathrm{MJLTE}}) =&\sigma^2(I - (k + d)^2A^{-2})(I - (k + d)A^{-1})\Lambda^{-1} \\ &\mathrm{x}(I - (k + d)A^{-1})(I - (k + d)^2A^{-2}) + (k + d)^2w_{lt}A^{-1}\gamma\gamma^\top A^{-1}w_{lt},\end{aligned}$$

where $w_{lt} = (I + (k + d)A^{-1} - (k + d)^2 A^{-2})$.

## 2.2. The generalized linear model estimator and the MSE properties of the estimators

In this section, bias vectors, covariance matrices, and MMSEs of the listed estimators are derived. Before starting to derive these functions, following [2,8,54,55], the canonical form of the model and the estimators are obtained. For this purpose, consider the transformation $G^\top X^\top \hat{W} X G = Z^\top \hat{W} Z = \Lambda_{ML}$ and $Z = XG$ where $\Lambda_{ML} = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_p)$ such that $G$ is a $pxp$ dimensional orthogonal matrix whose columns are the normalized eigenvectors corresponding to the ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p > 0$ of the matrix $X^\top \hat{W} X$. The GLMs of $\hat{\beta}_{ML}$ can be written as

$$\hat{\beta}_{ML} = (X^\top \hat{W} X)^{-1} X^\top \hat{W} \hat{z},$$

where $\hat{z}$ is a vector of adjusted response such that $\hat{z} = \eta_i + (y - \mu)/\mu$ where $\mu = \exp(X\hat{\beta}_{ML})$, $\eta = \hat{X}\hat{\beta}_{ML}$ and $\hat{W}$, $\mathrm{diag}(\mu_i)$ for PR, $\mathrm{diag}(\mu_i/(1+\alpha\mu_i))$ for NBR, $\mathrm{diag}[(\mu_i)^2]$ for GR models. The maximum likelihood estimator (MLE) of $\beta$ is asymptotically normally distributed with a covariance matrix

$$\mathrm{COV}(\hat{\beta}_{ML}) = (X^\top \hat{W} X)^{-1}. \tag{2.4}$$

Thus, the vector of coefficients of the Jackknifed models can be expressed as $\hat{\alpha}_{\mathrm{ML}} = G^\top \hat{\beta}_{ML}$. Therefore, we handle the canonical forms of JRE, MJRE, JLE, MJLE, JLTE and MJLTE.

The JRE and the MJRE proposed by [54,55] in the PR and NBR as

$$\hat{\alpha}_{\mathrm{JRE}} = (I + kC^{-1})\hat{\alpha}_{\mathrm{RE}} = (I - k^2 C^{-2})\hat{\alpha}_{\mathrm{ML}},$$

where $C = (\Lambda_{\mathrm{ML}} + kI) = (Z^\top \hat{W} Z + kI)$ and $k > 0$. This estimator, as an almost unbiased ridge estimator, was addressed in the GR by [10] and in the NBR by [39].

Moreover, the MJRE proposed by [54,55] in the PR and NBR as

$$\hat{\alpha}_{\mathrm{MJRE}} = (I - k^2 C^{-2})(I - kC^{-1})\hat{\alpha}_{\mathrm{ML}}.$$

Akdeniz Duran and Akdeniz [2] proposed the JLE and the MJLE as follows respectively

$$\hat{\alpha}_{\mathrm{JLE}} = (\Lambda_{\mathrm{ML}} + I)^{-1}(Z^\top \hat{W}\hat{s} + d\hat{\alpha}_{\mathrm{ML}}) = (I - (1-d)^2 D^{-2})\hat{\alpha}_{\mathrm{ML}},$$

$$\hat{\alpha}_{\mathrm{MJLE}} = (I - (1-d)^2 D^{-2})(I - (1-d)D^{-1})\hat{\alpha}_{\mathrm{ML}},$$

where $D = (\Lambda_{\mathrm{ML}} + I)$ and $0 < d < 1$.

The JLTE in the PR proposed by [8] as follows:

$$\hat{\alpha}_{\mathrm{JLTE}} = (I + (k + d)C^{-1})\hat{\alpha}_{\mathrm{LTE}} = (I - (k + d)^2 C^{-2})\hat{\alpha}_{\mathrm{ML}},$$

where $k > 0$ and $0 < d < 1$.

From here, we obtain the MJLTE as

$$\hat{\alpha}_{\mathrm{MJLTE}} = (I - (k + d)^2 C^{-2})(I - (k + d)C^{-1})\hat{\alpha}_{\mathrm{ML}},$$

where $k > 0$ and $0 < d < 1$.

In order to obtain the MMSE functions of the estimators, both the bias vectors and the covariance matrices of the estimators are needed. It is known that $\hat{\alpha}_{\mathrm{ML}}$ is asymptotically unbiased and its covariance is obtained by Equation (2.4) as $\mathrm{COV}(\hat{\beta}_{ML}) = \Lambda_{ML}^{-1}$. Note that the matrix $\Lambda_{ML}^{-1}$ is positive definite, since all eigenvalues are assumed to be positive. Now, for GLMs the bias vectors and the covariance matrices of estimators are obtained

$$\mathrm{bias}(\hat{\alpha}_{\mathrm{JRE}}) = -k^2 C^{-2}\alpha,$$

$$\mathrm{COV}(\hat{\alpha}_{\mathrm{JRE}}) = \phi(I - k^2 C^{-2})\Lambda_{ML}^{-1}(I - k^2 C^{-2}),$$

$$\mathrm{bias}(\hat{\alpha}_{\mathrm{MJRE}}) = -kC^{-1} w_R \alpha,$$

$$\mathrm{COV}(\hat{\alpha}_{\mathrm{MJRE}}) = \phi(2I - R_k)R_k^2 \Lambda_{ML}^{-1}[(2I - R_k)R_k^2]^\top,$$

where $w_R = (I + kC^{-1} - k^2C^{-2})$, $R_k = I - kC^{-1}$ and $\phi = (n-p)^{-1}\sum_{j=1}^{p}(y_i - \hat{\mu_i})^2/\hat{\mu_i}^2$.

$$\text{bias}(\hat{\alpha}_{\text{JLE}}) = -(1-d)^2D^{-2}\alpha,$$

$$\text{COV}(\hat{\alpha}_{\text{JLE}}) = \phi(I - (1-d)^2D^{-2})\Lambda_{ML}^{-1}(I - (1-d)^2D^{-2}),$$

$$\text{bias}(\hat{\alpha}_{\text{MJLE}}) = -(1-d)D^{-1}w_L\alpha,$$

$$\text{COV}(\hat{\alpha}_{\text{MJLE}}) = \phi(2I - F_d)F_d^2\Lambda_{ML}^{-1}[(2I - F_d)F_d^2]^\top,$$

where $w_L = (I + (1-d)D^{-1} - (1-d)^2D^{-2})$ and $F_d = I - (1-d)D^{-1}$.

$$\text{bias}(\hat{\alpha}_{\text{JLTE}}) = -(k+d)^2C^{-2}\alpha,$$

$$\text{COV}(\hat{\alpha}_{\text{JLTE}}) = \phi(I - (k+d)^2C^{-2})\Lambda_{ML}^{-1}(I - (k+d)^2C^{-2}),$$

$$\text{bias}(\hat{\alpha}_{\text{MJLTE}}) = -(k+d)C^{-1}w_{LT}\alpha,$$

$$\text{COV}(\hat{\alpha}_{\text{MJLTE}}) = \phi(2I - H_d)H_d^2\Lambda_{ML}^{-1}[(2I - H_d)H_d^2]^\top,$$

where $w_{LT} = (I + (k+d)C^{-1} - (k+d)^2C^{-2})$ and $H_d = I - (k+d)C^{-1}$.

MMSE of the MLE is given as

$$\text{MMSE}(\hat{\alpha}_{\text{ML}}) = \phi\Lambda_{ML}^{-1},$$

Thus, using both the bias vectors and the covariance matrices, MMSEs of estimators are obtained respectively as

$$\text{MMSE}(\hat{\alpha}_{\text{JRE}}) = \phi(I - k^2C^{-2})\Lambda_{ML}^{-1}(I - k^2C^{-2}) + k^4C^{-2}\alpha\alpha^\top C^{-2},$$

$$\text{MMSE}(\hat{\alpha}_{MJRE}) = \phi(I - k^2C^{-2})(I - kC^{-1})\Lambda_{ML}^{-1}(I - kC^{-1})(I - k^2C^{-2}) + k^2w_RC^{-1}\alpha\alpha^\top C^{-1}w_R,$$

$$\text{MMSE}(\hat{\alpha}_{\text{JLE}}) = \phi(I - (1-d)^2D^{-2})\Lambda_{ML}^{-1}(I - (1-d)^2D^{-2}) + (1-d)^4D^{-2}\alpha\alpha^\top D^{-2},$$

$$\begin{aligned}\text{MMSE}(\hat{\alpha}_{\text{MJLE}}) =& \phi(I - (1-d)^2D^{-2})(I - (1-d)D^{-1})\Lambda_{ML}^{-1} \\ & \text{x}(I - (1-d)D^{-1})(I - (1-d)^2D^{-2}) + (1-d)^2w_LD^{-1}\alpha\alpha^\top D^{-1}w_L,\end{aligned}$$

$$\text{MSE}(\hat{\alpha}_{\text{JLTE}}) = \phi(I - (k+d)^2C^{-2})\Lambda_{ML}^{-1}(I - (k+d)^2C^{-2}) + (k+d)^4C^{-2}\alpha\alpha^\top C^{-2},$$

$$\begin{aligned}\text{MSE}(\hat{\alpha}_{\text{MJLTE}}) =& \phi(I - (k+d)^2C^{-2})(I - (k+d)C^{-1})\Lambda_{ML}^{-1} \\ & \text{x}(I - (k+d)C^{-1})(I - (k+d)^2C^{-2}) + (k+d)^2w_{LT}C^{-1}\alpha\alpha^\top C^{-1}w_{LT}.\end{aligned}$$

**2.2.1. Proposed Jackknifed two-parameter estimators.** In this study, the JTPE and the MJTPE were derived from the GLMs to cope with the multicollinearity problem and to reduce biasing of the TPE. We improve a new TPE for the PR, NBR and GR models by the following studies of [46] TPE as follows:

$$\hat{\beta}_{k,d} = (X^\top \hat{W} X + kI)^{-1}(X^\top \hat{W} X + kdI)\hat{\beta}_{ML},$$

where $k > 0$ and $0 < d < 1$.

The Jackknifed method on an estimator is applied a removal systematically of each one observation from a data set and calculating of the estimate, then finding average of these. To apply this idea, let $s_{-i}$, $Z_{-i}$ and $W_{[-i]}$ denote, respectively, the vector $s$ with its $i$th row deleted, the matrix $Z$ with the $i$th row deleted, and the matrix $W$ with the $i$th row and column deleted. The TPE in the GLMs with the $i$th observation deleted is given by

$$\hat{\alpha}_{-i} = (Z_{-i}^\top \hat{W}_{[-i]} Z_{-i} + kI)^{-1}(Z_{-i}^\top \hat{W}_{[-i]} Z_i + kdI)Z_{-i}^\top \hat{W}_{[-i]}\hat{s}_{-i}, \tag{2.5}$$

where since $Z_{-i}^\top \hat{W}_{[-i]} Z_{-i} = Z^\top \hat{W} Z - z_i^\top w_{[i]}\hat{z}_i$, and $Z_{-i}^\top \hat{W}_{[-i]}\hat{s}_{-i} = Z^\top \hat{W}\hat{s} - z_i^\top w_{[i]}\hat{s}_i$. Then, we can write Eq.(2.5) as follows

$$\hat{\alpha}_{-i} = (Z^\top \hat{W} Z + kI - z_i^\top w_{[i]}\hat{z}_i)^{-1}(Z^\top \hat{W} Z + kdI - z_i^\top w_{[i]}\hat{z}_i)(Z^\top \hat{W}\hat{s} - z_i^\top w_{[i]}\hat{s}_i), \tag{2.6}$$

and $(Z^\top \hat{W} Z + kI - z_i^\top w_{[i]} \hat{z}_i)^{-1}(Z^\top \hat{W} Z + kdI - z_i^\top w_{[i]} \hat{z}_i)$ is obtained from the ShermanMorrison Woodbury theorem as [17]

$$\begin{aligned}(Z_{-i}^\top \hat{W}_{[-i]} Z_{-i} + kI)^{-1}(Z_{-i}^\top \hat{W}_{[-i]} Z_i + kdI) = {} & (Z^\top \hat{W} Z + kI)^{-1}(Z^\top \hat{W} Z + kdI) \\ & + (Z^\top \hat{W} Z + kI)^{-1}(Z^\top \hat{W} Z + kdI)\hat{z}_i w_{[i]} \\ & x[I - z_i^\top (Z^\top \hat{W} Z + kI)^{-1}(Z^\top \hat{W} Z + kdI)\hat{z}_i w_{[i]}]^{-1} \\ & x z_i^\top (Z^\top \hat{W} Z + kI)^{-1}(Z^\top \hat{W} Z + kdI).\end{aligned} \tag{2.7}$$

Writing (2.7) in (2.6) and after multiplying the terms in parenthesis, we get

$$\hat{\alpha}_{-i} = \hat{\alpha}_{TPE} - \frac{K^{-1} z_i w_{[i]} \hat{s}_i}{1 - w_{kd-ii}} + \frac{K^{-1} \hat{z}_i z_i^\top}{1 - w_{kd-ii}} \hat{\alpha}_{TPE},$$

where $\hat{\alpha}_{TPE} = (Z^\top \hat{W} Z + kI)^{-1}(Z^\top \hat{W} Z + kdI) Z^\top \hat{W} \hat{s}$, $K^{-1} = (Z^\top \hat{W} Z + kI)^{-1}(Z^\top \hat{W} Z + kdI)$ and $w_{kd-ii} = z_i^\top K^{-1} \hat{z}_i w_{[i]}$. It can be shown, after algebraic simplifications, that

$$\hat{\alpha}_{-i} = \hat{\alpha}_{TPE} - (Z^\top \hat{W} Z + kI)^{-1}(Z^\top \hat{W} Z + kdI)\hat{z}_i w_{[i]} \left[ \frac{\hat{s}_i - z_i^\top \hat{\alpha}_{TPE}}{1 - w_{kd-ii}} \right]. \tag{2.8}$$

From [22] are proposed the weighted pseudo-values as

$$P_i = \hat{\alpha}_{TPE} + n(1 - w_{kd-ii})(\hat{\alpha}_{TPE} - \hat{\alpha}_{-i}), \tag{2.9}$$

and the corresponding weighted Jackknifed estimator as

$$\hat{\alpha}_{JTPE} = \overline{P} = n^{-1} \sum P_i.$$

The JTPE is obtained by using Eqs. (2.8), (2.9) and $\sum_{i=1}^n z_i w_{[i]} \hat{s}_i = Z^\top \hat{W} \hat{s}$, $\sum_{i=1}^n \hat{z}_i w_{[i]} z_i^\top = Z^\top \hat{W} Z$ the equations as

$$\hat{\alpha}_{JTPE} = (I + k(1-d)C^{-1})\hat{\alpha}_{TPE}, \tag{2.10}$$

where $C = (Z^\top \hat{W} Z + kI), k > 0$ and $0 < d < 1$. After writting $\hat{\alpha}_{JTPE}$ as a linear transformation of the OLS estimator, $\hat{\alpha}_{TPE} = (I - k(1-d)C^{-1})\hat{\alpha}_{ML}$, $\hat{\alpha}_{JTPE}$ in Equation (2.10) becomes

$$\hat{\alpha}_{JTPE} = (I - k^2(1-d)^2 C^{-2})\hat{\alpha}_{ML}. \tag{2.11}$$

The proposed estimator includes the special cases of the $\hat{\alpha}_{ML}$ and the $\hat{\alpha}_{JRE}$.

Case 1: When $k = 0$ or $d = 1$ the $\hat{\alpha}_{JTPE}$ will becomes the $\hat{\alpha}_{ML}$, $\hat{\alpha}_{\text{JTPE}}(0, d) = \hat{\alpha}_{\text{JTPE}}(k, 1) = \hat{\alpha}_{\text{ML}}$.

Case 2: When $d = 0$ the $\hat{\alpha}_{\text{JTPE}}$ will becomes JRE, $\hat{\alpha}_{\text{JTPE}}(k, 0) = \hat{\alpha}_{\text{JRE}}$.

From (2.11), proposed the MJTPE, based on [2,8,12] as

$$\hat{\alpha}_{MJTPE} = (I - k^2(1-d)^2 C^{-2})(I - k(1-d)C^{-1})\hat{\alpha}_{ML}.$$

Now, the bias vectors and the covariance matrices of TPE, JTPE and MJTPE are obtained respectively by

$$\begin{aligned} \text{bias}(\hat{\alpha}_{\text{TPE}}) &= -k(1-d)C^{-1}\alpha, \\ \text{COV}(\hat{\alpha}_{\text{TPE}}) &= \phi(I - k(1-d)C^{-1})\Lambda_{ML}^{-1}(I - k(1-d)C^{-1}), \\ \text{bias}(\hat{\alpha}_{\text{JTPE}}) &= -k^2(1-d)^2 C^{-2}\alpha, \\ \text{COV}(\hat{\alpha}_{\text{JTPE}}) &= \phi(I - k^2(1-d)^2 C^{-2})\Lambda_{ML}^{-1}(I - k^2(1-d)C^{-2}), \\ \text{bias}(\hat{\alpha}_{\text{MJTPE}}) &= -k(1-d)C^{-1} w_{TP}\alpha, \\ \text{COV}(\hat{\alpha}_{\text{MJTPE}}) &= \phi(2I - K_d)K_d^2 \Lambda_{ML}^{-1}[(2I - K_d)K_d^2]^\top, \end{aligned}$$

where $w_{TP} = (I + k(1-d)C^{-1} - k^2(1-d)^2C^{-2}$ and $K_d = I - k(1-d)C^{-1}$. Thus, using the bias vectors and the covariance matrices of TPE, JTPE and MJTPE, MMSEs of TPE, JTPE and MJTPE are obtained respectively as

$$\begin{aligned}\mathrm{MMSE}(\hat{\alpha}_{\mathrm{TPE}}) &= \mathrm{Var}(\hat{\alpha}_{\mathrm{TPE}}) + \mathrm{Bias}(\hat{\alpha}_{\mathrm{TPE}})\big(\mathrm{Bias}(\hat{\alpha}_{\mathrm{TPE}})\big)^\top \\ &= \phi(I - k(1-d)C^{-1})\Lambda_{ML}^{-1}(I - k(1-d)C^{-1}) + k^2(1-d)^2C^{-1}\alpha\alpha^\top C^{-1},\end{aligned}$$

$$\begin{aligned}\mathrm{MMSE}(\hat{\alpha}_{\mathrm{JTPE}}) &= \mathrm{Var}(\hat{\alpha}_{\mathrm{JTPE}}) + \mathrm{Bias}(\hat{\alpha}_{\mathrm{JTPE}})\big(\mathrm{Bias}(\hat{\alpha}_{\mathrm{JTPE}})\big)^\top \\ &= \phi(I - k^2(1-d)^2C^{-2})\Lambda_{ML}^{-1}(I - k^2(1-d)^2C^{-2}) + k^4(1-d)^4C^{-2}\alpha\alpha^\top C^{-2},\end{aligned}$$

$$\begin{aligned}\mathrm{MMSE}(\hat{\alpha}_{\mathrm{MJTPE}}) =& \mathrm{Var}(\hat{\alpha}_{\mathrm{MJTPE}}) + \mathrm{Bias}(\hat{\alpha}_{\mathrm{MJTPE}})\big(\mathrm{Bias}(\hat{\alpha}_{\mathrm{MJTPE}})\big)^\top \\ &= \phi(I - k^2(1-d)^2C^{-2})(I - k(1-d)C^{-1})\Lambda_{ML}^{-1} \\ &\mathrm{x}(I - k(1-d)C^{-1})(I - k^2(1-d)^2C^{-2}) + k^2(1-d)^2w_{TP}C^{-1}\alpha\alpha^\top C^{-1}w_{TP}.\end{aligned}$$

## 3. The performance of the estimators

We have already seen in the previous section that, the JTPE and the MJTPE are biased and hence the appropriate criterion to measure the performance of these estimators are bias vectors and covariance matrices. To do so, will be proved of theorems that provide conditions that JTPE is superior to the MLE, JRE, JLE and JLTE by this criterion.

### 3.1. Bias comparisons of the estimators

The aim of proposing the JTPE in PR, NBR and GR models was to reduce the bias of the ridge and Liu parameters. Hence, bias comparisons are presented in the following theorem. According to Theorems 3.1-3.2, both the total squared bias and the absolute bias of each individual parameter of the JTPE are less than those of the others estimators respectively.

**Theorem 3.1:** In the PR, NBR and GR models, the total squared bias of JTPE is always less than the squared bias of TPE, namely, $\|\mathrm{bias}(\hat{\alpha}_{\mathrm{JTPE}})\|^2 < \|\mathrm{bias}(\hat{\alpha}_{\mathrm{TPE}})\|^2$ holds for all $k > 0$ and $0 < d < 1$.

**Proof:** From the bias vectors of TPE and JTPE, we obtain

$$\begin{aligned}\|\mathrm{bias}(\hat{\alpha}_{\mathrm{TPE}})\|^2 - \|\mathrm{bias}(\hat{\alpha}_{\mathrm{JTPE}})\|^2 &= \sum_{j=1}^{p}\frac{k^2(1-d)^2\alpha_j^2}{(\lambda_j+k)^2} - \sum_{j=1}^{p}\frac{k^4(1-d)^4\alpha_j^2}{(\lambda_j+k)^4} \\ &= \sum_{j=1}^{p}\frac{k^2(1-d)^2\alpha_j^2[(\lambda_j+k)^2 - k^2(1-d)^2]}{(\lambda_j+k)^4},\end{aligned} \tag{3.1}$$

where $\lambda_j$ is the eigenvalue of $\Lambda$, which is always positive for $k > 0$ and $0 < d < 1$ hence, it is easily seen that Equation (3.1) is positive.

**Theorem 3.2:** The absolute value of the $j$th component of the bias of the JTPE is always smaller than the $j$th component of the bias of the TPE for all $k > 0$ and $0 < d < 1$.

**Proof:** The difference between the absolute values of the $j$th component of the bias vectors

$$|\mathrm{bias}(\hat{\alpha}_{\mathrm{TPE}})| - |\mathrm{bias}(\hat{\alpha}_{\mathrm{JTPE}})| = \frac{k(1-d)(\lambda_j+kd)}{(\lambda_j+k)^2}|\alpha_j|$$

which is strictly positive for all values of $k > 0$ and $0 < d < 1$ such that $|\alpha_j|$ is the absolute value of the $j$th element of $\hat{\alpha}$. Thus, the proof is finished.

According to Theorem 3.1, that is, $\|\text{bias}(\hat{\alpha}_{\text{JTPE}})\|^2 < \|\text{bias}(\hat{\alpha}_{\text{JRE}})\|^2$ is applicable to all $0 < d < 1$. From the bias vectors of JRE and JTPE, we obtain

$$\begin{aligned}\|\text{bias}(\hat{\alpha}_{\text{JRE}})\|^2 - \|\text{bias}(\hat{\alpha}_{\text{JTPE}})\|^2 &= \sum_{j=1}^{p} \frac{k^4\alpha_j^2}{(\lambda_j+k)^4} - \frac{k^4(1-d)^4\alpha_j^2}{(\lambda_j+k)^4} \\ &= \sum_{j=1}^{p} \frac{k^4\alpha_j^2(1-(1-d)^4)}{(\lambda_j+k)^4},\end{aligned}$$

which is always positive for $0 < d < 1$.

According to Theorem 3.2, the difference between the absolute values of the $j$th component of the bias vectors

$$|\text{bias}(\hat{\alpha}_{\text{JRE}})| - |\text{bias}(\hat{\alpha}_{\text{JTPE}})| = \frac{k^2(1-(1-d)^2)}{(\lambda_j+k)^2}|\alpha_j|,$$

which is strictly positive for all values of $0 < d < 1$ such that $|\alpha_j|$ is the absolute value of the $j$th element of $\alpha$.

According to Theorem 3.1, namely, $\|\text{bias}(\hat{\alpha}_{\text{JTPE}})\|^2 < \|\text{bias}(\hat{\alpha}_{\text{JLE}})\|^2$ holds for all $0 < k < 1$. From the bias vectors of JLE and JTPE, we obtain

$$\begin{aligned}\|\text{bias}(\hat{\alpha}_{\text{JLE}})\|^2 - \|\text{bias}(\hat{\alpha}_{\text{JTPE}})\|^2 &= \sum_{j=1}^{p} \frac{(1-d)^4\alpha_j^2}{(\lambda_j+1)^4} - \frac{k^4(1-d)^4\alpha_j^2}{(\lambda_j+k)^4} \\ &= \sum_{j=1}^{p} \frac{(1-d)^4\alpha_j^2[(\lambda_j+k)^4 - k^4(\lambda_j+1)^4]}{(\lambda_j+1)^4(\lambda_j+k)^4}\end{aligned}$$

where $\lambda_j$ is the eigenvalue of $\Lambda$, which is always positive for $0 < k < 1$.

According to Theorem 3.2, the difference between the absolute values of the $j$th component of the bias vectors

$$|\text{bias}(\hat{\alpha}_{\text{JLE}})| - |\text{bias}(\hat{\alpha}_{\text{JTPE}})| = \frac{(1-d)^2[(\lambda_j+k)^2 - k^2(\lambda_j+1)^2]}{(\lambda_j+1)^2(\lambda_j+k)^2}|\alpha_j|,$$

where $\lambda_j$ is the eigenvalue of $\Lambda$, which is always positive for $0 < k < 1$ such that $|\alpha_j|$ is the absolute value of the $j$th element of $\alpha$.

According to Theorem 3.1, $\|\text{bias}(\hat{\alpha}_{\text{JTPE}})\|^2 < \|\text{bias}(\hat{\alpha}_{\text{JLTE}})\|^2$ holds for all $k > 0$ and $0 < d < 1$. From the bias vectors of JLTE and JTPE, we obtain

$$\begin{aligned}\|\text{bias}(\hat{\alpha}_{\text{JLTE}})\|^2 - \|\text{bias}(\hat{\alpha}_{\text{JTPE}})\|^2 &= \sum_{j=1}^{p} \frac{(k+d)^4\alpha_j^2}{(\lambda_j+k)^4} - \frac{k^4(1-d)^4\alpha_j^2}{(\lambda_j+k)^4} \\ &= \sum_{j=1}^{p} \frac{\alpha_j^2[(k+d)^4 - k^4(1-d)^4]}{(\lambda_j+k)^4},\end{aligned}$$

which is always positive for $k > 0$ and $0 < d < 1$.

According to Theorem 3.2, the difference between the absolute values of the $j$th component of the bias vectors

$$|\text{bias}(\hat{\alpha}_{\text{JLTE}})| - |\text{bias}(\hat{\alpha}_{\text{JTPE}})| = \frac{[(k+d)^2 - k^2(1-d)^2]}{(\lambda_j+k)^2}|\alpha_j|,$$

which is strictly positive for all values of $k > 0$ and $0 < d < 1$ such that $|\alpha_j|$ is the absolute value of the $j$th element of $\alpha$.

### 3.2. Variance comparisons of the estimators

The total variance of an estimator $\hat{\beta}^*$ is defined as the trace of its covariance matrix $\mathrm{Var}(\hat{\beta}^*) = \mathrm{tr}[\mathrm{Cov}(\hat{\beta}^*)]$. In the following theorems, the total variances of the estimators are compared.

**Theorem 3.3:** The total variance of the JTPE is always less than the total variance of MLE for all values of $k > 0$ and $0 < d < 1$, namely, $\mathrm{Var}(\hat{\alpha}_{ML})$-$\mathrm{Var}(\hat{\alpha}_{JTPE}) > 0$.

**Proof:** Since the total variance of JTPE is given as

$$Var(\hat{\alpha}_{JTPE}) = tr(Cov(\hat{\alpha}_{JTPE})) = \phi \sum_{j=1}^{p} \frac{\lambda_j[\lambda_j + 2k(1-d)^2]}{(\lambda_j + k)^4},$$

the difference of the variances of MLE and JTPE is obtained as

$$\begin{aligned} Var(\hat{\alpha}_{ML}) - Var(\hat{\alpha}_{JTPE}) &= \phi \sum_{j=1}^{p} \frac{1}{\lambda_j} - \phi \sum_{j=1}^{p} \frac{\lambda_j[\lambda_j + 2k(1-d)^2]}{(\lambda_j + k)^4} \\ &= \phi \sum_{j=1}^{p} \frac{(2kd\lambda_j + k)(2\lambda_j^2 + 4k\lambda_j + k^2 - 2kd\lambda_j)}{\lambda_j(\lambda_j + k)^4} \end{aligned}$$

which is positive for all $k > 0$ and $0 < d < 1$. Thus, the proof ends.

According to Theorem 3.3, $\mathrm{Var}(\hat{\alpha}_{JRE})$-$\mathrm{Var}(\hat{\alpha}_{JTPE}) > 0$ for $k > \frac{2d-\lambda_j}{2}$ and $0 < d < \frac{\lambda_j + 2k}{2}$.

$$\begin{aligned} Var(\hat{\alpha}_{JRE}) - Var(\hat{\alpha}_{JTPE}) &= \phi \sum_{j=1}^{p} \frac{\lambda_j(\lambda_j + 2k)^2}{(\lambda_j + k)^4} - \phi \sum_{j=1}^{p} \frac{\lambda_j[\lambda_j + 2k(1-d)^2]}{(\lambda_j + k)^4} \\ &= \phi \sum_{j=1}^{p} \frac{4d\lambda_j[\lambda_j + k(2-d)]}{(\lambda_j + k)^4} \end{aligned}$$

which is positive for all $k > \frac{2d-\lambda_j}{2}$ and $0 < d < \frac{\lambda_j+2k}{2}$.

According to Theorem 3.3, $\mathrm{Var}(\hat{\alpha}_{JLE})$-$\mathrm{Var}(\hat{\alpha}_{JTPE})$>0 for $k > 0$ and $0 < d < 1$.

$$\begin{aligned} Var(\hat{\alpha}_{JLE}) - Var(\hat{\alpha}_{JTPE}) &= \phi \sum_{j=1}^{p} \frac{\lambda_j[\lambda_j + 2(1-d)^2]}{(\lambda_j + 1)^4} - \phi \sum_{j=1}^{p} \frac{\lambda_j[\lambda_j + 2k(1-d)^2]}{(\lambda_j + k)^4} \\ &= \phi \sum_{j=1}^{p} \frac{2(k-1)[d\lambda_j^2 + k(1-d) + \lambda_j(k^2+1)]}{(\lambda_j + 1)^4(\lambda_j + k)^4} \end{aligned}$$

which is positive for all $k > 0$ and $0 < d < 1$.

According to Theorem 3.3, namely, $\mathrm{Var}(\hat{\alpha}_{JLTE})$-$\mathrm{Var}(\hat{\alpha}_{JTPE})$>0 for $k > 0$ and $0 < d < 1$.

$$\begin{aligned} Var(\hat{\alpha}_{JLTE}) - Var(\hat{\alpha}_{JTPE}) &= \phi \sum_{j=1}^{p} \frac{\lambda_j[\lambda_j + 2(k+d)^2]}{(\lambda_j + k)^4} - \phi \sum_{j=1}^{p} \frac{\lambda_j[\lambda_j + 2k(1-d)^2]}{(\lambda_j + k)^4} \\ &= \sum_{j=1}^{p} \frac{4d\lambda_j[\lambda_j(1+k) + 2k + d(1-k)^2]}{(\lambda_j + k)^4} \end{aligned}$$

which is positive for all $k > 0$ and $0 < d < 1$.

## 4. Selection of biasing parameter

There is no definite rule for estimating the ridge parameter, $k$, which is a positive constant, and the Liu parameter, $d$, which is between zero and one. We adapt ridge parameters which Kibria [28], Alkhamisi et al. [9] and Muniz et al. [37] proposed the best values of $k$ for the ridge regression model. The most classical ridge parameter estimator is $\hat{k} = \frac{\hat{\sigma}^2}{\hat{\alpha}^2_{max}}$ proposed by [23]. Based on this, Alkhamisi et al. [9] take harmonic average of the ridge parameter developed it as follows

$$\hat{k}_1 = \frac{p\hat{\varphi}^2}{\sum_{j=1}^{p} \hat{\alpha}_j^2}, j = 1, ..., p.$$

We estimate it by $\hat{\alpha}_j^2$ is the $j$th element of $\hat{\alpha} = \gamma^\top \hat{\beta}_{ML}$ and $\gamma$ is the eigenvector of the $X^\top \hat{W} X$. Practically, we used different values $\hat{W}$ for the PR, the NBR and the GR models.

Kibria [28] estimated the ridge parameter using geometric mean and median in his study. The prediction he suggested for the geometric mean, which has the best performance among the parameters adapted for GLM as follows:

$$\hat{k}_2 = \frac{\hat{\varphi}^2}{\left(\prod_{j=1}^{p} \hat{\alpha}_j^2\right)^{1/p}}, j = 1, ..., p.$$

The best estimate chosen among the estimates suggested by [9] is adapted to GLM as follows:

$$\hat{k}_3 = \max\left(\sqrt{\frac{\hat{\varphi}^2}{\hat{\alpha}_j^2}}\right), j = 1, ..., p.$$

In addition, the following the estimator selected from some new estimators proposed by [37], we modify the estimator for the GLM as

$$\hat{k}_4 = \max\left(\sqrt{\frac{t_{\max}\hat{\varphi}^2}{(n-p)\hat{\varphi}^2 + t_{\max}\hat{\alpha}_j^2}}\right), j = 1, ..., p,$$

where $t_{\max}$ is defined as the maximum eigenvalue of the $X^\top \hat{W} X$.

In addition, we adapt $d$ values which Månsson [33] proposed the best values of $d$ for the LR model

$$\hat{d}_1 = \min\left\{\frac{t_j\hat{\alpha}_j^2}{2(1 + t_j\hat{\alpha}_j^2)}\right\}, j = 1, ..., p,$$

$$\hat{d}_2 = \max\left[0, \min\left(\frac{\hat{\alpha}_j^2 - \hat{\varphi}^2}{1/t_j + \hat{\alpha}_j^2}\right)\right], j = 1, ..., p,$$

where $t_j$ is defined as the $j$th eigenvalue of the $X^\top \hat{W} X$.

## 5. The Monte-Carlo simulation

In this section, the simulated data sets are used to evaluate the performance of the new proposed JTPE on MLE, JRE, JLE and JLTE. Similarly, we show the performance of MJTPE on the MLE, MJRE, MJLE, and MJLTE. We start by describing how the data are generated and which factors have been varied in the design of the experiment. Performance evaluations of these Monte Carlo simulation studies are discussed.

### 5.1. The algorithm of the simulation experiment

In this subsection, simulation studies are conducted under various schemes, since the distribution shape parameter, the degree of multicollinearity, the number of explanatory variables, the number of sample sizes and the ridge and the Liu parameters affect the superiority of the estimators. The Monte Carlo simulation study is carried out using R software [14]. We consider the following settings.

(1) The sample sizes are taken as $n = 30, 75, 100$ and $250$ to observe the effect of the number of observations, and the number of explanatory variables are taken as $p = 3$ and $5$.

(2) Following [34], the explanatory variables $x_i^\top = (x_{i1}, x_{i2}, ..., x_{ip})$ have been generated from the following formula

$$x_{ij} = (1-\rho^2)^{1/2} w_{ij} + \rho w_{i(p+1)}, i = 1,2,...,n, j = 1,2,...,p,$$

where $w_{ij}$ are explanatory standard normal pseudo-random numbers generated using the standard normal distribution and $\rho^2$ represents the degree of correlation between any two explanatory variables. In the design of the experiment, we obtain three different values of $\rho^2$ corresponding to 0.90, 0.95, and 0.99. The explanatory variables are then standardized using unit length scaling so that $X^\top X$ is a matrix of correlations.

(3) The parameter vector $\beta$ is taken as the normalized eigenvector corresponding to the largest eigenvalue of $X^\top X$ by following [49] for each set of explanatory variables.

(4) We generate the response variable of the observations $n$ from the PR$y_i \sim \mathrm{P}(\mu_i)$, NBR $y_i \sim \mathrm{NB}(n, s = \frac{1}{\alpha}, \frac{s}{s+\mu_i})$ where the parameter $s = shape$ is chosen as $s = 10$ and GR $y_i \sim \mathrm{Gamma}(n, rate = s/\mu_i)$, where the parameter $s = shape$ is chosen as $s = 5$ with log-link function $\mu_i = \exp(\beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_p x_{ip})$, $i = 1,2,...,n$ and $j = 1,2,...,p$.

(5) We set $\mu_i = \exp(X\hat{\beta}_L)$, where $\hat{\beta}_L = (X^\top X)^{-1} X^\top y$ is the OLS estimator.

(6) We determine the adjusted response variable $z$ with the $i$th observation set to $\hat{z}_i = \log(\hat{\mu}_i) + (y_i - \hat{\mu}_i)/(\hat{\mu}_i)$ for the Poisson, negative binomial, and gamma distribution. We calculate the weight matrix $\hat{W}$ with diagonal elements of $\hat{\mu}_i$ for PR, $\frac{\hat{\mu}_i}{1+\alpha\hat{\mu}_i}$ for the NBR and $(\mu_i)^2$ for the GR.

(7) For explanatory variables and each sample size, the dispersion parameter is estimated as $\hat{\varphi}^2 = (n-p)^{-1} \sum_{i=1}^{n} (y_i - \hat{\mu}_i)^2/(\hat{\mu}_i)^2$ by using the Pearson method for the GR and taken as $\hat{\varphi}^2 = 1$ for the PR and the NBR.

(8) The number of replications is chosen as 2000 in the Monte-Carlo simulation study. The superiority of the estimators is examined in terms of the simulated MSE (SMSE)

$$\mathrm{SMSE}(\hat{\alpha}) = \frac{1}{2000} \sum_{i=1}^{5000} (\hat{\alpha}_{(i)} - \alpha)^\top (\hat{\alpha}_{(i)} - \alpha),$$

where the subscript(i) refers to the $i$th replication and $\hat{\alpha}_{(i)}$ is defined as

a) $\hat{\alpha}_{\mathrm{ML}} = \Lambda_{ML}^{-1} Z^\top \hat{W} \hat{s}$,
b) $\hat{\alpha}_{\mathrm{JRE}} = (I - k^2 C^{-2}) \hat{\alpha}_{\mathrm{ML}}$,
c) $\hat{\alpha}_{\mathrm{MJRE}} = (I - k^2 C^{-2})(I - kC^{-1}) \hat{\alpha}_{\mathrm{ML}}$,
d) $\hat{\alpha}_{\mathrm{JLE}} = (I - (1-d)^2 D^{-2}) \hat{\alpha}_{\mathrm{ML}}$,
e) $\hat{\alpha}_{\mathrm{MJLE}} = (I - (1-d)^2 D^{-2})(I - (1-d) D^{-1}) \hat{\alpha}_{\mathrm{ML}}$,
f) $\hat{\alpha}_{\mathrm{JLTE}} = (I - (k+d)^2 C^{-2}) \hat{\alpha}_{\mathrm{ML}}$,
g) $\hat{\alpha}_{\mathrm{MJLTE}} = (I - (k+d)^2 C^{-2})(I - (k+d) C^{-1}) \hat{\alpha}_{\mathrm{ML}}$,
h) $\hat{\alpha}_{JTPE} = (I - k^2 (1-d)^2 C^{-2}) \hat{\alpha}_{ML}$,
) $\hat{\alpha}_{MJTPE} = (I - k^2 (1-d)^2 C^{-2})(I - k(1-d) C^{-1}) \hat{\alpha}_{ML}$.

We first use SMSE plots generated using ***"ggplot2"*** package [56] in ***R*** software [14] to obtain the bias parameter with the minimum SMSE one among the ridge and Liu bias parameters given in Section 4. These parameters are placed into JRE, JLE, JLTE and JTPE when the response variable is Poisson, negative binomial, and gamma. Our aim here is to select the best parameter from biasing parameters that have been suggested numerous times in the literature and to find the best Jackknifed estimator by using the best parameter. Thus, we obtain the minimum bias estimator using biasing parameters with the minimum SMSE value. While $n = 30, 75, 100, 250$, $p = 3$ and $\rho^2 = 0.99$, their performance as a function of the correlation among explanatory variables of MLE, JRE, JLE, JLTE and

JTPE respectively is presented in Figures 1-3, PR, NBR and GR. According to Figures 1-2, as $n$ increases, it appears that with minimum SMSE values the estimators of parameters $k_3$ and $d_2$. According to Figure 3, as $n$ increases, it appears that with minimum SMSE values there are estimators of the parameters $k_4$ and $d_1$.

The results of the simulation study are given in Tables 1-3, which are presented in Appendix. The main results of the simulation study are as follows:

(1) When $n$ is constant, with increasing correlation the degree of correlation also increases SMSE values of all estimates. This is a natural result because with the increase of the multicollinearity problem, the weighted cross-matrix problem is also increasing. This increase of SMSE is particularly large for MLE and is particularly decreases when applying biased estimators. Tables 1-3. show that for all biased estimators it can be seen that it performs better than MLE.

(2) As the number of explanatory variables increases, the SMSE value of MLE also increases. Therefore, the negative impact of increasing the number of explanatory variables is greater for MLE than biased estimators are too much. However, the estimator least affected by this increase was JTPE and MJTPE.

(3) It can be said that sample sizes have a decreasing effect on SMSE values. Working with more very sample sizes will further reduce the value of estimated MSE (EMSE).

(4) Compared to Poisson, the negative binomial and gamma distribution is the response variable in GLM, gamma has lower SMSE values.

(5) Generally, in the case of multicollinearity from tables, using the Jackknifed technique reduced SMSE values and modified Jackknifed technique further improved EMSEs comment can be made. It can be said that this is supported by the results of many authors in the literature on these results [11, 54, 55, 57].

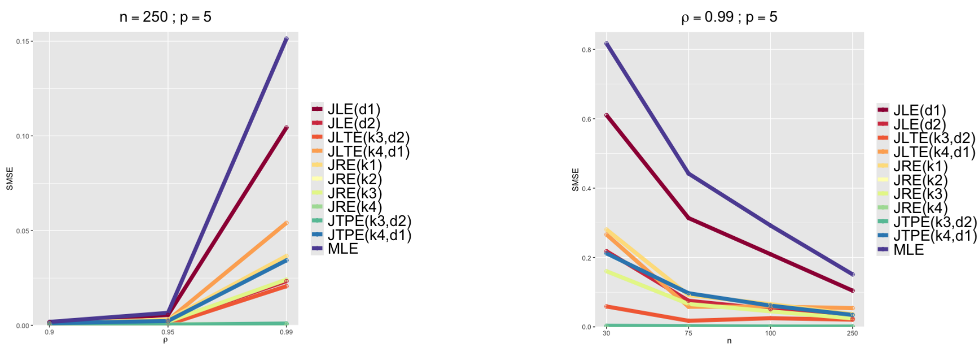


**Figure 1.** SMSE values of the estimators in the PR for different $\rho$ and $n$ values.

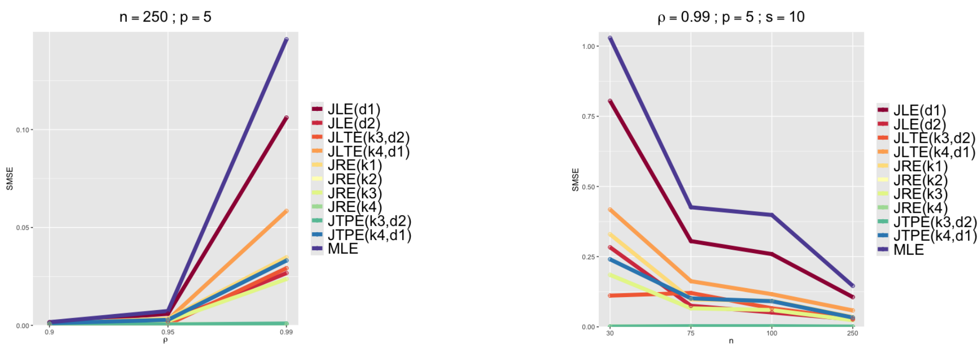


**Figure 2.** SMSE values of the estimators in the NBR for different $\rho$ and $n$ values.

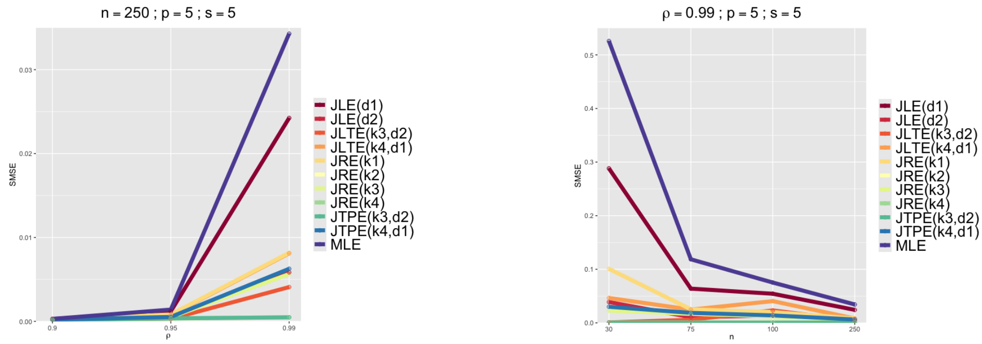


**Figure 3.** SMSE values of the estimators in the GR for different $\rho$ and $n$ values.

## 6. Numerical examples

In this section, we illustrate the benefits of new estimators using real data sets.

### 6.1. Poisson regression: Aircraft damage data set

We use the aircraft damage data set for PR presented by [38] that dates back to the Vietnam War. Considerable resources were deployed against the A-4 and A-6, including small arms, AAA or anti-aircraft artillery, and surface-to-air missiles. It contains data from 30 strike missions that involved these two types of aircraft. The regressor $X_1$ is an indicator variable (A-4 $= 0$ and A-6 $= 1$), and the other regressors $X_2$ and $X_3$ are the bomb load (in tons) and the total of months of experience of aircrew. The response variable, $Y$ is the number of locations where damage was inflicted on the aircraft. Firstly, the Pearson $\chi^2$ goodness-of-fit test is used before performing the Poisson distribution on the response variable. The chi-squared statistics and the corresponding p-value are obtained as 3.1864 and 0.2033, respectively. Since this p-value exceeds the significance level 0.05, we conclude that the data set follows the Poisson distribution. This result is also supported by the graphical presentations shown in Figure 4. Theoretical and empirical density and distribution plots show a good fitting to the Poisson distribution. The goodness of fit test and the plots are performed using the ***fitdistrplus*** package [16] in ***R*** software [14].

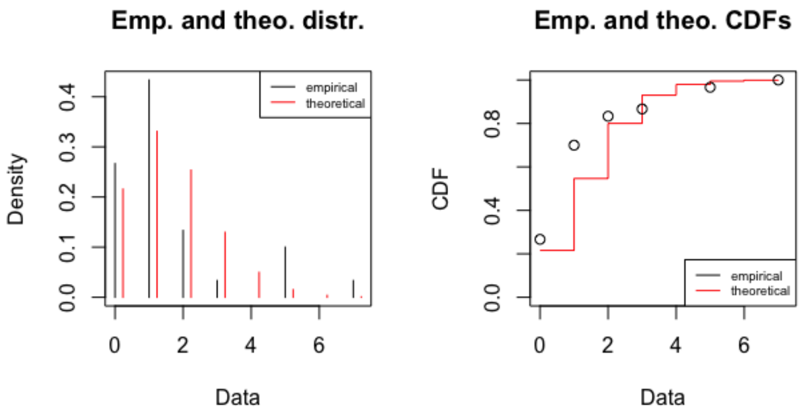


**Figure 4.** Diagnostics for goodness of fit to the Poisson distribution for the aircraft damage dataset.

The data matrix $X$ is centered and standardized so that $X^\top \hat{W} X$ is in correlation form. Myers et al. [38] claim that this data set exhibits multicollinearity. Including the constant term has been obtained as $\lambda_1 = 354.136144$, $\lambda_2 = 10.436062$, $\lambda_3 = 7.371267$ and $\lambda_4 = 1.775029$ the eigenvalues of the matrix $X^\top \hat{W} X$. The condition number computed $\mathrm{CN} = \dfrac{\lambda_1}{\lambda_4} \cong 119.51006$, which is a measure of multicollinearity ($\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of $X \top \hat{W} X$, respectively), which shows that there exists severe multicollinearity. We present the correlation matrix of the data in Figure 5. Figure 5 one can see the bivariate correlations and we also see that there is a high correlation (0.836) between the variables $X_1$ and $X_3$. In this case, biased estimators should

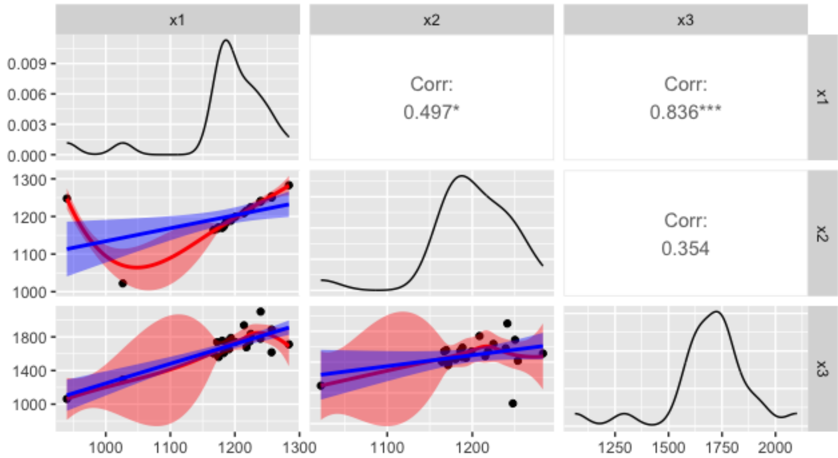


**Figure 5.** Correlation plots of the aircraft damage dataset.

be used as an alternative to MLE. We find the ridge parameter as $k_3 = 0.007463192$ and the Liu parameter as $d_2 = 0.04269131$ for PR.

Estimated coefficients, estimated MSE values and biases of these methods are presented in Table 4 which is provided in Appendix. According to the results, one can conclude that the signs and magnitudes of the MLE coefficients are negatively affected by collinearity compared to the biased estimators. The bias values of MLE are quite higher than those of biased estimators. The estimated theoretical MSE of the MLE is also inflated. However, the signs and magnitudes of the biased estimators are similar to each other and seem to be more stable than the MLE. The bias values of the JTPE are lower than those of the MLE, JRE, JLE and JLTE and this is an expected situation since the variance of the JTPE is always lower than that of the biased estimators according to Theorem 3.3. The bias values of the JTPE are lower than those of the biased estimators. This situation is again consistent with Theorems 3.1-3.2. According to Table 4, MSE of the JTPE is always less than MSE of the MLE, JRE, JLE, and JLTE. Moreover, modified Jackknifed estimators improved the magnitudes of all biased estimators.

## 6.2. Negative binomial regression: Swedish traffic data set

In this subsection, we illustrate for NBR using a real data set taken from the Department of Transport Analysis in Sweden (www.trafa.se). A regression model is estimated, where the dependent variable ($Y$) is the number of pedestrians killed, and the explanatory variables are the number of kilometers driven by cars ($X_1$), buses ($X_2$), trucks ($X_3$) and motorcycles ($X_4$), respectively. In this application, we try to investigate the effect of changing the usage of cars and trucks on the number of pedestrians killed. Data are pooled for different counties (the total number is 21) in Sweden during 2017. Firstly, the Pearson $\chi^2$ goodness-of-fit test is used before performing the negative binomial distribution on the response variable. The chi-squared statistics and the corresponding p-value are obtained as 0.7669 and 0.3812, respectively. Since this p-value exceeds the significance level 0.05, we conclude that the data set follows the negative binomial distribution. This result is also supported by the graphical presentations shown in Figure 6. Theoretical and empirical density and distribution plots show a good fitting to the negative binomial distribution.

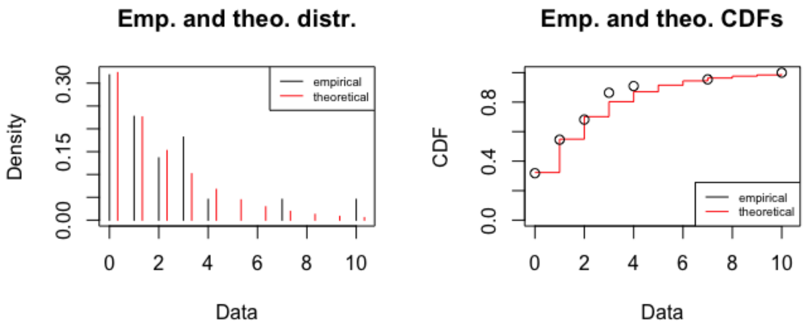


**Figure 6.** Diagnostics for goodness of fit to the negative binomial distribution for the Swedish traffic dataset.

The data matrix $X$ is centered and standardized so that $X^{\top}\hat{W}X$ is in correlation form. Including the constant term has been obtained as $\lambda_1 = 661.428437$, $\lambda_2 = 76.2894$, $\lambda_3 = 16.049592$, $\lambda_4 = 6.161682$ and $\lambda_5 = 2.124143$ the eigenvalues of the matrix $X^{\top}\hat{W}X$. The computed condition number $\text{CN} = \dfrac{\lambda_1}{\lambda_5} \cong 311.386021$, which is a measure of multicollinearity ($\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of $X^{\top}\hat{W}X$, respectively), which shows that there exists severe multicollinearity. In addition, we present the correlation matrix of the data in Figure 5. Figure 5 one can see that the bivariate correlations and we also see that there is a high correlation (0.831) between the variables $X_1$ and $X_2$ and (0.701) between $X_1$ and $X_3$.

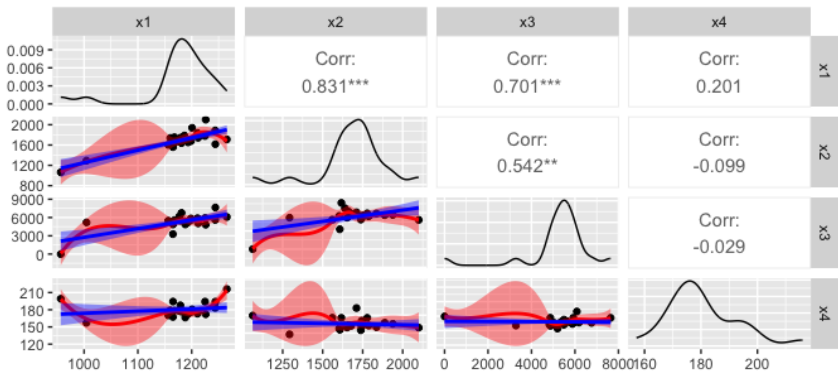


**Figure 7.** Correlation plots of the Swedish traffic dataset.

In this case, biased estimators should be used as an alternative to MLE. We find the ridge parameter as $k_3 = 0.4709132$ and the Liu parameter as $d_2 = 0.01160828$ for NBR. We present the estimated coefficients, the estimated MSE values and the biases of the estimators considered in Tables 5, which is provided in Appendix. According to results, we observe that the coefficients of the JRE, JLE, JLTE and JTPE have signs similar to those of MLE. When we compare the biases values of estimators, it is observed that the JTPE have lower biases values than the biased estimators, which makes them more stable. Thus, the JTPE should be preferred since they are compared with the coefficients of the biased estimators and the MLE. This situation is again consistent with Theorems 3.1-3.2. According to Table 5, the MSE of JTPE is always less than the MSE of MLE, JRE, JLE, and JLTE. Moreover, modified Jackknifed estimators improved the magnitudes of all biased estimators.

### 6.3. Gamma regression: Weather data set

This subsection illustrates the results with a real-life data set, first analyzed by [13]. The data correspond to the weather factors and nitrogen dioxide concentrations ($Y$), in parts per hundred million (p.p.h.m.), for 26 days in September 1984 measured at a monitoring station in the San Francisco Bay area.There are four explanatory variables. The variables considered in the study are mean wind speed in miles per hour ($X_1$) in mph, maximum temperature ($X_2$) in, insolation ($X_3$) in langleys per day and stability factor ($X_4$). First, the Cramer-von-Mises (CvM) goodness-of-fit test is used before performing the gamma distribution on the response variable. The test statistics and the corresponding p-value are obtained as 0.0499 and 0.4344, respectively. Since this p-value exceeds the significance level 0.05, we conclude that the data set follows the gamma distribution. This result is also supported by the graphical presentations shown in Figure 8. The histogram with density line and quantile plot shows a good fitting to the gamma distribution. In addition, Kurtoglu and Özkale [30] and Çetinkaya et al. [15] computations are shown that were performed using [13] proposed that this data set has a gamma distribution. In several air pollution models, the random disturbance appears to have a gamma distribution.

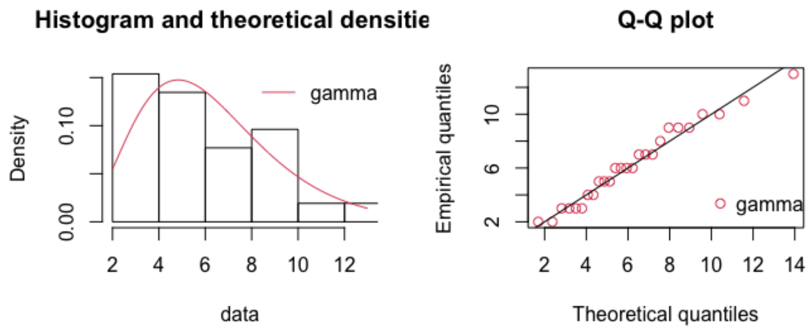


**Figure 8.** Diagnostics for goodness of fit to the gamma distribution for the aircraft damage dataset.

The data matrix $X$ is centered and standardized so that $X^\top \hat{W} X$ is in correlation form. Kurtoglu and Özkale [30] and Çetinkaya et al. [15] also showed that including the constant term of the matrix $X \top \hat{W} X$ is an ill-conditioned matrix. We present the correlation matrix of the data in Figure 9. Figure 9 one can see the bivariate correlations and we also see that there is a high correlation $(-0.720)$ between the variables $X_1$ and $X_2$.

In this case, biased estimators should be used as an alternative to MLE. We find the ridge parameter as $k_4 = 0.5574405$ and the Liu parameter as $d_1 = 0.0005390112$ for the NBR model. We present the estimated coefficients, the estimated MSE values and the biases of the estimators considered in Tables 6, which is provided in Appendix. According to the results, we observe that the coefficients of the JRE, JLE , JLTE and JTPE have signs similar to those of the MLE. When we compare the bias values of the estimators, it is observed that the JTPEs have lower bias values than the biased estimators, making them more stable. Thus, the JTPE should be preferred since they are compared with the coefficients of other estimators and the MLE. This situation is again consistent with Theorems 3.1-3.2. According to Table 6, MSE of the JTPE is always less than MSE of the MLE, JRE, JLE, and JLTE. Moreover, modified Jackknifed estimators improved the magnitudes of all biased estimators.

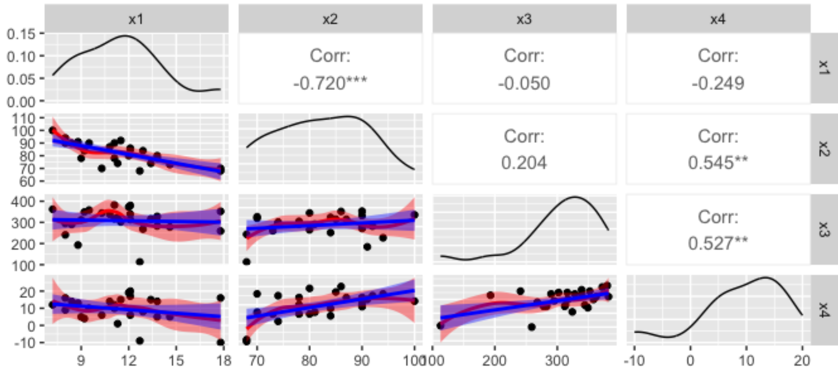


**Figure 9.** Correlation plots of the weather dataset.

## 7. Conclusive remarks

In this paper, new JTPE and MJTPE are proposed in the PR, NBR and GR models in order to overcome the effects of the multicollinearity problem. The purpose of the Jackknife procedure is to reduce the bias, hence it is proved both theoretically and numerically that the JTPE and MJTPE has a lower bias than that of the JRE, JLE and JLTE. Likewise, the MJTPE has a lower bias than that of the MJRE, MJLE and MJLTE. The investigation has been carried out with the variation of the degree of correlation, the number of observations, and the number of independent variables. In addition, some estimators of the biasing parameters k and d are proposed so that the performance of the new method becomes better than that of the others in terms of both MSE and bias values. One can always find some values of the biasing parameter so that the JTPE and MJTPE have a lower MSE and bias. Moreover, real data applications are considered to illustrate benefits of using the new estimators in the context of Jackknifed models. In conclusion, the use of JTPE and MJTPE is recommended when multicollinearity is present in the Jackknifed PR, NBR and GR models.

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**Data availability.** No data was used for the research described in the article.

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# APPENDIX

**Table 1.** SMSE values of the estimators in the PR when $p = 3$ and $p = 5$.

| $p$ | $q$ | $n$ | **MLE** | **JRE** | **MJRE** | **JLE** | **MJLE** | **JLTE** | **MJLTE** | **JTPE** | **MJTPE** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.90 | 30 | 0.003620 | 0.002558 | 0.000436 | 0.003192 | 0.000900 | 0.002215 | 0.000420 | 0.002188 | 0.000350 |
| | | 75 | 0.002596 | 0.001884 | 0.000409 | 0.002338 | 0.000849 | 0.001801 | 0.000401 | 0.001658 | 0.000375 |
| | | 100 | 0.001942 | 0.001430 | 0.000414 | 0.001804 | 0.000821 | 0.001399 | 0.000350 | 0.001260 | 0.000370 |
| | | 250 | 0.000728 | 0.000624 | 0.000302 | 0.000709 | 0.000446 | 0.000646 | 0.000274 | 0.000594 | 0.000292 |
| | 0.95 | 30 | 0.031373 | 0.017600 | 0.001024 | 0.025025 | 0.005226 | 0.016007 | 0.004876 | 0.010945 | 0.000888 |
| | | 75 | 0.009775 | 0.005832 | 0.000499 | 0.008052 | 0.001783 | 0.005243 | 0.000829 | 0.004495 | 0.000477 |
| | | 100 | 0.005750 | 0.003814 | 0.000539 | 0.005030 | 0.001604 | 0.003730 | 0.000582 | 0.002609 | 0.000537 |
| | | 250 | 0.002908 | 0.002125 | 0.000476 | 0.002674 | 0.001107 | 0.002091 | 0.000490 | 0.001781 | 0.000421 |
| | 0.99 | 30 | 0.763557 | 0.492415 | 0.025566 | 0.618556 | 0.186083 | 0.507971 | 0.064515 | 0.320814 | 0.001272 |
| | | 75 | 0.283370 | 0.145475 | 0.006485 | 0.228595 | 0.046789 | 0.198240 | 0.069743 | 0.136649 | 0.000713 |
| | | 100 | 0.216102 | 0.110641 | 0.004504 | 0.179721 | 0.042133 | 0.110947 | 0.057487 | 0.106634 | 0.000650 |
| | | 250 | 0.065759 | 0.037441 | 0.002015 | 0.055124 | 0.013558 | 0.045758 | 0.044919 | 0.033949 | 0.000649 |
| 5 | 0.90 | 30 | 0.066528 | 0.023897 | 0.001200 | 0.049223 | 0.011726 | 0.027009 | 0.008407 | 0.015827 | 0.001029 |
| | | 75 | 0.004949 | 0.002501 | 0.000525 | 0.004242 | 0.001722 | 0.002251 | 0.000454 | 0.002070 | 0.000408 |
| | | 100 | 0.003295 | 0.001885 | 0.000472 | 0.002845 | 0.001240 | 0.001751 | 0.000421 | 0.001645 | 0.000390 |
| | | 250 | 0.001840 | 0.001161 | 0.000384 | 0.001670 | 0.000850 | 0.001171 | 0.000357 | 0.001102 | 0.000354 |
| | 0.95 | 30 | 0.087298 | 0.031696 | 0.002351 | 0.061855 | 0.014346 | 0.121458 | 0.798195 | 0.021980 | 0.001029 |
| | | 75 | 0.024132 | 0.008743 | 0.000676 | 0.017924 | 0.004694 | 0.006865 | 0.001411 | 0.006844 | 0.000597 |
| | | 100 | 0.017078 | 0.006550 | 0.000780 | 0.012494 | 0.003332 | 0.005015 | 0.000697 | 0.004661 | 0.000527 |
| | | 250 | 0.006632 | 0.003083 | 0.000571 | 0.005508 | 0.002083 | 0.002713 | 0.000494 | 0.002331 | 0.000440 |
| | 0.99 | 30 | 0.816987 | 0.280590 | 0.003195 | 0.610411 | 0.218085 | 0.266101 | 0.058826 | 0.211960 | 0.000985 |
| | | 75 | 0.442043 | 0.091521 | 0.001845 | 0.313740 | 0.075324 | 0.097012 | 0.017452 | 0.057902 | 0.000829 |
| | | 100 | 0.292087 | 0.065587 | 0.001125 | 0.209365 | 0.051061 | 0.061182 | 0.024962 | 0.058688 | 0.000838 |
| | | 250 | 0.151313 | 0.036635 | 0.000995 | 0.104388 | 0.023378 | 0.054093 | 0.020695 | 0.034338 | 0.000802 |

**Table 2.** SMSE values of the estimators in the NBR when $p = 3$, $p = 5$ and $s = 5$, $s = 10$.

| $p$ | $q$ | $n$ | **MLE** | **JRE** | **MJRE** | **JLE** | **MJLE** | **JLTE** | **MJLTE** | **JTPE** | **MJTPE** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.90 | 30 | 0.009826 | 0.006780 | 0.001018 | 0.004655 | 0.000756 | 0.008616 | 0.002826 | 0.003754 | 0.000406 |
| | | 75 | 0.003610 | 0.002678 | 0.000532 | 0.002195 | 0.000429 | 0.003326 | 0.001239 | 0.002132 | 0.000378 |
| | | 100 | 0.002241 | 0.001759 | 0.000475 | 0.001583 | 0.000381 | 0.002085 | 0.000941 | 0.001502 | 0.000377 |
| | | 250 | 0.000877 | 0.000762 | 0.000331 | 0.000728 | 0.000275 | 0.000854 | 0.000518 | 0.000722 | 0.000265 |
| | 0.95 | 30 | 0.036226 | 0.022866 | 0.001797 | 0.080743 | 0.025320 | 0.029410 | 0.006458 | 0.014297 | 0.000457 |
| | | 75 | 0.016689 | 0.010283 | 0.000699 | 0.007881 | 0.000785 | 0.014445 | 0.004145 | 0.005487 | 0.000453 |
| | | 100 | 0.006904 | 0.004606 | 0.000541 | 0.003518 | 0.000553 | 0.006114 | 0.001945 | 0.003357 | 0.000417 |
| | | 250 | 0.003223 | 0.002279 | 0.000485 | 0.001908 | 0.000429 | 0.002949 | 0.001161 | 0.001878 | 0.000399 |
| | 0.99 | 30 | 0.631091 | 0.421117 | 0.000705 | 0.379686 | 0.184116 | 0.624914 | 0.148348 | 0.245646 | 0.000472 |
| | | 75 | 0.274621 | 0.133208 | 0.000566 | 0.107687 | 0.129488 | 0.228813 | 0.054223 | 0.107315 | 0.000446 |
| | | 100 | 0.191900 | 0.093773 | 0.000703 | 0.107390 | 0.086020 | 0.157584 | 0.035202 | 0.078117 | 0.000460 |
| | | 250 | 0.078714 | 0.040605 | 0.000715 | 0.080244 | 0.072166 | 0.065238 | 0.015066 | 0.034328 | 0.000459 |
| 5 | 0.90 | 30 | 0.024312 | 0.011009 | 0.001269 | 0.007129 | 0.000742 | 0.018810 | 0.005546 | 0.005041 | 0.000442 |
| | | 75 | 0.005615 | 0.002840 | 0.000632 | 0.002466 | 0.000517 | 0.004703 | 0.001885 | 0.002016 | 0.000424 |
| | | 100 | 0.004975 | 0.002657 | 0.000562 | 0.002402 | 0.000468 | 0.004311 | 0.001806 | 0.002008 | 0.000387 |
| | | 250 | 0.001645 | 0.001049 | 0.000394 | 0.001064 | 0.000360 | 0.001532 | 0.000873 | 0.000990 | 0.000355 |
| | 0.95 | 30 | 0.042712 | 0.015785 | 0.001390 | 0.036202 | 0.080250 | 0.031418 | 0.008344 | 0.007468 | 0.000448 |
| | | 75 | 0.027411 | 0.010133 | 0.000890 | 0.006892 | 0.000926 | 0.020944 | 0.006010 | 0.005539 | 0.000484 |
| | | 100 | 0.014869 | 0.005825 | 0.000753 | 0.004630 | 0.000645 | 0.011140 | 0.003175 | 0.003260 | 0.000474 |
| | | 250 | 0.007338 | 0.003450 | 0.000599 | 0.002950 | 0.000505 | 0.005922 | 0.002064 | 0.002244 | 0.000437 |
| | 0.99 | 30 | 1.029248 | 0.329811 | 0.001608 | 0.417958 | 0.111262 | 0.804664 | 0.283456 | 0.185234 | 0.000476 |
| | | 75 | 0.425977 | 0.097588 | 0.001288 | 0.162632 | 0.120659 | 0.305399 | 0.075074 | 0.066035 | 0.000471 |
| | | 100 | 0.398441 | 0.092521 | 0.001312 | 0.116343 | 0.066139 | 0.259131 | 0.051273 | 0.059913 | 0.000487 |
| | | 250 | 0.146095 | 0.034752 | 0.000804 | 0.058429 | 0.029230 | 0.106123 | 0.026713 | 0.023690 | 0.000478 |

**Table 3.** SMSE values of the estimators in the GR when $p = 3$, $p = 5$ and $s = 5$.

| $p$ | $q$ | $n$ | **MLE** | **JRE** | **MJRE** | **JLE** | **MJLE** | **JLTE** | **MJLTE** | **JTPE** | **MJTPE** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.90 | 30 | 0.001635 | 0.001170 | 0.000338 | 0.001303 | 0.000459 | 0.001069 | 0.000377 | 0.000887 | 0.000398 |
| | | 75 | 0.000760 | 0.000619 | 0.000285 | 0.000671 | 0.000304 | 0.000567 | 0.000299 | 0.000491 | 0.000339 |
| | | 100 | 0.000452 | 0.000392 | 0.000234 | 0.000427 | 0.000226 | 0.000372 | 0.000238 | 0.000338 | 0.000282 |
| | | 250 | 0.000183 | 0.000169 | 0.000144 | 0.000178 | 0.000123 | 0.000165 | 0.000146 | 0.000153 | 0.000174 |
| | 0.95 | 30 | 0.006438 | 0.003789 | 0.000411 | 0.004642 | 0.000908 | 0.002856 | 0.000524 | 0.002772 | 0.000444 |
| | | 75 | 0.003035 | 0.002068 | 0.000401 | 0.002464 | 0.000691 | 0.001580 | 0.000461 | 0.001647 | 0.000402 |
| | | 100 | 0.001806 | 0.001299 | 0.000374 | 0.001609 | 0.000607 | 0.001128 | 0.000392 | 0.001090 | 0.000411 |
| | | 250 | 0.000727 | 0.000571 | 0.000298 | 0.000669 | 0.000314 | 0.000515 | 0.000313 | 0.000477 | 0.000335 |
| | 0.99 | 30 | 0.201042 | 0.107405 | 0.003108 | 0.167552 | 0.040500 | 0.121579 | 0.008858 | 0.092365 | 0.007440 |
| | | 75 | 0.043465 | 0.022233 | 0.000555 | 0.034686 | 0.007344 | 0.036322 | 0.036506 | 0.020020 | 0.000671 |
| | | 100 | 0.054110 | 0.028259 | 0.000683 | 0.045137 | 0.010775 | 0.074802 | 0.056023 | 0.025725 | 0.000890 |
| | | 250 | 0.016353 | 0.008899 | 0.000577 | 0.013795 | 0.003601 | 0.016891 | 0.011946 | 0.008329 | 0.000536 |
| 5 | 0.90 | 30 | 0.002334 | 0.001161 | 0.000454 | 0.001710 | 0.000587 | 0.001052 | 0.000433 | 0.000651 | 0.000446 |
| | | 75 | 0.001246 | 0.000751 | 0.000273 | 0.001066 | 0.000459 | 0.000706 | 0.000265 | 0.000498 | 0.000345 |
| | | 100 | 0.000988 | 0.000646 | 0.000269 | 0.000866 | 0.000405 | 0.000617 | 0.000263 | 0.000440 | 0.000299 |
| | | 250 | 0.000302 | 0.000242 | 0.000162 | 0.000285 | 0.000183 | 0.000237 | 0.000158 | 0.000207 | 0.000172 |
| | 0.95 | 30 | 0.011527 | 0.003506 | 0.000348 | 0.007759 | 0.001760 | 0.003099 | 0.000361 | 0.001831 | 0.000446 |
| | | 75 | 0.004544 | 0.001750 | 0.000414 | 0.003245 | 0.000968 | 0.001617 | 0.000413 | 0.001087 | 0.000455 |
| | | 100 | 0.003232 | 0.001450 | 0.000417 | 0.002265 | 0.000710 | 0.001342 | 0.000403 | 0.000861 | 0.000455 |
| | | 250 | 0.001408 | 0.000821 | 0.000314 | 0.001202 | 0.000530 | 0.000770 | 0.000301 | 0.000533 | 0.000354 |
| | 0.99 | 30 | 0.525697 | 0.100791 | 0.000494 | 0.288181 | 0.038878 | 0.046417 | 0.000887 | 0.029972 | 0.000473 |
| | | 75 | 0.118578 | 0.025197 | 0.000530 | 0.063828 | 0.010519 | 0.024677 | 0.005758 | 0.018659 | 0.000507 |
| | | 100 | 0.075191 | 0.020236 | 0.001127 | 0.054290 | 0.013761 | 0.040534 | 0.023236 | 0.013858 | 0.000482 |
| | | 250 | 0.034284 | 0.008170 | 0.000495 | 0.024244 | 0.005846 | 0.008112 | 0.004089 | 0.006269 | 0.000485 |

**Table 4.** Estimated Coefficients and EMSE values for Aircraft Damage Dataset (biases in parentheses)

| Coefficients | MLE | JRE | MJRE | JLE | MJLE | JLTE | MJLTE | JTPE | MJTPE |
|---|---|---|---|---|---|---|---|---|---|
| $\beta_0$ | -1.729286 | -4.479915 | -5.315273 | -4.489423 | -5.026032 | -4.448213 | -5.893657 | -4.492178 | -4.700262 |
| | | (0.013685) | (0.002417) | (0.050740) | (0.001006) | (0.014004) | (0.007103) | (0.001653) | (0.000296) |
| $\beta_1$ | 28.334667 | 0.499495 | 4.652421 | 1.434855 | 3.836325 | 2.039869 | 2.497286 | 1.481828 | 2.665730 |
| | | (0.439764) | (0.004120) | (0.059760) | (0.005561) | (0.023875) | (0.012109) | (0.021326) | (0.000505) |
| $\beta_2$ | -23.974502 | -16.953684 | -10.154604 | 17.974183 | 13.805739 | -13.882162 | -3.422819 | 18.141877 | 16.374623 |
| | | (-0.038937) | (-0.229842) | (-0.187919) | (-0.085123) | (-0.331784) | (-0.067546) | (-0.028156) | (-0.020693) |
| $\beta_3$ | -11.311900 | -2.326816 | -1.241808 | -2.538167 | 2.174835 | -1.721971 | -0.567603 | -2.560796 | -2.313777 |
| | | (-0.090859) | (-0.038843) | (-0.392441) | (-0.015774) | (-0.225069) | (-0.114151) | (-0.007919) | (-0.004758) |
| $\beta_4$ | 15.158746 | 27.224213 | 21.239399 | 27.789595 | 23.929749 | 25.376162 | 15.840496 | 27.939116 | 26.402068 |
| | | (-0.249333) | (-0.130407) | (-0.189756) | (-0.053090) | (-0.455623) | (-0.383238) | (-0.015975) | (-0.014058) |
| MSE | 1.237853 | 0.006797 | 0.005975 | 0.006693 | 0.006594 | 0.006902 | 0.002453 | 0.006633 | 0.001639 |

**Table 5.** Estimated Coefficients and EMSE values for Swedish Traffic Dataset (biases in parentheses)

| Coefficients | MLE | JRE | MJRE | JLE | MJLE | JLTE | MJLTE | JTPE | MJTPE |
|---|---|---|---|---|---|---|---|---|---|
| $\beta_0$ | 2.662794 | 8.775604 | 10.784610 | 8.798111 | 10.174860 | 8.770634 | 10.894710 | 8.779133 | 10.620170 |
| | | (0.081038) | (-0.002151) | (0.055392) | (-0.003412) | (0.085567) | (-0.002404) | (-0.041679) | (-0.001098) |
| $\beta_1$ | -54.148234 | -46.875308 | -20.867860 | -47.508120 | -30.886110 | -46.745803 | -19.415380 | -46.967267 | -23.031620 |
| | | (0.152710) | (-0.081958) | (0.142909) | (-0.108172) | (0.088855) | (-0.091602) | (0.078171) | (-0.041835) |
| $\beta_2$ | 42.385735 | 149.510347 | 126.214990 | 150.544160 | 135.699440 | 149.298655 | 124.677770 | 149.660665 | 128.307540 |
| | | (-0.358558) | (-0.083688) | (-0.235710) | (-0.119087) | (-0.227738) | (-0.093536) | (-0.110540) | (-0.042719) |
| $\beta_3$ | -4.500517 | -20.300366 | -22.337990 | -20.319875 | -21.537130 | -20.295853 | -22.449130 | -20.303570 | -22.171480 |
| | | (-0.069168) | (0.000287) | (-0.038369) | (0.000754) | (-0.074222) | (0.000320) | (-0.031076) | (0.000146) |
| MSE | 0.774950 | 0.003220 | 0.003186 | 0.003191 | 0.003162 | 0.003225 | 0.003191 | 0.003180 | 0.003157 |

**Table 6.** Estimated Coefficients and EMSE values for Weather Dataset (biases in parentheses)

| Coefficients | MLE | JRE | MJRE | JLE | MJLE | JLTE | MJLTE | JTPE | MJTPE |
|---|---|---|---|---|---|---|---|---|---|
| $\beta_0$ | -58.1797 | -15410.1400 | -15450.2600 | -15410.1300 | -15438.6100 | -15410.1700 | -15478.9500 | -15410.1300 | -15420.1600 |
| | | (0.000183) | (0.000180) | (0.000421) | (0.000157) | (0.000481) | (0.000136) | (0.000191) | (0.000140) |
| $\beta_1$ | 1477.6179 | -62456.5800 | -61387.8800 | -62458.9800 | -61698.8200 | -62447.1600 | -60615.2200 | 62460.2200 | 62193.0400 |
| | | (-0.000109) | (-0.000107) | (-0.00123)4 | (-0.000790) | (-0.001037) | (-0.001030) | (-0.000197) | (-0.000134) |
| $\beta_2$ | -1541.9455 | 243195.4100 | 242171.0600 | 243197.7000 | 242469.0400 | 243186.3500 | 241430.3800 | 243198.9200 | 242942.8300 |
| | | (-0.003029) | (-0.001086) | (-0.000154) | (-0.000707) | (-0.001755) | (-0.000187) | (-0.001396) | (-0.001054) |
| $\beta_3$ | 363.8507 | 50637.5500 | 50385.4200 | 50637.3900 | 50458.0800 | 50638.1600 | 50205.9400 | 50637.3100 | 50574.2800 |
| | | (0.001322) | (0.001035) | (0.000163) | (0.000125) | (0.001646) | (0.001023) | (0.000355) | (0.000174) |
| $\beta_4$ | -987.6520 | -60817.9700 | -61501.8700 | -60819.2500 | -61305.4900 | -60812.9500 | -61985.3700 | -60819.9100 | -60990.8800 |
| | | (0.002321) | (0.001991) | (0.001176) | (0.001704) | (0.001345) | (0.001008) | (0.000256) | (0.000149) |
| MSE | 0.003895 | 0.003834 | 0.003824 | 0.003831 | 0.003824 | 0.003849 | 0.003824 | 0.003824 | 0.003804 |