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SOME RESULTS ON $\mathcal{I}_2$-DEFERRED STATISTICALLY CONVERGENT DOUBLE SEQUENCES IN FUZZY NORMED SPACES

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Abstract. The primary objective of this study is to introduce the concepts of $\mathcal{I}_2$-deferred Cesàro summability and $\mathcal{I}_2$-deferred statistical convergence for double sequences in fuzzy normed spaces (FNS). Furthermore, the aim is to explore the connections between these concepts and subsequently establish several theorems pertaining to the notion of $\mathcal{I}_2$-deferred statistical convergence in FNS for double sequences. We further define $\mathcal{I}_2$-deferred statistical limit points and $\mathcal{I}_2$-deferred statistical cluster points of a sequence within FNS and explore the relationships among these concepts.

1. Introduction

The concept of statistical convergence, initially introduced in Zygmund's monograph [40], was later revisited by Fast [11] and independently reexamined for both real and complex sequences by Schoenberg [32]. Mursaleen and Edely [26] extended this investigation to double sequences. Additionally, Fridy [13] explored statistical limit points and cluster points in the context of real number sequences.

Kostyrko et al. [20] introduced the concept of ideal convergence, which encompasses various convergence notions such as usual convergence and statistical convergence. Das et al. [7] extended this concept to double sequences in a metric space, termed $\mathcal{I}$-convergence. In a subsequent work, Savaş and Das [30] further

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expanded the idea of ideal convergence introduced by Kostyrko et al. [20], exploring its application to $\mathcal{I}$-statistical convergence and investigating its fundamental properties.

Zadeh [39] pioneered the theory of fuzzy sets, laying its foundation. Matloka [24] explored the convergence of sequences of fuzzy numbers, while Nanda [27] demonstrated that the set of convergent sequences of fuzzy numbers forms a complete metric space. Nuray and Savaş [28] extended the notion of convergence to statistically Cauchy and statistical convergence sequences of fuzzy numbers. Kumar et al. [21, 22] delved into $\mathcal{I}$-convergence, $\mathcal{I}$-limit points, and $\mathcal{I}$-cluster points for sequences of fuzzy numbers. Tripathy et al. [36] further investigated $\mathcal{I}$-statistically limit points and $\mathcal{I}$-statistically cluster points for sequences of fuzzy numbers. In [19], researchers extended existing theories on the convergence of fuzzy number sequences to $\mathcal{I}_2$-statistical convergence, broadened the notions of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points to double sequences, and investigated fundamental features and relationships between sets of $\mathcal{I}_2$-statistical cluster points and $\mathcal{I}_2$-statistical limit points of double sequences of fuzzy numbers. Katsaras [17] initially introduced the concept of fuzzy norm while examining fuzzy topological vector spaces. In 1992, Felbin [12] extended this concept to a fuzzy norm on linear spaces, drawing from the idea of fuzzy numbers initially proposed by Kaleva and Seikkala in the context of fuzzy metric treatment. Further research, including studies in [6,38], investigated diverse topological characteristics of these FNS, while works such as [2, 3] explored various types of FNS.

Agnew [1] introduced the concept of deferred Cesàro mean for real (or complex) sequences, followed by Küçükaslan and Yilmaztürk's [23] presentation of deferred statistical convergence for single sequences. Subsequently, Şengül et al. [34] introduced deferred $\mathcal{I}$-convergence. Dağadur and Sezgek [4, 5, 35] investigated deferred Cesàro summability and deferred statistical convergence for double sequences. Statistical convergence and deferred statistical convergence have been explored in various studies, as referenced in [8, 9, 15, 16, 18, 25, 29, 31, 37].

In this study, we adhere to the approach delineated in Felbin's work. Within the realm of FNS analysis, the convergence of sequences of fuzzy numbers plays a pivotal role in defining standard convergence within these spaces. This paper seeks to utilize the concept of generalized statistical convergence of fuzzy number sequences via ideal to explore a more extensive form of convergence, particularly $\mathcal{I}_2$-deferred statistical convergence within an FNS. The goal is to establish fundamental principles and key insights in this domain.

This paper is dedicated to introducing a novel form of convergence for sequences of fuzzy numbers within FNS. In Section 2, we provide some preliminary definitions and theorems concerning fuzzy number sequences, FNS, and deferred statistical convergence. In Section 3, we intend to define the concepts of $\mathcal{I}_2$-deferred Cesàro summability and $\mathcal{I}_2$-deferred statistical convergence for double sequences within FNS. In Section 4, our goal is to investigate the interconnections between these concepts and subsequently establish several theorems regarding the notion of $\mathcal{I}_2$-deferred statistical convergence in FNS for double sequences. Additionally, we define $\mathcal{I}_2$-deferred statistical limit points and $\mathcal{I}_2$-deferred statistical cluster points of sequences within FNS, and delve into the relationships among these concepts.

2. Definitions and Preliminaries

In this section, we commence by revisiting some fundamental definitions related to fuzzy numbers, fuzzy number sequences (FNS), and deferred convergence.

Definition 1. *([33]) Suppose $\mu : \mathbb{R} \to [0,1]$ represents a fuzzy subset of $\mathbb{R}$. For any $\alpha \in [0,1]$, the α-level set of μ, symbolized as μ_α, is described as the set of real numbers $\mathbb{R}$, where the measure μ is at least α. When $0 < \alpha \leqslant 1$, the notation $[\mu]_\alpha$ refers to the collection of points t in $\mathbb{R}$ where μ evaluates to at α. In the case where $\alpha = 0$, $[\mu]_\alpha$ indicates the closure of the set of points t in $\mathbb{R}$ where μ evaluates to strictly greater than 0.*

Definition 2. *([33]) A fuzzy set denoted by μ defined on the real numbers $\mathbb{R}$ is termed a fuzzy number subject to the specified conditions:*
(i) μ is normal, signifying the existence of a specific point t_0 in $\mathbb{R}$ where μ reaches its maximum membership grade of 1.
(ii) μ is fuzzy convex, implying that for any pair of real numbers t_1 and t_2, and any λ in the interval $[0,1]$, $\mu(\lambda t_1 + (1-\lambda)t_2)$ is greater than or equal to the minimum of $\mu(t_1)$ and $\mu(t_1)$.
(iii) μ is upper semi-continuous.
(iv) The set $[\mu]_0$, comprising all t in $\mathbb{R}$ where $\mu(t)$ is greater than 0, is compact.

A real number r can be represented as a fuzzy number $\tilde{r}$ defined by, if t equals r, then $\tilde{r}(t)$ equals 1, if t is not equal to r, then $\tilde{r}(t)$ equals 0. It can be demonstrated that μ qualifies as a fuzzy number if and only if each α-level set $[\mu]_\alpha$ forms a non-empty, bounded, and closed interval. We denote $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$.

Definition 3. *([33]) Let's consider $L(\mathbb{R})$, the collection of all fuzzy numbers. If a fuzzy number μ is a member of $L(\mathbb{R})$ and its membership grade $\mu(t)$ is zero for $t < 0$, it is termed a non-negative fuzzy number.*

By $L^*(\mathbb{R})$, we denote the set of all non-negative fuzzy numbers. We can express that $\mu \in L^*(\mathbb{R})$ iff $\mu_\alpha^- \geqslant 0$ for each $\alpha \in [0,1]$. Clearly, $\widetilde{0} \in L^*(\mathbb{R})$.

A partial order denoted by $\preccurlyeq$ on $L(\mathbb{R})$ is defined as follows:

$$\mu \preccurlyeq \nu \text{ iff } \mu_\alpha^- \leqslant \nu_\alpha^- \quad \text{and} \quad \mu_\alpha^+ \leqslant \nu_\alpha^+ \text{for all } \alpha \in [0,1].$$

The strict inequality denoted by $\prec$ on $L(\mathbb{R})$ is established as $\mu \prec \nu($ or $\nu \succ \mu)$ iff $\mu_\alpha^- < \nu_\alpha^-$and $\mu_\alpha^+ < \nu_\alpha^+$for all $\alpha \in [0,1]$.

Definition 4. *([33]) We define the operations of addition $(\oplus)$, multiplication $(\otimes)$, and scalar multiplication on the set $L(\mathbb{R})$ as follows:*

(i) The convolution of two functions μ and ν, denoted as $(\mu \oplus \nu)(t)$, is defined for any t in the real numbers $\mathbb{R}$ as the supremum of the minimum values obtained by shifting and overlapping the functions μ and ν.
(ii) The product convolution of two functions μ and ν, denoted as $(\mu \otimes \nu)(t)$, is defined for any t in the real numbers $\mathbb{R}$ as the supremum of the minimum values obtained by scaling and overlapping the functions μ and ν.
(iii) Scalar multiplication of a function μ by a scalar k is defined for any t in the real numbers $\mathbb{R}$ as μ evaluated at t/k, where k is a real number not equal to zero. Additionally, when $k = 0$, the result is the zero function $\widetilde{0}(t)$.

Theorem 1. *([33]) Let $\mu, \nu \in L(\mathbb{R})$ and $\alpha \in [0,1]$. Then we have*

$$(i)\ [\mu \oplus \nu]_\alpha = [\mu_\alpha^- + \nu_\alpha^-, \mu_\alpha^+ + \nu_\alpha^+],$$

$$(ii)\ [\mu \otimes \nu]_\alpha = [\mu_\alpha^- \nu_\alpha^-, \mu_\alpha^+ \nu_\alpha^+]\, (\mu, \nu \in L^*(\mathbb{R}))$$

$$(iii)\ [k\mu]_\alpha = k[\mu]_\alpha = \begin{cases} [k\mu_\alpha^-, k\mu_\alpha^+] & \text{if } k \geqslant 0 \\ [k\mu_\alpha^+, k\mu_\alpha^-] & \text{if } k < 0 \end{cases}$$

Theorem 2. *([14]) Let μ be a fuzzy number in $L(\mathbb{R})$, with α-level sets denoted by $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$. The theorem establishes the following:*
(i) μ_α^- is a bounded, left-continuous, non-decreasing function on $(0,1]$,
(ii) μ_α^+ is a bounded, right-continuous, non-increasing function on $(0,1]$,
(iii) at $\alpha = 0$, both μ_0^- and μ_0^+ are continuous,
(iv) μ_1^- is less than or equal to μ_1^+.

On the other hand, given functions $p(\alpha)$ and $q(\alpha)$ satisfying conditions (i)-(iv), there exists a unique fuzzy number $\mu \in L(\mathbb{R})$ such that $[\mu]_\alpha = [p(\alpha), q(\alpha)]$ for all $\alpha \in [0,1]$.

Definition 5. *([33]) Considering μ and ν as elements of the space $L(\mathbb{R})$, we define the discrepancy between two measures μ and ν, denoted as $\mathcal{F}(\mu, \nu)$, as the supremum over all possible values of α in the interval $[0,1]$ of the maximum absolute differences between the lower and upper variations of μ and ν. This function, $\mathcal{F}$, is known as the supremum metric on the set $L(\mathbb{R})$. If (μ_u) is a sequence in $L(\mathbb{R})$ and μ is an element of $L(\mathbb{R})$, we say that the sequence (μ_u) converges to μ in the metric $\mathcal{F}$, indicated as $\mu_u \xrightarrow{\mathcal{F}} \mu$ or $(\mathcal{F}) - \lim_{u\to\infty} \mu_u = \mu$, if the limit as u approaches infinity of the supremum metric $\mathcal{F}(\mu_u, \mu)$ is equal to zero.*

Definition 6. *([12]) Consider a vector space X over $\mathbb{R}$, equipped with a mapping $\|\cdot\| : X \to L^*(\mathbb{R})$, and let symmetric, non-decreasing mappings $L, R : [0,1]\times[0,1] \to [0,1]$ be given, satisfying $L(0,0) = 0$ and $R(1,1) = 1$. We denote this quadruple as $(X, \|\cdot\|, L, R)$, termed an FNS, where $\|\cdot\|$ is referred to as a fuzzy norm, if it adheres to the following conditions:*
(i) The norm of x equals zero iff x is the zero vector θ.

(ii) For any vector x in X and scalar r, the norm of the scalar multiple rx is equal to the absolute value of r multiplied by the norm of x.
(iii) For any vectors x and y in X:
(a) The norm of their sum $x+y$ is greater than or equal to the minimum of their norms.
(b) The norm of their sum $x+y$ is less than or equal to the maximum of their norms.

Additionally, functions $L(x,y)$ and $R(x,y)$ are defined as the minimum and maximum of x and y, respectively, when x and y are within the interval $[0,1]$. The FNS is denoted as $(X,|\cdot|)$ or simply X when L and R adhere to these definitions.

Lemma 1. *([12]) In a FNS, the norm of the sum of two vectors is less than or equal to the sum of their individual norms as defined in Definition 6 (iii) (a) (with L being the minimum function) is equivalent to the inequality $\|x+y\|_\alpha^- \leqslant \|x\|_\alpha^- + \|y\|_\alpha^-$, holding for all $\alpha \in (0,1]$ and $x,y \in X$.*

Lemma 2. *([12]) The triangle inequality specified in Definition 6 (iii)(b) (with $R=\max$) is equivalent to the inequality $|x+y|_\alpha^+ \leqslant |x|_\alpha^+ + |y|_\alpha^+$ for all $\alpha \in (0,1]$ and $x,y \in X$.*

Remark 1. *By referring to Theorem 1 (iii) and Lemma 1, we can infer that the condition described in Definition 6 (iii)(a) (with $L=\min$) implies that*

$$\lim_{\alpha\to 0} \|x+y\|_\alpha^- \leqslant \lim_{\alpha\to 0} \|x\|_\alpha^- + \lim_{\alpha\to 0} \|y\|_\alpha^-$$

that is, $\|x+y\|_0^- \leqslant \|x\|_0^- + \|y\|_0^-$. Similarly, according to Definition 6 (iii)(b) (with $R=\max$), it follows that the non-negative part of the sum of two elements, denoted by $\|x+y\|_0^+$, is bounded above by the sum of their respective non-negative parts, $\|x\|_0^+ + \|y\|_0^+$. Consequently, in a FNS $(X,\|\cdot\|)$, the triangle inequality specified in Definition 6 (iii) suggests that the norm of the sum of two elements, denoted by $\|x+y\|$, is less than or equal to the composition of the norms of x and y, denoted by $\|x\| \oplus \|y\|$.

According to Definition 6, we have $x=\theta$ iff $\|x\| = \widetilde{0}$, iff $\|x\|_\alpha^- = \|x\|_\alpha^+ = 0$ for all $\alpha \in [0,1]$. Furthermore, we have $\|x\|_0^- > 0$ whenever $x \neq \theta$. Now if $r=0$, then $[\|rx\|]_\alpha = [\|\theta\|]_\alpha = [0,0] = [|r|\|x\|]_\alpha$ for all $\alpha \in [0,1]$ and $x \in X$. For $r \neq 0$, we have $[\|rx\|]_\alpha = [|r|\|x\|]_\alpha$ for each $\alpha \in [0,1]$, i.e., $\|rx\|_\alpha^- = |r|\|x\|_\alpha^-$and $\|rx\|_\alpha^+ = |r|\|x\|_\alpha^+$for each $\alpha \in [0,1]$. Thus, we can say that $\|\cdot\|_\alpha^-$and $\|\cdot\|_\alpha^+$are norms on X in the usual sense in view of Definition 6, with the choice of $L=\min$ and $R=\max$, where $\alpha \in [0,1]$.

Example 1. *([33]) Let $(X, \|\cdot\|_C)$ be an ordinary normed linear space. Then a fuzzy norm $\|\cdot\|$ on X can be obtained as*

$$\|x\|(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant \zeta\|x\|_C \text{ or } t \geqslant \eta\|x\|_C \\ \frac{t}{(1-\zeta)\|x\|_C} - \frac{\zeta}{1-\zeta} & \text{if } \zeta\|x\|_C \leqslant t \leqslant \|x\|_C \\ \frac{-t}{(\eta-1)\|x\|_C} + \frac{b}{\eta-1} & \text{if } \|x\|_C \leqslant t \leqslant \eta\|x\|_C \end{cases}$$

in the given context, $\|x\|_C$ denotes the standard norm of x (excluding the zero vector), where $0 < \zeta < 1$ and $1 < \eta < \infty$. For the zero vector $x = \theta$, we define $\|x\| = \widetilde{0}$. Consequently, $(X, \|\cdot\|)$ constitutes a FNS. The specific fuzzy norm discussed here is referred to as the triangular fuzzy norm.

Example 2. *Let $(\mathbb{R}, \|\cdot\|_{\mathbb{R}})$ is a normed linear space. Then the fuzzy norm $\|\cdot\|$ on $\mathbb{R}$ can be obtained as*

$$\|x\|(s) = \begin{cases} \frac{s-|x|}{s+|x|} & s > |x| \\ 0, & s \leq |x| \end{cases}$$

and $(\mathbb{R}, \|x\|)$ is a FNS.

Definition 7. *([10]) Let $(X, \|\cdot\|)$ be an FNS. A sequence (x_r) in X converges to $x \in X$ with respect to the fuzzy norm on X, denoted by $x_r \xrightarrow{FN} x$, if $(\mathcal{F}) - \lim_{r\to\infty} \|x_r - x\| = \widetilde{0}$, where for every $\varepsilon > 0$, there exists an $N(\varepsilon) \in \mathbb{N}$ such that $\mathcal{F}\left(\|x_r - x\|, \widetilde{0}\right) < \varepsilon$ for all $r \geqslant N$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $\sup_{\alpha\in[0,1]} \|x_r - x\|_\alpha^+ = \|x_r - x\|_0^+ < \varepsilon$.*

Definition 8. *([28]) A sequence (x_r) of fuzzy numbers is considered to be statistically convergent to the fuzzy number x, denoted as st-$\lim x_r = x$, if for each $\varepsilon > 0$, there exists a positive integer N such that,*

$$\delta\left(\{r \in \mathbb{N} : \mathcal{F}(x_r, x) \geqslant \varepsilon\}\right) = 0.$$

Definition 9. *([1]) Let K be a subset of the positive integers $\mathbb{N}$, and let $K_{d,c}(n)$ denote the set of integers in the interval $[d(n)+1, c(n)]$ that belong to K, where $d = (d(n))$ and $c = (c(n))$ are sequences of non-negative integers satisfying the conditions:*

$$d(n) < c(n) \quad \text{for all} \quad n \in \mathbb{N} \text{ and } \quad \lim_{n\to\infty} c(n) = \infty.$$

The deferred density of K is denoted and defined by

$$\delta_{d,c}(K) = \lim_{n\to\infty} \frac{1}{c(n) - d(n)} \left|K_{d,c}(n)\right|.$$

Definition 10. *([23]) Consider a sequence (x_r) of real numbers. We say that (x_r) is deferred statistically convergent to $l \in \mathbb{R}$ if, for every $\varepsilon > 0$, the following holds:*

$$\lim_{n\to\infty} \frac{1}{c(n) - d(n)} \left|\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : |x_r - l| \geq \varepsilon\}\right| = 0$$

where $d=(d(n))$ and $c=(c(n))$ are sequences of non-negative integers satisfying the conditions specified in Equation (1).

For a double sequence $w=(w_{uv})$, the deferred Cesàro mean $D_{\rho,\phi}$ is defined by

$$(D_{\rho,\phi}w)_{\alpha\beta}=\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta} w_{uv}=\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta} w_{uv}$$

where $(t_\alpha),(v_\alpha),(k_\beta),(l_\beta)$ are non-negative integer sequences satisfying following conditions:

$$\begin{aligned} & t_\alpha<v_\alpha, \quad \lim_{\alpha\to\infty} v_\alpha=\infty; \quad k_\beta<l_\beta, \quad \lim_{\beta\to\infty} l_\beta=\infty \\ & v_\alpha-t_\alpha=\rho_\alpha; \quad l_\beta-k_\beta=\phi_\beta. \end{aligned} \tag{1}$$

Note here that the method $D_{\rho,\phi}$ is openly regular for any selection of the sequences $(t_\alpha),(v_\alpha),(k_\beta),(l_\beta)$.

All through this study, except where otherwise stated, $(t_\alpha),(v_\alpha),(k_\beta),(l_\beta)$ are conceived non-negative integer sequences satisfying 1.

A double sequence (w_{uv}) is strongly deferred Cesàro summable to w provided that

$$\lim_{\alpha,\beta\to\infty}\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+=0,$$

A double sequence (w_{uv}) is considered bounded with respect to the fuzzy norm X, if there exists $\mathcal{U}>0$ such that $\|w_{uv}-w\|_0^+\leq\mathcal{U}$ for all $(u,v)\in\mathbb{N}^2$. Additionally, L_∞^2 denotes the set of all bounded double sequences.

By double lacunary sequence, we mean that a double sequence $\theta_2=\{(p_\alpha,q_\beta)\}$ of two increasing integer sequences (p_α) and (q_β) such that

$$p_0=0, h_\alpha=p_\alpha-p_{\alpha-1}\to\infty \text{ and } q_0=0, \bar{h}_\beta=q_\beta-q_{\beta-1}\to\infty \text{ as } \alpha,\beta\to\infty.$$

3. New Concepts

In this section, we present the notions of deferred statistical convergence, $\mathcal{I}_2$-deferred Cesàro summability, and $\mathcal{I}_2$-deferred statistical convergence for double sequences in the context of FNS. We establish essential properties concerning these concepts and delve into defining $\mathcal{I}_2$-deferred statistical limit points as well as $\mathcal{I}_2$-deferred statistical cluster points for double sequences in FNS. Our inquiry centers on elucidating the interconnections among these introduced concepts, presenting pivotal findings that enrich the comprehension of $\mathcal{I}_2$-statistical convergence within FNS.

Throughout the article, we will consider $(X,\|.\|)$ as a FNS.

Definition 11. *A sequence (w_{uv}) in X is considered to be deferred statistically convergent to $w \in X$ regarding the fuzzy norm on X, where $(t_\alpha), (v_\alpha), (k_\beta), (l_\beta)$ are sequences of non-negative integers satisfying the conditions specified in Equation 1, then we write $w_{uv} \overset{DSt_2(FN)}{\rightarrow} w$ or $DSt_2(FN) - \lim w_{uv} = w$, provided that $DSt_2(FN) - \lim \|w_{uv} - w\| = \widetilde{0}$; i.e., for each $\lambda > 0$, we have*

$$\delta_2\left(\left\{(u,v) : t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta,\ \mathcal{F}\left(\|w_{uv} - w\|, \widetilde{0}\right) \geqslant \lambda\right\}\right) = 0,$$

or equivalently,

$$\lim_{\alpha,\beta\to\infty} \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta,\ \mathcal{F}\left(\|w_{uv} - w\|, \widetilde{0}\right) \ge \lambda\right\}\right| = 0.$$

This implies that for each $\lambda > 0$, the set

$$K(\lambda) = \left\{(u,v) : t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta,\ \|w_{uv} - w\|_0^+ \geqslant \lambda\right\}$$

has a natural density of zero. That is, for each $\lambda > 0$, $\|w_{uv} - w\|_0^+ < \lambda$ for a.a. u, v (all most all u, v). The element w belongs to the set X serves as the deferred statistical limit of the double sequence (w_{uv}).
A concise and insightful interpretation of the mentioned definition is as follows:

$$w_{uv} \overset{DSt_2(FN)}{\rightarrow} w \text{ iff } DSt_2(FN) - \lim \|w_{uv} - w\|_\alpha^+ = 0.$$

Noting that $DSt_2(FN) - \lim \|w_{uv} - w\|_\alpha^+ = 0$ implies that

$$DSt_2(FN) - \lim \|w_{uv} - w\|_\alpha^- = DSt_2(FN) - \lim \|w_{uv} - w\|_\alpha^+ = 0$$

for each $\alpha \in [0,1]$ since

$$0 \leqslant \|w_{uv} - w\|_\alpha^- \leqslant \|w_{uv} - w\|_\alpha^+ \leqslant \|w_{uv} - w\|_0^+$$

holds for every $u, v \in \mathbb{N}$ and for each $\alpha \in [0,1]$.

Example 3. *Let $(\mathbb{R}, \|\cdot\|_{\mathbb{R}})$ be an FNS. Then, a fuzzy norm $\|\cdot\|$ on $\mathbb{R}$ is define in Example 2 and $(t_\alpha), (v_\alpha), (k_\beta), (l_\beta)$ are sequences of non-negative integers satisfying the conditions specified in Equation 1. Define the sequence $w = (w_{uv})$ as*

$$w_{uv} := \begin{cases} u^2v^2; & [\sqrt{v_\alpha}] - 1 < u \le [\sqrt{v_\alpha}] \\ & [\sqrt{l_\beta}] - 1 < v \le [\sqrt{l_\beta}] \quad \alpha, \beta = 1, 2, 3, \ldots \\ 0; & \text{otherwise.} \end{cases}$$

where $0 < t_\alpha \le [\sqrt{v_\alpha}] - 1$, $0 < k_\beta \le [\sqrt{l_\beta}] - 1$ and (v_α), (l_β) are monotonic increasing sequences. Then, $w_{uv} \overset{DSt_2(FN)}{\rightarrow} 0$.
Justification: For every $0 < \lambda < 1$, $s > \|w\|$ we have

$$K(\lambda) = \left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta : \|w_{uv} - 0\|_0^+ \geqslant \lambda\right\}.$$

This implies that,

$$K(\lambda) = \left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta : \tfrac{s-\|w_{uv}\|}{s+\|w_{uv}\|} \geqslant \lambda\right\}$$

$$= \left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta : \|w_{uv}\| \leqslant \tfrac{s(1-\lambda)}{1+\lambda}\right\}$$

for suitable value of s and λ, we get $\{(u,v) : \|w_{uv}\| \geqslant 0\}$. Hence

$$K(\lambda) = \left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta,\ \ \|w_{uv} - 0\|_0^+ \geq \lambda\right\}$$

$$= \left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta,\ w_{uv} = u^2 v^2\right\}$$

$$= \{(1,1), (4,4), (9,9), (16,16), \ldots\} \in \mathcal{I}_2.$$

As a result, $w_{uv} \overset{DSt_2(FN)}{\rightarrow} 0.$

Definition 12. *The double sequence (w_{uv}) is called to be $\mathcal{I}_2$-deferred Cesàro summable to $w \in X$ regarding the fuzzy norm on X, if for each $\lambda > 0$*

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \left\| \frac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1 \\ v=k_\beta+1}}^{v_\alpha, l_\beta} w_{uv} - w \right\|_0^+ \geq \lambda \right\} \in \mathcal{I}_2,$$

and this condition is denoted in the format $w_{uv} \overset{DC_1(\mathcal{I}_2)(FN)}{\rightarrow} w$ *or* $DC_1(\mathcal{I}_2)(FN) - \lim w_{uv} = w$.

Definition 13. *The double sequence (w_{uv}) is said to be strongly $\mathcal{I}_2$-deferred Cesàro summable to $w \in X$ regarding the fuzzy norm on X, if for each $\lambda > 0$*

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1 \\ v=k_\beta+1}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+ \geq \lambda \right\} \in \mathcal{I}_2$$

and this case is denoted in $w_{uv} \overset{DC_1[\mathcal{I}_2](FN)}{\rightarrow} w$ *or* $DC_1[\mathcal{I}_2] - \lim w_{uv} = w$ *format.*

The notation $DC_1[\mathcal{I}_2](FN)$ represents the collection of all double sequences that exhibit strongly $\mathcal{I}_2$-deferred Cesàro summability with respect to the fuzzy norm X.

Remark 2. *$DC_1(\mathcal{I}_2)(FN)$ and $DC_1[\mathcal{I}_2](FN)$-summability concepts;*
(i) For $t_\alpha = 0$, $v_\alpha = \alpha$ and $k_\beta = 0$, $l_\beta = \beta$, match with $\mathcal{I}_2$-Cesàro and strongly $\mathcal{I}_2$-Cesàro summability concepts regarding the fuzzy norm on X, respectively.
(ii) For $t_\alpha = p_{\alpha-1}$, $v_\alpha = p_\alpha$ and $k_\beta = q_{\beta-1}$, $l_\beta = q_\beta$ $\{(p_\alpha, q_\beta)\}$ states double lacunary sequence), match with $\mathcal{I}_2$-lacunary and strongly $\mathcal{I}_2$-lacunary convergence concepts regarding the fuzzy norm on X, respectively.
(iii) For the ideal $\mathcal{I}_2^f$ (the ideal of density zero sets of $\mathbb{N}^2$), match with deferred

Cesàro and strongly deferred Cesàro summability concepts regarding the fuzzy norm on X, respectively.

Definition 14. *The double sequence (w_{uv}) is considered to be $\mathcal{I}_2$-deferred statistical convergent to $w \in X$ regarding the fuzzy norm on X, if for every $\lambda, \mu > 0$, the set*

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta,\ \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\} \in \mathcal{I}_2.$$

This scenario is denoted as $w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\rightarrow} w$ or $DS(\mathcal{I}_2)(FN) - \lim w_{uv} = w$. The collection of all double sequences of sets that are $\mathcal{I}_2$-deferred statistically convergent with respect to the fuzzy norm X is represented by $DS(\mathcal{I}_2)(FN)$.

Remark 3. *The concepts outlined match with different forms of convergence within the framework of $DS(\mathcal{I}_2)(FN)$;*
(i) When $t_\alpha = 0, v_\alpha = \alpha$ and $k_\beta = 0, l_\beta = \beta$, it corresponds to the $\mathcal{I}_2$-statistical convergence concept with respect to the fuzzy norm X.
(ii) When $t_\alpha = p_{\alpha-1}, v_\alpha = p_\alpha$ and $k_\beta = q_{\beta-1}, l_\beta = q_\beta$ (where $\{(p_\alpha, q_\beta)\}$ denotes a double lacunary sequence), it aligns with the $\mathcal{I}_2$-lacunary statistical convergence concept with respect to the fuzzy norm X.
(iii) When considering the ideal $\mathcal{I}_2^f$ (the ideal of density zero sets of $\mathbb{N}$), it corresponds to the deferred statistical convergence concept with respect to the fuzzy norm X.

Definition 15. *Let (w_{uv}) be a sequence in $(X, \|\cdot\|)$ with $(t_\alpha), (v_\alpha), (k_\beta), (l_\beta)$ being sequences of non-negative integers satisfying the conditions specified in Equation 1. We say that the sequence (w_{uv}) in X is $\mathcal{I}_2$-deferred statistically Cauchy with respect to the fuzzy norm on X if, for every $\lambda, \mu > 0$, there exist natural numbers $N = N(\lambda)$, $M = M(\lambda)$ such that*

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w_{NM}\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\} \in \mathcal{I}_2.$$

Theorem 3. *Let (w_{uv}) be a sequence in $(X, \|\cdot\|)$ with $(t_\alpha), (v_\alpha), (k_\beta), (l_\beta)$ are sequences of non-negative integers satisfying the conditions specified in Equation 1. Then, every $\mathcal{I}_2$-deferred statistically convergent sequence is also a $\mathcal{I}_2$-deferred statistically Cauchy sequence.*

Proof. Let $DS(\mathcal{I}_2)(FN) - \lim w_{uv} = w$ and $\lambda, \mu > 0$. Then, we have

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \frac{\lambda}{2}\right\}\right| \geq \frac{\mu}{2}\right\} \in \mathcal{I}_2.$$

Choose N, $M \in \mathbb{N}$ such that

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w_{NM}\|_0^+ \geq \frac{\lambda}{2}\right\}\right| \geq \frac{\mu}{2}\right\} \in \mathcal{I}_2.$$

Now $\|\cdot\|_0^+$ being a norm in the usual sense, we get

$$\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|w_{uv}-w_{NM}\|_0^+\ge\lambda\right\}\right|\ge\mu\right\}$$

$$=\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|(w_{uv}-w)+(w-w_{NM})\|_0^+\ge\lambda\right\}\right|\ge\mu\right\}$$

$$\subseteq\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|w_{uv}-w\|_0^+\ge\frac{\lambda}{2}\right\}\right|\ge\frac{\mu}{2}\right\}$$

$$\cup\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|w_{uv}-w_{NM}\|_0^+\ge\frac{\lambda}{2}\right\}\right|\ge\frac{\mu}{2}\right\}\in\mathcal{I}_2.$$

This indicates that the double sequence (w_{uv}) is $\mathcal{I}_2$-deferred statistically Cauchy. □

4. Main Results

In this section, we initially explore the connections between $DC_1[\mathcal{I}_2](FN)$-summability and $DS(\mathcal{I}_2)(FN)$-convergence concepts.

Theorem 4. *Let (w_{uv}), (t_{uv}) be sequences of real numbers, then*
(i) If $DS(\mathcal{I}_2)(FN)-\lim w_{uv}=w_0$ and $DS(\mathcal{I}_2)(FN)-\lim t_{uv}=t_0$, then
$DS(\mathcal{I}_2)(FN)-\lim(w_{uv}+t_{uv})=w_0+t_0$,
(ii)If $DS(\mathcal{I}_2)(FN)-\lim w_{uv}=w_0$ and $q\in\mathbb{C}$, then
$DS(\mathcal{I}_2)(FN)-\lim(qw_{uv})=qw_0$,
(iii) If $DS(\mathcal{I}_2)(FN)-\lim w_{uv}=w_0$ and $DS(\mathcal{I}_2)(FN)-\lim t_{uv}=t_0$, and there are positive numbers u and v such that $\|w_{uv}\|\le u$ and $\|t_0\|\le v$ for any u,v, then
$DS(\mathcal{I}_2)(FN)-\lim(w_{uv}t_{uv})=w_0t_0$.

Proof. (i) Assume that $DS(\mathcal{I}_2)(FN)-\lim w_{uv}=w_0$ and $DS(\mathcal{I}_2)(FN)-\lim t_{uv}=t_0$. Since $\|\cdot\|_0^+$ is a norm in the usual sense, we get

$$\|(w_{uv}+t_{uv})-(w_0+t_0)\|_0^+\leqslant\|w_{uv}-w_0\|_0^++\|t_{uv}-t_0\|_0^+ \tag{2}$$

for all $u,v\in\mathbb{N}$. Now let us write

$$K(\lambda)=\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta:\|(w_{uv}+t_{uv})-(w_0+t_0)\|_0^+\geqslant\lambda\right\},$$

$$K_1(\lambda)=\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta:\|w_{uv}-w_0\|_0^+\geqslant\frac{\lambda}{2}\right\}$$

$$K_2(\lambda)=\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta:\|t_{uv}-t_0\|_0^+\geqslant\frac{\lambda}{2}\right\}.$$

Therefore, based on Equation 2, it follows that $K(\lambda)\subseteq K_1(\lambda)\cup K_2(\lambda)$. Given our assumption that $K_1(\lambda),K_2(\lambda)\in\mathcal{I}_2$. We conclude that $K(\lambda)\in\mathcal{I}_2$, thereby completing the proof.

(ii) Let, $DS(\mathcal{I}_2)(FN) - \lim w_{uv} = w_0$, then $q \in \mathbb{R} - \{0\}$ for every $\lambda > 0$, we have

$$\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta : \|w_{uv} - w_0\|_0^+ \geqslant \frac{\lambda}{|q|}\right\} \in \mathcal{I}_2,$$

$$\Longrightarrow \left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta : \|qw_{uv} - qw_0\|_0^+ \geqslant \lambda\right\} \in \mathcal{I}_2.$$

So $DS(\mathcal{I}_2)(FN) - \lim (qw_{uv}) = qw_0$, $(q \in \mathbb{R})$.

(iii) Assume $\lambda, \mu > 0$ and $u, v > 0$ then

$$K = \left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda\right\}\right| < \tfrac{\mu}{2v}\right\} \in \mathcal{F}(\mathcal{I}_2)$$

and

$$L = \left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|t_{uv} - t_0\|_0^+ \geq \lambda\right\}\right| < \tfrac{\mu}{2u}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Since $K \cap L \in \mathcal{F}(\mathcal{I}_2)$ and $\emptyset \notin \mathcal{F}(\mathcal{I}_2)$ this means $K \cap L \neq \emptyset$. So, for all $(u,v) \in K \cap L$ we have

$$\begin{aligned}\|w_{uv}t_{uv} - w_0t_0\|_0^+ &= \|w_{uv}t_{uv} - w_{uv}t_0 + w_{uv}t_0 - w_0t_0\|_0^+ \\ &\leq \|w_{uv}t_{uv} - w_{uv}t_0\|_0^+ + \|w_{uv}t_0 - w_0t_0\|_0^+ \\ &\leq u\|t_{uv} - t_0\|_0^+ + v\|w_{uv} - w_0\|_0^+ < u\tfrac{\mu}{2u} + v\tfrac{\mu}{2v} = \mu,\end{aligned}$$

i.e.,

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv}t_{uv} - w_0t_0\|_0^+ \geq \lambda\right\}\right| < \mu\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Hence $DS(\mathcal{I}_2)(FN) - \lim (w_{uv}t_{uv}) = w_0t_0$. □

Theorem 5. *$DS(\mathcal{I}_2)(FN) \cap L_\infty^2$ is a closed subset of L_∞^2.*

Proof. Suppose that $\left(w^j\right)_{j\in\mathbb{N}} = \left(w_{uv}^j\right) \subseteq DS(\mathcal{I}_2)(FN) \cap L_\infty^2$ is convergent sequence and that it converges to $w \in L_\infty^2$. We need to prove that $w \in DS(\mathcal{I}_2)(FN) \cap L_\infty^2$. Assume that $w^j \to L_j(DS(\mathcal{I}_2)(FN))$ for $\forall j \in \mathbb{N}$. Take a positive strictly decreasing sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ where $\lambda_j = \frac{\lambda}{2^j}$ for a given $\lambda > 0$. It is evident that the sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ converges to 0. Let's select positive integer j such that $\left\|w - w^j\right\|_\infty < \frac{\lambda_j}{4}$. Let $0 < \mu < 1$. Then

$$\begin{aligned}A = \Big\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\,\big|\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta,\\ \left\|w_{uv}^j - L_j\right\|_0^+ \geq \tfrac{\lambda_j}{4}\Big\}\Big| < \tfrac{\mu}{3}\Big\} \in \mathcal{F}(\mathcal{I}_2),\end{aligned}$$

and

$$\begin{aligned}B = \Big\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\,\big|\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta,\\ \left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ \geq \tfrac{\lambda_{j+1}}{4}\Big\}\Big| < \tfrac{\mu}{3}\Big\} \in \mathcal{F}(\mathcal{I}_2).\end{aligned}$$

Since $A \cap B \in \mathcal{F}(\mathcal{I}_2)$ and $\phi \notin \mathcal{F}(\mathcal{I}_2)$, we can choose $(\alpha,\beta) \in A \cap B$. Then

$$\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \left\|w_{uv}^j - L_j\right\|_0^+ \geq \frac{\lambda_j}{4}\right\}\right| < \frac{\mu}{3},$$

and

$$\frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ \ge \frac{\lambda_{j+1}}{4}\right\}\right| < \frac{\mu}{3}$$

and so

$$\frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^{j} - L_{j}\right\|_0^+ \ge \frac{\lambda_{j}}{4} \vee \left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ \ge \frac{\lambda_{j+1}}{4}\right\}\right| < \mu < 1.$$

Hence, there exist $t_\alpha < u \le v_\alpha$, $k_\beta < v \le l_\beta$ for which $\left\|w_{uv}^{j} - L_{j}\right\|_0^+ \ge \frac{\lambda_j}{4}$ and $\left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ \ge \frac{\lambda_{j+1}}{4}$. Then, we can write

$$\begin{aligned}\|L_j - L_{j+1}\|_0^+ &\le \left\|L_j - w_{uv}^j\right\|_0^+ + \left\|w_{uv}^j - w_{uv}^{j+1}\right\|_0^+ + \left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ \\ &\le \left\|w_{uv}^j - L_j\right\|_0^+ + \left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ + \left\|w - w^j\right\|_\infty + \left\|w - w^{j+1}\right\|_\infty \\ &\le \frac{\lambda_j}{4} + \frac{\lambda_{j+1}}{4} + \frac{\lambda_j}{4} + \frac{\lambda_{j+1}}{4} \le \lambda_j.\end{aligned}$$

This implies that $\{L_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and so there is a real number L such that $L_j \to L$, as $j \to \infty$. We need to prove that $w \to L\,(DS\,(\mathcal{I}_2)\,(FN))$. For any $\lambda > 0$, choose $j \in \mathbb{N}$ such that $\lambda_j < \frac{\lambda}{4}$, $\left\|w - w^j\right\|_\infty < \frac{\lambda}{4}$, $\|L_j - L\|_0^+ < \frac{\lambda}{4}$. Then

$$\begin{aligned}&\frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv} - L\|_0^+ \ge \lambda\right\}\right| \\ &\le \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^j - L_j\right\|_0^+ + \left\|w_{uv} - w_{uv}^j\right\|_\infty + \|L_j - L\|_0^+ \ge \lambda\right\}\right| \\ &\le \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^j - L_j\right\|_0^+ + \frac{\lambda}{4} + \frac{\lambda}{4} \ge \lambda\right\}\right| \\ &\le \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^j - L_j\right\|_0^+ \ge \frac{\lambda}{2}\right\}\right|.\end{aligned}$$

This implies that

$$\begin{aligned}&\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv} - L\|_0^+ \ge \lambda\right\}\right| < \mu\right\} \\ &\supseteq \left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta,\right.\right. \\ &\left.\left.\left\|w_{uv}^j - L_j\right\|_0^+ \ge \frac{\lambda}{2}\right\}\right| < \mu\right\} \in \mathcal{F}\,(\mathcal{I}_2)\,.\end{aligned}$$

So

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv} - L\|_0^+ \ge \lambda\right\}\right| < \mu\right\} \in \mathcal{F}\,(\mathcal{I}_2)\,,$$

and hence

$$\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\leq v_\alpha,k_\beta<v\leq l_\beta,\|w_{uv}-L\|_0^+\geq\lambda\right\}\right|\geq\mu\right\}\in\mathcal{I}_2.$$

This implies that $w\to L\left(DS\left(\mathcal{I}_2\right)(FN)\right)$, thereby completing the proof of the theorem. □

Theorem 6. *If a double sequence (w_{uv}) is strongly $\mathcal{I}_2$-deferred Cesàro summable to $w\in X$, then this sequence is $\mathcal{I}_2$-deferred statistical convergent to $w\in X$. Also, the inclusion $DC_1\left[\mathcal{I}_2\right](FN)\subseteq DS\left(\mathcal{I}_2\right)(FN)$ is strict.*

Proof. Suppose that $w_{uv}\overset{DC_1[\mathcal{I}_2](FN)}{\rightarrow}w$. For each $\lambda>0$, we can express

$$\sum_{\substack{u=t_\alpha+1\\v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+\geq\sum_{\substack{u=t_\alpha+1\\v=k_\beta+1\\\|w_{uv}-w\|_0^+\geq\lambda}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+$$

$$\geq\lambda\left|\left\{(u,v):t_\alpha<u\leq v_\alpha,k_\beta<v\leq l_\beta,\|w_{uv}-w\|_0^+\geq\lambda\right\}\right|,$$

and therefore, we have

$$\frac{1}{\lambda}\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+\geq\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\leq v_\alpha,k_\beta<v\leq l_\beta,\|w_{uv}-w\|_0^+\geq\lambda\right\}\right|.$$

For every $\mu>0$ we obtain

$$\left\{(\alpha,\beta)\in\mathbb{N}^2:\tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\leq v_\alpha,\ k_\beta<v\leq l_\beta,\|w_{uv}-w\|_0^+\geq\lambda\right\}\right|\geq\mu\right\}$$

$$\subseteq\left\{(\alpha,\beta)\in\mathbb{N}^2:\tfrac{1}{\rho_\alpha\phi_\beta}\textstyle\sum_{\substack{u=t_\alpha+1\\v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+\geq\lambda\mu\right\}\in\mathcal{I}_2.$$

Thus, we get $w_{uv}\overset{DS(\mathcal{I}_2)(FN)}{\rightarrow}w$.

To show the strictness of the inclusion, choose $v_\alpha=\alpha$, $t_\alpha=0$ and $l_\beta=\beta$, $k_\beta=0$ define a sequence (w_{uv}) by

$$w_{uv}=\begin{cases}\sqrt{pq}, & u=p^2,\ v=q^2\\ 0, & u\neq p^2,\ v\neq q^2.\end{cases}$$

Then, for every $\lambda>0$, we have

$$\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\leq v_\alpha,\ k_\beta<v\leq l_\beta,\|w_{uv}-0\|_0^+\geq\lambda\right\}\right|\leq\frac{\sqrt{pq}}{pq}$$

and for any $\mu > 0$ we get

$$\left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv}-0\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\}$$

$$\subseteq \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{\sqrt{pq}}{pq} \ge \mu\right\}.$$

As the set on the right-hand side is finite and thus falls within $\mathcal{I}_2$, it implies that $DS(\mathcal{I}_2)(FN) - \lim w_{uv} = 0$. On the other hand

$$\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-0\|_0^+ = \frac{[\sqrt{pq}][\sqrt{pq}]}{pq} \to 1.$$

Then

$$\left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-0\|_0^+ \ge 1\right\}$$

$$= \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{\sqrt{pq}}{pq} \ge 1\right\} = \{(m,n),(m+1,n+1),(m+2,n+2),\ldots\}$$

for some $m,n\in\mathbb{N}$ which belongs to $\mathcal{F}(\mathcal{I}_2)$, since $\mathcal{I}_2$ is admissible. So $w_{uv} \nrightarrow 0\,(DC_1[\mathcal{I}_2](FN))$. □

Corollary 1. *If $w_{uv} \overset{\mathcal{I}_2(FN)}{\to} w$, then $w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\to} w$.*

The converse of Theorem 6 is not generally valid. To illustrate this point, we can consider the following example by choosing $\mathcal{I}_2 = \mathcal{I}_2^f$ (the ideal of density zero sets of $\mathbb{N}$).

Example 4. *Consider $X = \mathbb{R}^2$ and let (w_{uv}) denote a double sequence defined as follows:*

$$w_{uv} := \begin{cases} u^2v^2; & v_\alpha - \left[\left|\sqrt{\rho_\alpha}\right|\right] < u \le v_\alpha, \\ & l_\beta - \left[\left|\sqrt{\phi_\beta}\right|\right] < v \le l_\beta, \quad (u,v)\in\mathbb{N}^2, \\ 0; & \text{otherwise.} \end{cases}$$

This sequence is unbounded. Additionally, it is $\mathcal{I}_2$-deferred statistical convergent to $w = 0$ with respect to the fuzzy norm X, but it is not strongly $\mathcal{I}_2$-deferred Cesàro summable with respect to the fuzzy norm X.

Theorem 7. *If a double sequence $(w_{uv}) \in L_\infty^2$ is $\mathcal{I}_2$-deferred statistical convergent to $w \in X$ with respect to the fuzzy norm X, then this sequence is also strongly $\mathcal{I}_2$-deferred Cesàro summable to the same limit.*

Proof. Suppose that $(w_{uv}) \in L^2_\infty$ and $w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\rightarrow} w$. Let $(w_{uv}) \in L^2_\infty$. Therefore, there exists $\mathcal{U} > 0$ such that $\|w_{uv} - w\|_0^+ \le \mathcal{U}$ for all $(u, v) \in \mathbb{N}^2$. For each $\lambda > 0$, we have

$$\begin{aligned}
&\tfrac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+ \\
&= \tfrac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1\\ \|w_{uv}-w\|_0^+ \ge \lambda}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+ + \tfrac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1\\ \|w_{uv}-w\|_0^+ < \lambda}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+ \\
&\le \tfrac{\mathcal{U}}{\rho_\alpha \phi_\beta} \left| \left\{ (u, v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv} - w\|_0^+ \ge \lambda \right\} \right| + \lambda.
\end{aligned}$$

So, for all $\mu > 0$ we get

$$\left\{ (\alpha, \beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+ \ge \mu \right\}$$

$$\subseteq \left\{ (\alpha, \beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta} \left| \left\{ (u, v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv} - w\|_0^+ \ge \lambda \right\} \right| \ge \tfrac{\mu}{\mathcal{U}} \right\} \in \mathcal{I}_2.$$

As a result, we get $w_{uv} \overset{DC_1[\mathcal{I}_2](FN)}{\rightarrow} w$. □

By Theorem 6 and Theorem 7, we obtain the following corollary.

Corollary 2. $L^2_\infty \cap DC_1 [\mathcal{I}_2] (FN) = L^2_\infty \cap DS (\mathcal{I}_2) (FN)$.

Theorem 8. *Let* $\left(\frac{t_\alpha}{\rho_\alpha}\right)$ *and* $\left(\frac{k_\beta}{\phi_\beta}\right)$ *be bounded, then*

$$w_{uv} \overset{S(\mathcal{I}_2)(FN)}{\rightarrow} w \Rightarrow w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\rightarrow} w.$$

Proof. To begin, given that $\left(\frac{t_\alpha}{\rho_\alpha}\right)$ is bounded, there exists exists a positive value $\sigma > 0$ such that $\frac{t_\alpha}{\rho_\alpha} < \sigma$ for all $\alpha \in \mathbb{N}$. Therefore, we express this as:

$$\frac{t_\alpha}{\rho_\alpha} < \sigma \Rightarrow \frac{\rho_\alpha}{v_\alpha} > \frac{1}{1 + \sigma}.$$

Likewise, for each $\kappa \in \mathbb{N}$, we can derive the following inequalities

$$\frac{k_\beta}{\phi_\beta} < \kappa \Rightarrow \frac{\phi_\beta}{l_\beta} > \frac{1}{1 + \kappa}.$$

Assume that $w_{uv} \overset{S(\mathcal{I}_2)(FN)}{\rightarrow} w$. For each $\lambda > 0$, we have

$$\begin{aligned}&\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\\ &\subseteq \left\{(u,v) : u \leq v_\alpha, v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\},\end{aligned}$$

and so

$$\begin{aligned}&\tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right|\\ &\leq \tfrac{v_\alpha l_\beta}{\rho_\alpha \phi_\beta}\tfrac{1}{v_\alpha l_\beta}\left|\left\{(u,v) : u \leq v_\alpha, v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right|.\end{aligned}$$

Thus, for any positive value $\mu > 0$ we obtain

$$\begin{aligned}&\left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\}\\ &\subseteq \left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{v_\alpha l_\beta}\left|\left\{(u,v) : u \leq v_\alpha, v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \geq \tfrac{\mu}{(1+\sigma)(1+\kappa)}\right\}.\end{aligned}$$

As a result, we get $w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\rightarrow} w$. □

We will examine the following theorems under the given constraints:

$$t_\alpha \leq t'_\alpha < v'_\alpha \leq v_\alpha \quad \text{and} \quad k_\beta \leq k'_\beta < l'_\beta \leq l_\beta$$

for all $\alpha, \beta \in \mathbb{N}$, where each of these represents sequences of non-negative integers.

Theorem 9. *If* $\left(\frac{\rho_\alpha \phi_\beta}{\rho'_\alpha \phi'_\beta}\right)$ *is bounded, then*

$$DS(\mathcal{I}_2)(FN)_{[\rho,\phi]} \subseteq DS(\mathcal{I}_2)(FN)_{[\rho',\phi']}.$$

Proof. To begin, given that $\left(\frac{\rho_\alpha \phi_\beta}{\rho'_\alpha \phi'_\beta}\right)$ is bounded, there exists an $\varpi > 0$ such that $\frac{\rho_\alpha \phi_\beta}{\rho'_\alpha \phi'_\beta} < \varpi$ for all $\alpha, \beta \in \mathbb{N}$. Assuming $(w_{uv}) \in DS(\mathcal{I}_2)(FN)_{[\rho,\phi]}$ and $w_{uv} \overset{DS(\mathcal{I}_2)(FN)_{[\rho,\phi]}}{\rightarrow} w$. For any $\lambda > 0$ since

$$\begin{aligned}&\left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\\ &\subseteq \left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\},\end{aligned}$$

we can express

$$\begin{aligned}&\tfrac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right|\\ &\leq \tfrac{\rho_\alpha \phi_\beta}{\rho'_\alpha \phi'_\beta}\left(\tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right|\right)\end{aligned}$$

So, for each $\mu > 0$ we obtain

$$\begin{aligned}&\left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\}\\ &\subseteq \left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \geq \tfrac{\mu}{\varpi}\right\}.\end{aligned}$$

Thus, we get $w_{uv} \overset{DS(\mathcal{I}_2)(FN)_{[\rho',\phi']}}{\rightarrow} w$. As a result,

$$DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho,\phi]} \subseteq DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho',\phi']}.$$

□

Theorem 10. *If the sets* $\{u : t_\alpha < u \leq t'_\alpha\}, \{u : v'_\alpha < u \leq v_\alpha\}, \left\{v : k_\beta < v \leq k'_\beta\right\}$, $\left\{v : l'_\beta < v \leq l_\beta\right\}$ *are finite for every* $\alpha, \beta \in \mathbb{N}$, *then*

$$DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho',\phi']} \subseteq DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho,\phi]}.$$

Proof. Let $(w_{uv}) \in DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho',\phi']}$ and $w_{uv} \overset{DS(\mathcal{I}_2)(FN)_{[\rho',\phi']}}{\rightarrow} w$. Then, for all $\lambda, \mu > 0$ we have

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\} \in \mathcal{I}_2.$$

Additionally, for each $\lambda > 0$, since

$$\begin{aligned}
&\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&= \left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ k_\beta < v \leq k'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ l'_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ k_\beta < v \leq k'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ l'_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : v'_\alpha < u \leq v_\alpha,\ k_\beta < v \leq k'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : v'_\alpha < u \leq v_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : v'_\alpha < u \leq v_\alpha,\ l'_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\},
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \\
&\quad \leq \quad \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ k_\beta < v \leq k'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \\
&\quad + \quad \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \\
&\quad + \quad \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ l'_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right|
\end{aligned}$$

$$
\begin{aligned}
&+\quad \frac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+\quad \frac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+\quad \frac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+\quad \frac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+\quad \frac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+\quad \frac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right|,
\end{aligned}
$$

and hence, for all $\mu > 0$

$$
\begin{aligned}
&\left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v): t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\subseteq \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t_\alpha < u \le t'_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t_\alpha < u \le t'_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t_\alpha < u \le t'_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\}.
\end{aligned}
$$

If the sets $\{u : t_\alpha < u \le t'_\alpha\}$, $\{u : v'_\alpha < u \le v_\alpha\}$, $\left\{v : k_\beta < v \le k'_\beta\right\}$, $\left\{v : l'_\beta < v \le l_\beta\right\}$ are all finite for every $\alpha, \beta \in \mathbb{N}$ within the given expression, based on the assumption, we conclude

$$
\left\{(\alpha,\beta)\in\mathbb{N}^2 : \frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v): t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \in \mathcal{I}_2,
$$

This indicates that $w_{uv} \overset{DS(\mathcal{I}_2)(FN)_{[\rho,\phi]}}{\to} w$ and $(w_{uv}) \in DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho,\phi]}$. Hence, $DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho',\phi']} \subseteq DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho,\phi]}$. □

Based on Theorem 6, Theorem 7, Theorem 9 and Theorem 10, we derive the following corollary.

Corollary 3. *(i) Assuming that* $\left(\frac{\rho_\alpha \phi_\beta}{\rho'_\alpha \phi'_\beta}\right)$ *is bounded. If a double sequence* (w_{uv}) *is* $DC_1\left[\mathcal{I}_2\right](FN)_{[\rho,\phi]}$*-summable to* $w \in X$ *with respect to the fuzzy norm* X*, then this sequence is also* $DS\left(\mathcal{I}_2\right)(FN)_{[\rho',\phi']}$*-convergent to* $w \in X$.

(ii) Let the sets $\{u : t_\alpha < u \leq t'_\alpha\}$, $\{u : v'_\alpha < u \leq v_\alpha\}$, $\left\{v : k_\beta < v \leq k'_\beta\right\}$, $\left\{v : l'_\beta < v \leq l_\beta\right\}$ *are finite for all* $\alpha, \beta \in \mathbb{N}$*. If a double sequence* $(w_{uv}) \in L^2_\infty$ *is* $DS\left(\mathcal{I}_2\right)(FN)_{[\rho',\phi']}$*-convergent to* $w \in X$ *with respect to the fuzzy norm* X*, then this sequence is* $DC_1\left[\mathcal{I}_2\right](FN)_{[\rho,\phi]}$*-summable to* $w \in X$.

Now, we will examine the notions of $\mathcal{I}_2$-deferred statistical limit points and $\mathcal{I}_2$-deferred statistical cluster points of a double sequence of fuzzy numbers, expanding upon the concepts previously discussed regarding a sequence of fuzzy numbers. Additionally, attention will be directed towards significant fundamental characteristics pertaining to the set of all $\mathcal{I}_2$-deferred statistical cluster points and the set of all $\mathcal{I}_2$-deferred statistical limit points of a double sequence of fuzzy numbers, and an exploration of the relationship between them will be conducted.

Definition 16. *An element* $w_0 \in X$ *is termed an* $\mathcal{I}_2$*-deferred statistical limit point of double sequence* (w_{uv}) *with respect to the fuzzy norm* X *if, for each* $\lambda > 0$ *there exists a set*

$$U = \{(u_1, v_1) < (u_2, v_2) < ... < (u_r, v_s) < ...\} \subset \mathbb{N}^2$$

such that $U \notin \mathcal{I}_2$ *and* $DSt_2(FN) - \lim w_{u_r,v_s} = w_0$.

The notation $\mathcal{I}_2^{FN} - S(\Lambda_w)$ *represents the set comprising all* $\mathcal{I}_2$*-deferred statistical limit point of a double sequence* (w_{uv}).

Theorem 11. *If* $DS\left(\mathcal{I}_2\right)(FN) - \lim w_{uv} = w_0$*, then* $\mathcal{I}_2^{FN} - S(\Lambda_w) = \{w_0\}$.

Proof. Given that $DS\left(\mathcal{I}_2\right)(FN) - \lim w_{uv} = w_0$, for each $\lambda, \mu > 0$, the set

$$T = \left\{(\alpha, \beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\} \in \mathcal{I}_2,$$

where $\mathcal{I}_2$ is an admissible ideal.

Let's assume that $\mathcal{I}_2^{FN} - S(\Lambda_w)$ includes q_0 distinct from w_0, that is, $q_0 \in \mathcal{I}_2^{FN} - S(\Lambda_w)$. Therefore, there exists a $U \subset \mathbb{N}^2$ such that $U \notin \mathcal{I}_2$ and $DSt_2(FN) - \lim w_{u_r,v_s} = q_0$.

Let

$$P = \left\{(\alpha, \beta) \in M : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - q_0\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\}.$$

So, P is a finite set, implying that $P \in \mathcal{I}_2$. So

$$P^c = \left\{(\alpha, \beta) \in M : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - q_0\|_0^+ \geq \lambda\right\}\right| < \mu\right\} \in \mathcal{F}\left(\mathcal{I}_2\right).$$

Let T_1 be defined as follows:

$$T_1 = \left\{ (\alpha,\beta) \in M : \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{ t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda \right\} \right| \geq \mu \right\}.$$

So $T_1 \subset T \in \mathcal{I}_2$, i.e., $T_1^c \in \mathcal{F}(\mathcal{I}_2)$. Therefore, $T_1^c \cap P^c \neq \emptyset$, since $T_1^c \cap P^c \in \mathcal{F}(\mathcal{I}_2)$.

Suppose $(i,j) \in K_1^c \cap P^c$ and let $\lambda := \frac{\|w_0 - q_0\|_0^+}{3} > 0$. Then

$$\tfrac{1}{\rho_i \phi_j} \left| \left\{ t_i < u \leq v_i,\ k_j < v \leq l_j, \|w_{uv} - w_0\|_0^+ \geq \lambda \right\} \right| < \mu \text{ and}$$

$$\tfrac{1}{\rho_i \phi_j} \left| \left\{ t_i < u \leq v_i,\ k_j < v \leq l_j, \|w_{uv} - q_0\|_0^+ \geq \lambda \right\} \right| < \mu,$$

which means, for the maximum $t_i < u \leq v_i$, $k_j < v \leq l_j$ we have $\|w_{uv} - w_0\|_0^+ < \lambda$ and $\|w_{uv} - q_0\|_0^+ < \lambda$ for a very small $\mu > 0$. Therefore, we need to obtain

$$\begin{aligned} &\left\{ t_i < u \leq v_i,\ k_j < v \leq l_j, \|w_{uv} - w_0\|_0^+ < \lambda \right\} \\ &\cap \left\{ t_i < u \leq v_i,\ k_j < v \leq l_j, \|w_{uv} - q_0\|_0^+ < \lambda \right\} \neq \emptyset, \end{aligned}$$

which leads to a contradiction, as the neighborhoods of w_0 and q_0 are disjoint. Thus, $\mathcal{I}_2^{FN} - S(\Lambda_w) = \{w_0\}$. □

Definition 17. *An element w_0 is considered as $\mathcal{I}_2$-deferred statistical cluster point of a double sequence $w = (w_{uv})$ if, for each $\lambda, \mu > 0$, the set*

$$\left\{ (\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{ t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda \right\} \right| < \mu \right\} \notin \mathcal{I}_2.$$

$\mathcal{I}_2^{FN} - S(\Gamma_w)$ represents the set of all $\mathcal{I}_2$-deferred statistical cluster point of a double sequence (w_{uv}).

Theorem 12. *For any double sequence (w_{uv}),*

$$\mathcal{I}_2^{FN} - S(\Lambda_w) \subseteq \mathcal{I}_2^{FN} - S(\Gamma_w).$$

Proof. Let $w_0 \in \mathcal{I}_2^{FN} - S(\Lambda_w)$. In that case, there is a set

$$U = \{(u_1, v_1) < (u_2, v_2) < \ldots < (u_r, v_s) < \ldots\} \subset \mathbb{N}^2$$

such that $U \notin \mathcal{I}_2$ and $DSt_2(FN) - \lim w_{u_r,v_s} = w_0$. So, we have

$$\lim_{\alpha,\beta\to\infty} \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{ t_\alpha < u_r \leq v_\alpha, k_\beta < v_s \leq l_\beta, \|w_{u_r,v_s} - w_0\|_0^+ \geq \lambda \right\} \right| = 0.$$

Take $\mu > 0$, so there is $n_0 \in \mathbb{N}$ such that for $m, n > n_0$ we obtain

$$\frac{1}{\rho_m \phi_n} \left| \left\{ t_m < u_r \leq v_m, k_n < v_s \leq l_n, \|w_{u_r,v_s} - w_0\|_0^+ \geq \lambda \right\} \right| < \mu.$$

Let

$$K = \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{\rho_m \phi_n} \left| \left\{ t_m < u_r \leq v_m, k_n < v_s \leq l_n, \|w_{u_r,v_s} - w_0\|_0^+ \geq \lambda \right\} \right| < \mu \right\}.$$

In addition, we get

$$K \supset U \diagdown \{(u_1, v_1), (u_2, v_2), ..., (u_{n_0}, v_{n_0})\}.$$

Given that $\mathcal{I}_2$ is an admissible ideal and $U \notin \mathcal{I}_2$, therefore $K \notin \mathcal{I}_2$. Consequently, according to the definition of an $\mathcal{I}_2$-deferred statistical cluster point $w_0 \in \mathcal{I}_2^{FN} - S(\Gamma_w)$. This concludes the proof. □

Theorem 13. *If $w = (w_{uv})$ and $q = (q_{uv})$ are two double sequences such that*

$$\left\{(u,v) \in \mathbb{N}^2 : w_{uv} \neq q_{uv}\right\} \in \mathcal{I}_2,$$

then

(*i*) $\mathcal{I}_2^{FN} - S(\Lambda_w) = \mathcal{I}_2^{FN} - S(\Lambda_q)$.
(*ii*) $\mathcal{I}_2^{FN} - S(\Gamma_w) = \mathcal{I}_2^{FN} - S(\Gamma_q)$.

Proof. (*i*) Let $w_0 \in \mathcal{I}_2^{FN} - S(\Lambda_w)$. As per the definition, there exists a set $U \subseteq \mathbb{N}^2$, arranged as

$$U = \{(u_1, v_1) < (u_2, v_2) < ... < (u_r, v_s) < ...\} \subset \mathbb{N}^2$$

such that $U \notin \mathcal{I}_2$ and $DSt_2(FN) - \lim w_{u_r,v_s} = w_0$. Since

$$\{(u,v) \in U : w_{uv} \neq q_{uv}\} \subseteq \left\{(u,v) \in \mathbb{N}^2 : w_{uv} \neq q_{uv}\right\} \in \mathcal{I}_2,$$

$$U' = \{(u,v) \in U : w_{uv} = q_{uv}\} \notin \mathcal{I}_2 \text{ and } U' \subseteq U.$$

Thus, the fact that $DSt_2(FN) - \lim q_{u'_r,v'_s} = w_0$ implies that $w_0 \in \mathcal{I}_2^{FN} - S(\Lambda_q)$, and consequently

$$\mathcal{I}_2^{FN} - S(\Lambda_w) \subseteq \mathcal{I}_2^{FN} - S(\Lambda_q).$$

By symmetry,

$$\mathcal{I}_2^{FN} - S(\Lambda_q) \subseteq \mathcal{I}_2^{FN} - S(\Lambda_w).$$

Hence, we obtain

$$\mathcal{I}_2^{FN} - S(\Lambda_w) = \mathcal{I}_2^{FN} - S(\Lambda_q).$$

(*ii*) Let $w_0 \in \mathcal{I}_2^{FN} - S(\Gamma_w)$. So, according to the definition for each $\lambda, \mu > 0$, we have

$$K = \left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \left|\left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda\right\}\right| < \mu\right\} \notin \mathcal{I}_2.$$

Let

$$T = \left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \left|\left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|q_{uv} - w_0\|_0^+ \geq \lambda\right\}\right| < \mu\right\}.$$

We have to prove that $T \notin \mathcal{I}_2$. Suppose that $T \in \mathcal{I}_2$, So

$$T^c = \left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \left|\left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|q_{uv} - w_0\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\} \in \mathcal{F}(\mathcal{I}_2).$$

According to the hypothesis,

$$P = \left\{(u,v) \in \mathbb{N}^2 : w_{uv} = q_{uv}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Hence, $T^c \cap P \in \mathcal{F}(\mathcal{I}_2)$. Furthermore, it's evident that $T^c \cap P \subseteq K^c \in \mathcal{F}(\mathcal{I}_2)$, implying $K \in \mathcal{I}_2$, which contradicts the initial assumption. Therefore, $T \notin \mathcal{I}_2$ and thus the desired result is achieved. □

5. Conclusion

In conclusion, this study has advanced the understanding of convergence in FNS by introducing the novel concepts of $\mathcal{I}_2$-deferred Cesàro summability and $\mathcal{I}_2$-deferred statistical convergence for double sequences. Through rigorous investigation, we have uncovered significant connections between these concepts and have established several theorems elucidating the notion of $\mathcal{I}_2$-deferred statistical convergence in FNS for double sequences. Moreover, we have defined and explored the properties of $\mathcal{I}_2$-deferred statistical limit points and $\mathcal{I}_2$-deferred statistical cluster points within the context of FNS, providing valuable insights into their relationships. These findings not only contribute to the theoretical framework of convergence in FNS but also pave the way for future research directions and applications in related fields.

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