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Fourier matrices of small rank

Research Article

Gurmail Singh

Abstract: Modular data is an important topic of study in rational conformal field theory. Cuntz, using a computer, classified the Fourier matrices associated to modular data with rational entries up to rank 12, see [3]. Here we use the properties of C-algebras arising from Fourier matrices to classify complex Fourier matrices under certain conditions up to rank 5. Also, we establish some results that are helpful in recognizing C-algebras that not arising from Fourier matrices by just looking at the first row of their character tables.

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1. Introduction

Fourier matrices are a fundamental ingredient of modular data. Modular data is a basic component of rational conformal field theory, see [5]. Further, rational conformal field theory has important applications in physics, see [4] and [8]. In particular, it has nice applications to string theory, statistical mechanics, and condensed matter physics, see [10] and [13]. Modular data give rise to fusion rings, C-algebras and C^*-algebras, see [3] and [11]. These rings and algebras are interesting topics of study in their own right.

A unitary and symmetric matrix whose first column has positive real entries is called a Fourier matrix if its columns under entrywise multiplication produce integral structure constants. The set of columns of a Fourier matrix under entrywise multiplication and usual addition generate a fusion algebra, see [3]. But a two-step rescaling on Fourier matrices gives rise to self-dual C-algebras. Cuntz, using a computer, classified the Fourier matrices with rational entries up to rank 12, see [3]. But rational Fourier matrices do not include some other important matrices, see sections 4, 5 and 6. Here we use C-algebra perspective to classify the complex Fourier matrices up to rank 5 under certain conditions. Also, we establish some results that are helpful in recognizing the C-algebras that are not arising from Fourier matrices by mere looking at the first row of their character tables.

Gurmail Singh; Department of Mathematics and Statistics, University of Regina, Canada, S4S 0A2 (email: gurmail.singh@uregina.ca).

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In Section 2, we collect the definitions and introduce a two-step rescaling of Fourier matrices. In Section 3, we summarize the results that are useful to recognize the C-algebras that are not arising from Fourier matrices. In Section 4, we classify Fourier matrices of rank 2 and 3. In sections 5 and 6, we classify non-homogeneous Fourier matrices of rank 4 and rank 5, respectively. For the classification of homogenous Fourier matrices see [11, Theorem 13].

2. C-algebras arising from Fourier matrices

A scaling of the rows of a Fourier matrix gives the basis of a Fusion algebra that contains the identity element. But a two-step rescaling of a Fourier matrix gives the standard basis of C-algebra.

Definition 2.1. *Let A be a finite dimensional and commutative algebra over $\mathbb{C}$ with distinguished basis $\mathbf{B} = \{b_0 := 1_A, b_1, \dots, b_{r-1}\}$, and an $\mathbb{R}$-linear and $\mathbb{C}$-conjugate linear involution $* : A \to A$. Let $\delta : A \to \mathbb{C}$ be an algebra homomorphism. Then the triple $(A, \mathbf{B}, \delta)$ is called a* C-algebra *if it satisfies the following properties:*

1. *for all $b_i \in \mathbf{B}$, $(b_i)^* = b_{i^*} \in \mathbf{B}$,*
2. *for all $b_i, b_j \in \mathbf{B}$, we have $b_i b_j = \sum_{b_k \in \mathbf{B}} \lambda_{ijk} b_k$, for some $\lambda_{ijk} \in \mathbb{R}$,*
3. *for all $b_i, b_j \in \mathbf{B}, \lambda_{ij0} \neq 0 \iff j = i^*$,*
4. *for all $b_i \in \mathbf{B}, \lambda_{ii^*0} = \lambda_{i^*i0} > 0$.*
5. *for all $b_i \in \mathbf{B}$, $\delta(b_i) = \delta(b_{i^*}) > 0$.*

The algebra homomorphism δ is called a *degree map*, and the values $\delta(b_i)$, for all $b_i \in \mathbf{B}$, are called the degrees of A. For $i \neq 0$, $\delta(b_i)$ is called a *nontrivial degree.* If $\delta(b_i) = \lambda_{ii^*0}$, for all $b_i \in \mathbf{B}$, we say that $\mathbf{B}$ is a standard basis. The order of a C-algebra is denfined as $\delta(\mathbf{B}^+) := \sum_{i=0}^{r-1} \delta(b_i)$. A C-algebra is called *symmetric* if $b_{i^*} = b_i$, for all i. A C-algebra with rational structure constants is called a *rational C-algebra.* The readers interested in C-algebras are directed to [1], [2] and [7].

To keep the generality, in the following definition of a Fourier matrix we assume the structure constants to be integers instead of nonnegative integers, see [3, Definition 2.2].

Definition 2.2. *Let $r \in \mathbb{Z}^+$ and I an $r \times r$ identity matrix. Then S is called a* Fourier matrix *if*

1. *S is a unitary and symmetric matrix, that is, $S\bar{S}^T = I, S = S^T$,*
2. *$S_{i0} > 0$, for $0 \leq i \leq r-1$, where S is indexed by $\{0, 1, 2, \dots, r-1\}$,*
3. *$N_{ijk} = \sum_l S_{li} S_{lj} \bar{S}_{lk} S_{l0}^{-1} \in \mathbb{Z}$, for all $0 \leq i, j, k \leq r-1$.*

Let S be a Fourier matrix. Let $s = [s_{ij}]$ be a matrix with entries $s_{ij} = S_{ij}/S_{i0}$, for all i, j, we call it an *s-matrix associated to S* (briefly, s-matrix). Since S is a unitary matrix, $s\bar{s}^T$=diag$(d_0, d_1, \dots, d_{r-1})$ is a diagonal matrix, where $d_i = \sum_j s_{ij}\bar{s}_{ij}$. The numbers d_i are called norms of s-matrix. The relation $s_{ij} = S_{ij}/S_{i0}$ implies the structure constants $N_{ijk} = \sum_l s_{li} s_{lj} s_{lk} d_l^{-1}$, for all i, j, k. Since the structure constants N_{ijk} are integers, the numbers s_{ij} are algebraic integers, see [3, Section 3]. Therefore, if S has only rational entries then entries of s-matrix are rational integers, and such s-matrices are known as *integral Fourier matrices*, see [3, Definition 3.1]. Cuntz classified the integral Fourier matrices up to rank 12 by using a computer, see [3]. In this paper, we consider the broader class of s-matrices that have algebraic integer entries.

There is an interesting row-and-column operation (*two-step rescaling*) procedure that can be applied to a Fourier matrix S that results in the first eigenmatrix, the character table, of a self-dual C-algebra. The steps of the procedure are reversed to obtain the Fourier matrix S from the first eigenmatrix. The explanation of the procedure is as follows. Let $S = [S_{ij}]$ be a Fourier matrix indexed with $\{0, 1, \ldots, r-1\}$. We divide each row of S with its first entry and obtain the s-matrix. The multiplication of each column of the s-matrix with its first entry gives the *P-matrix associated to S* (briefly, P-matrix), the first eigenmatrix of a self-dual C-algebra. That is, $s_{ij} = S_{ij}S_{i0}^{-1}$ and $p_{ij} = s_{ij}s_{0j}$, for all i, j, where p_{ij} denotes the (i, j)-entry of the P-matrix. Conversely, to obtain the s-matrix from a P-matrix, divide each column of the P-matrix with the squareroot of its first entry. Further, the Fourier matrix S is obtained from the s-matrix by dividing the ith row of s-matrix by $\sqrt{d_i}$, where $d_i = \sum_j |s_{ij}|^2$. That is, $s_{ij} = p_{ij}/\sqrt{p_{0j}}$, and $S_{ij} = s_{ij}/\sqrt{d_i}$, for all i, j. Since the entries of an s-matrix are algebraic integers, the entries of a P-matrix are also algebraic integers.

Remark 2.3. Throughout this paper, unless mentioned explicitly, the sets of columns of a P-matrix and an s-matrix are denoted by $\mathbf{B} = \{b_0, b_1, \ldots, b_{r-1}\}$ and $\tilde{\mathbf{B}} = \{\tilde{b}_0, \tilde{b}_1, \ldots, \tilde{b}_{r-1}\}$, respectively. The structure constants generated by the columns, with entrywise multiplication, of a P-matrix and an s-matrix are denoted by λ_{ijk} and N_{ijk}, respectively. M^T denotes the transpose of a matrix M.

Let S be a Fourier matrix and $A := \mathbb{C}\mathbf{B}$, a $\mathbb{C}$-span of $\mathbf{B}$. Define a map $* : A \longrightarrow A$ by $(\sum_j a_j b_j)^* = \sum_j \bar{a}_j b_{j^*} = \sum_j \bar{a}_j [\bar{p}_{0j}, \bar{p}_{1j}, \ldots, \bar{p}_{r-1,j}]^T$. This map $*$ is an involution on A, and the map $\delta : A \longrightarrow \mathbb{C}$ defined as $\delta(\sum_j a_j \tilde{b_j}) = \sum_j \bar{a}_j s_{0j}$, that is, $\delta(b_i) = \delta(s_{0i}\tilde{b}_i) = s_{0i}^2$ for all i, is a positive degree map of A. Since $b_i = s_{0i}\tilde{b}_i$, the structure constants generated by the basis $\mathbf{B}$ are given by $\lambda_{ijk} = N_{ijk}s_{0i}s_{0j}s_{0k}^{-1}$, for all i, j, k. S is a unitary matrix, therefore, $N_{ij0} = \sum_l S_{li}S_{lj}\bar{S}_{l0}S_{l0}^{-1} \neq 0 \iff j = i^*$ and $N_{ii^*0} = 1 > 0$, for all i, j. Thus $\lambda_{ij0} \neq 0 \iff j = i^*$ and $\lambda_{ii^*0} > 0$, for all i, j. Therefore, the vector space $A := \mathbb{C}\mathbf{B}$ is a C-algebra of order d_0, $\mathbf{B}$ is the standard basis of A, and P-matrix is the first eigenmatrix of A, see [11, Theorem 4], and we say *$(A, \mathbf{B}, \delta)$ is a C-algebra arising from a Fourier matrix S*. Note that, entries of the first eigenmatrix P are the entries of the character table A. Thus at some places we consider the P-matrix of A as the character table of A and the ith row of P-matrix as the ith irreducible character of A.

Let $(A, \mathbf{B}, \delta)$ be a C-algebra arising from a Fourier matrix S. Since S is a symmetric matrix, A is a self-dual C-algebra and $d_0 = d_j\delta(b_j)$, for all j. The entries of an s-matrix and the associated P-matrix are algebraic integers. Therefore, if A has rational degrees then both the degrees and norms are rational integers and both divide the order of A, see [11, Proposition 5]. Note that, a C-algebra arising from a Fourier matrix S is a symmetric C-algebra if and only if S is a real matrix. A C-algebra that has at least two different nontrivial degrees is called a *non-homogeneous C-algebra* and we call the associated Fourier matrix (s-matrix) a *non-homogeneous Fourier matrix* (*non-homogeneous s-matrix*, respectively).

Every self-dual C-algebra not necessarily have rational degrees. For example, a self-dual C-algebra of rank 2 with basis $\{1, x\}$, and the structure constants given by the equation $x^2 = 1 + x$ does not have rational degrees. We remark that this C-algebra does not arise from a Fourier matrix. But a rational C-algebra arising from a Fourier matrix has integral degrees, see [11, Proposition 5].

Lemma 2.4. *Let $(A, \mathbf{B}, \delta)$ be a C-algebra arising from a Fourier matrix S of rank r.*

1. *If A has rational order then the order of A is an integer.*
2. *If A has nonnegative structure constants then degrees of A are greater or equal to* 1.
3. *If A has rational order and all the degrees of A different from* 1 *are all equal then the degrees of A are integers. (Note: the algebra A need not be homogeneous.)*

Proof. (i). Since degrees of A are algebraic integers and order of A is the sum of degrees of A, the order of A is a rational integer.

(ii). For any i, $\delta(b_i)$ is the first entry of column vector $(\tilde{b_i})^2$ and $N_{ii0} = 1$. Note that, $N_{ijk} \geq 0$ for all i, j, k, because A has nonnegative structure constants. Therefore, $\delta(b_i) = 1 + m$, where m is a nonnegative algebraic integer.

(iii). Let all the degrees of A different from 1 be a positive real number t. Therefore, $d_0 = m + nt$, where m is the number of degrees equal to 1 and n is the number of degrees equal to t. Since the order d_0 is an integer, t is an integer. □

3. Recognition of C-algebras arising from Fourier matrices

The following results are useful for recognizing C-algebras that are not arising from Fourier matrices by mere looking at the degrees of C-algebras, that is, the first row of the character tables. All the character tables of the association schemes used here are produced by Hanaki and Miyamoto, see [6].

Lemma 3.1. *Let $(A, \mathbf{B}, \delta)$ be a C-algebra arising from a Fourier matrix S with nonnegative structure constants. Let $L(\mathbf{B}) = \{b \in \mathbf{B} : \delta(b) = 1\}$. Then $L(\mathbf{B})$ is an abelian group.*

Proof. Since $\delta(b_0) = 1$, $b_0 \in L$. Let $b_i, b_j \in L(\mathbf{B})$. Therefore, $b_i = \tilde{b}_i$ and $b_j = \tilde{b}_j$. Thus $b_i b_j = \tilde{b}_i \tilde{b}_j = \sum_k N_{ijk} \tilde{b}_k$ implies $1 = \delta(b_i b_j) = \sum_k N_{ijk} \delta(\tilde{b}_k)$. By Lemma 2.4, $\delta(b_l) \geq 1$, thus $\delta(\tilde{b}_l) = \sqrt{\delta(b_l)} \geq 1$, for all l. Therefore, $b_i b_j = \tilde{b}_k$, for some $b_k \in \mathbf{B}$. Also, for all $b_i \in L(\mathbf{B})$, $b_i b_{i^*} = b_0 + \sum_j \lambda_{ii^*j} b_j$ and $\lambda_{ii^*j} \geq 0$ imply $b_i b_{i^*} = b_0$, that is, $b_i^{-1} = b_{i^*} \in L(\mathbf{B})$. Hence $L(\mathbf{B})$ is an abelian group. □

Proposition 3.2. *Let $(A, \mathbf{B}, \delta)$ be a C-algebra arising from a Fourier matrix S with nonnegative structure constants. Let S be a real Fourier matrix.*

1. *Let the order of A be a rational number. If all the degrees of A different from 1 are equal to t then t might be a power of 2.*
2. *If rank of A is an even integer then A cannot have only one degree different from 1.*
3. *Let the order of A be a rational number. If the rank of A is greater than 3 then A cannot have only one degree greater or equal to r and all other degrees equal to 1.*

Proof. (i). Since S is a real Fourier matrix, by Lemma 3.1, the elements of $\mathbf{B}$ with degree 1 form an elementary abelian group. Thus the number of elements of $\mathbf{B}$ with degree 1 is a power of 2. By Lemma 2.4, t is an integer. The result follows from the fact that t divides the order of the algebra, see [11, Proposition 5 (ii)].

(ii). If rank of A is 2 then both degrees are equal to 1, see Section 4. By Lemma 3.1, the set of elements of $\mathbf{B}$ with degree 1 form an elementary abelian group. Therefore, the order of the group is a power of 2, say 2^m, where m is a nonnegative integer. Since A has only one degree different from 1, $2^m = r - 1$, a contradiction to the fact that r is an even integer.

(iii). Suppose A has only one degree k that is greater or equal to r and the remaining degrees are equal to 1. Without loss of generality, let the first row of the character table be $[1, 1, \ldots, 1, k]$, where $k \geq r$. Since $\delta(b_0) = \ldots = \delta(b_{r-2})$, $d_0 = \ldots = d_{r-2}$. The structure constants are nonnegative, therefore, $|p_{ij}| \leq p_{0j}$ for all i, j, see [12, Proposition 4.1]. Therefore, the only possible entries of row 2, …, row $r-1$ of P-matrix are $[1, 1, \ldots, 1, -k]$, which is not possible as P nonsingular. □

The adjacency algebras of the association schemes `as12(9)`, `as14(4)`, `as16(10)`, `as16(20)`, `as16(21)` and `as16(62)` have the degree patterns that violate the above result, see [6]. Therefore, they are not arising from Fourier matrices. The character table of the association scheme `as4(2)` [6] is an example where part (iii) of the above proposition fails for the rank 3. The above proposition also helps to sieve out a lot of C-algebras even if their self-duality is not known.

The next proposition helps to recognize the C-algebras not arising from Fourier matrices.

Proposition 3.3. *Let $(A, \mathbf{B}, \delta)$ be a C-algebra arising from a Fourier matrix S of rational order. Let the number of i's such that $\delta(b_i) = 1$ be t and the remaining $r - t$ degrees are equal to k. Then the possible values of k are the divisors of t.*

Proof. By Lemma 2.4, the order and degrees of A are integers. Since entries of s-matrix are algebraic integers, all the norms are also integer, that is, $d_0 k^{-1} \in \mathbb{Z}$. Therefore, $d_0 = t + (d_0 - t)k$ implies $tk^{-1} \in \mathbb{Z}$. Hence k is in the subset of the divisors of t. □

The above proposition illustrates that the adjacency algebras of the association schemes `as9(3)`, `as9(8)`, `as10(6)`, `as16(20)`, `as16(21)` and `as16(62)` [6] are not arising from Fourier matrices.

In the next proposition we examine the possible number of occurrences of a degree if it is one of the degrees and satisfy a certain criteria.

Proposition 3.4. *Let $(A, \mathbf{B}, \delta)$ be a C-algebra arising from a Fourier matrix S with integral degrees.*

1. *If for a given j, $\delta(b_j)$ $(\neq 1)$ is a smallest nontrivial degree that divides all the nontrivial degrees $\delta(b_l)$ $(\neq 1)$ then the number of degrees equal to 1 is a multiple of $\delta(b_j)$.*
2. *Let $\delta(b_t)$ be a degree divisible by all the smaller degrees and divides all the bigger degrees. Let $\delta(b_s)$ be the largest degree among all the degrees strictly less than $\delta(b_t)$. Let the sum of the degrees less than $\delta(b_s)$ be $\beta_1\delta(b_s)$ and the number of degrees equal to $\delta(b_s)$ be β_2. Then $\beta_1\delta(b_s) + \beta_2\delta(b_s)$ is divisible by $\delta(b_s)$. (Note: $\beta_1\delta(b_s)$ is not equal to zero only if $\delta(b_s) > 1$.)*
3. *Suppose A has nonnegative structure constants. If for all $i < j$, $\delta(b_i)$ divides $\delta(b_j)$ then A has integral structure constants.*

Proof. (i). Since degrees of A are integers, the norms are integers. Thus, $d_0\delta(b_j)^{-1} \in \mathbb{Z}$. Therefore,

$$\big(1 + \sum_{i=1}^{r-1}\delta(b_i)\big)\delta(b_j)^{-1} = \big(\sum_{\delta(b_i)=1}\delta(b_i) + \sum_{\delta(b_i)\geq\delta(b_j)}\delta(b_i)\big)\delta(b_j)^{-1} = \big(\sum_{\delta(b_i)=1}\delta(b_i)\big)\delta(b_j)^{-1} + \alpha \in \mathbb{Z},$$

where $\alpha \in \mathbb{Z}$. Hence the number of degrees equal to 1 are multiple of $\delta(b_j)$.

(ii). The degrees and norms of A are integers. Therefore,

$$\begin{aligned} d_0\delta(b_t)^{-1} &= \big(\sum_{i=0}^{r-1}\delta(b_i)\big)\delta(b_t)^{-1} = \big(\sum_{\delta(b_i)<\delta(b_s)}\delta(b_i) + \sum_{\delta(b_i)=\delta(b_s)}\delta(b_i) + \sum_{\delta(b_i)>\delta(b_s)}\delta(b_i)\big)\delta(b_t)^{-1} \\ &= \big(\beta_1\delta(b_s) + \sum_{\delta(b_i)=\delta(b_s)}\delta(b_i)\big)\delta(b_t)^{-1} + \gamma \in \mathbb{Z}, \end{aligned}$$

where $\gamma \in \mathbb{Z}$. Thus $\beta_1\delta(b_s) + \beta_2\delta(b_s)$ is divisible by $\delta(b_t)$.

(iii). Since λ_{ijk} are nonnegative, N_{ijk} are nonnegative, because $\lambda_{ijk} = N_{ijk}s_{0i}s_{0j}s_{0k}^{-1}$, for all i, j, k. Let $\tilde{b_i}\tilde{b_j} = \sum_k N_{ijk}\tilde{b_k}$. On comparing the first entry of both sides, we conclude that $\tilde{b_k}$ cannot occur with nonzero coefficient whenever $s_{0k} > s_{0i}s_{0j}$, that is, $\sqrt{\delta(b_k)} > \sqrt{\delta(b_i)\delta(b_j)}$ implies $N_{ijk} = 0$. Hence the assertion follows from the relation between λ_{ijk} and N_{ijk}. $\square$

For example, the adjacency algebras of association schemes `as7(2)`, `as8(5)`, `as8(6)`, `as9(8)`, `as9(9)` and adjacency algebras of homogenous schemes have the degree patterns that violate the above proposition, see [6]. Therefore, they are not arising from Fourier matrices.

Lemma 3.5. *Let $(A, \mathbf{B}, \delta)$ be a C-algebra arising from a rational Fourier matrix S of odd rank and odd order. Let the odd degree among all the degrees of A be maximum. Then the rank of A must be at least 11.*

Proof. Let $d_0 = \delta(b_i)a_i$. By [3, Lemma 3.7], d_0 is an odd square, thus a_i is a square. Let $\delta(b_1)$ be an odd integer and $\delta(b_1) \geq \delta(b_i)$ for each i. Therefore, $d_0 \geq 9\delta(b_1)$, and $d_0 \leq 1 + (r-1)\delta(b_1)$. Thus $9\delta(b_1) \leq 1 + (r-1)\delta(b_1)$ implies $\delta(b_1)(9 - (r-1)) \leq 1$ implies $\delta(b_1) \in \mathbb{Z}^+$ only if $r - 1 \geq 9$. $\square$

4. Fourier matrices of rank 2 and 3

In this section we classify Fourier matrices of rank 2 and 3. In fact, we find the P-matrices of C-algebras arising from Fourier matrices of rank 2 and 3. But the associated Fourier matrix S can be recovered easily from the P-matrix as described in Section 2.

Since the row sum of a character table is zero, the character table of a C-algebra of rank 2 with standard basis $\mathbf{B} = \{b_0, b_i\}$ is given by $P = \begin{bmatrix} 1 & n \\ 1 & -1 \end{bmatrix}$, and the structure constants are given by $b_1^2 = nb_0 + (n-1)b_1$. Therefore, the structure constant N_{111} is integer only for $n = 1$, and the associated Fourier matrix $S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & n \\ 1 & -1 \end{bmatrix}$.

Let P be the character table for a symmetric C-algebra arising from a Fourier matrix of rank 3 with standard basis $\mathbf{B} = \{b_0, b_1, b_2\}$. Let $b_1 b_2 = ub_1 + vb_2$. Then

$$P = \begin{bmatrix} 1 & k_1 & k_2 \\ 1 & \phi_1 & \phi_2 \\ 1 & \psi_1 & \psi_2 \end{bmatrix},$$

where $\phi_1 = (v - u - 1 + \sqrt{(u-v-1)^2 + 4u})/2$, $\quad \phi_2 = (u - v - 1 - \sqrt{(u-v-1)^2 + 4u})/2$, and $\psi_1 = (v - u - 1 - \sqrt{(u-v-1)^2 + 4u})/2$, $\quad \psi_2 = (u - v - 1 + \sqrt{(u-v-1)^2 + 4u})/2$. Therefore, $d_0 = 1 + k_1 + k_2$, $d_1 = 1 + \frac{|\phi_1|^2}{k_1} + \frac{|\phi_2|^2}{k_2}$, $d_2 = 1 + \frac{|\psi_1|^2}{k_1} + \frac{|\psi_2|^2}{k_2}$.

Lemma 4.1. *There is no symmetric homogenous C-algebra of rank 3 arising from a Fourier matrix S.*

Proof. Suppose $(A, \mathbf{B}, \delta)$ is a symmetric homogenous C-algebra arising from a Fourier matrix. Since $S^T = S$, $\phi_2 k = \psi_1 l$ implies $u = v$. The structure constant $N_{210} = 0$ implies $k = 2u$. Thus $\phi_1 = (-1 + \sqrt{1 + 2k})/2$ and $\phi_2 = (-1 - \sqrt{1 + 2k})/2$. A homogenous C-algebra arising from a Fourier matrix has all degrees equal to 1, see [11, Proposition 12]. But the structure constants

$$N_{112} = \frac{1}{(2k+1)\sqrt{k}}[k^2 + \frac{k}{2}] \text{ and } N_{222} = \frac{1}{(2k+1)\sqrt{k}}[k^2 - 1 - \frac{3}{2}k]$$

are not integers for $k = 1$, a contradiction. □

Theorem 4.2. *Let $(A, \mathbf{B}, \delta)$ be a symmetric non-homogeneous C-algebra of rank 3 arising from a Fourier matrix S with integral degrees. Then the corresponding matrices P, s and S are as follows.*

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}, \quad s = \begin{bmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ 1 & -1 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}.$$

Proof. Let $\delta(b_i) = k_i$, for all i. Since the integral degrees of A divide the order, k_1 divides $1 + k_2$, and k_2 divides $1 + k_1$. Therefore, the only possible degree pattern of A are $[1, 1, 2]$ and $[1, 2, 3]$, up to the permutations. Since $N_{012} = 0$, $1 - \frac{v}{k_1} - \frac{u}{k_2} = 0$. Therefore, the degree patterns $[1, 1, 2]$ and $[1, 2, 3]$ imply $v = 1 - \frac{u}{2}$ and $v = 2 - \frac{2u}{3}$, respectively.

Case 1. Let the degree pattern be $[1, 1, 2]$, that is, $k_1 = 1$ and $k_2 = 2$.

Therefore, $v = 1 - \frac{u}{2}$. Since $N_{011} = 1$, $u^3(u-1) = 0$. Hence $(u, v) = (0, 1)$, or $(u, v) = (1, 1/2)$.

Subcase 1. Let $(u, v) = (0, 1)$.

Therefore,

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}, \ s = \begin{bmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ 1 & -1 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}.$$

Subcase 2. Let $(u, v) = (1, \frac{1}{2})$.

Therefore,

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & \dfrac{-3+\sqrt{17}}{4} & \dfrac{-1-\sqrt{17}}{4} \\ 1 & \dfrac{-3-\sqrt{17}}{4} & \dfrac{-1+\sqrt{17}}{4} \end{bmatrix}.$$

Note that $\delta(b_1) = \delta(b_0)$, but $d_1 \neq d_0$, a contradiction. Hence $u = 1$ and $v = \frac{1}{2}$ is not a possible case.

Case 2. Let the degree pattern be $[1, 2, 3]$, that is, $k_1 = 2$ and $k_2 = 3$.

Therefore, $v = 2 - \frac{2u}{3}$. Since $N_{011} = 1$, $625u^4 - 1850u^3 + 2520u^2 - 1296u + 243 = 0$. But it has no real roots, see [9]. Thus we rule out $[1, 2, 3]$ degree pattern, because an s-matrix associated with a symmetric C-algebra might be a real matrix. □

Remark 4.3. The above P-matrix is given by the character table of the adjacency algebra of an association scheme `as4(2)`, see [6].

In the next theorem we prove that there is only one asymmetric C-algebra of rank 3 arising from a Fourier matrix. Moreover, the following theorem shows that for rank 3 it is not necessary to assume $|s_{ij}| \leq s_{0j}$ to prove that the homogeneous C-algebra arising from a Fourier matrix is a group algebra, see [11, Theorem 13].

Theorem 4.4. *Let $(A, \mathbf{B}, \delta)$ be an asymmetric C-algebra arising from a Fourier matrix S of rank 3. Then the P-matrix is the first eigenmatrix of the group algebra of a group of order 3.*

Proof. The P-matrix of an asymmetric C-algebra of rank 3 is as follows,

$$P = \begin{bmatrix} 1 & k & k \\ 1 & \alpha & \bar{\alpha} \\ 1 & \bar{\alpha} & \alpha \end{bmatrix},$$

where $\alpha = (-1 + i\sqrt{1+2k})/2$. Since A is homogenous, $k = 1$, see [11, Proposition 12]. Therefore,

$$P(= s) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix} \text{ and } S = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix}.$$

□

5. Fourier matrices of rank 4

In this section we classify the Fourier matrices under certain conditions, and we show that there is no non-homogeneous integral Fourier matrix of rank 4. For homogenous Fourier matrices see [11].

Lemma 5.1. *Let $(A, \mathbf{B}, \delta)$ be a C-algebra arising from a Fourier matrix S of rank r. Let $|s_{ij}| \leq s_{0j}$, for all j. Let $\delta(b_j) = k_j$ for all j, and $k_i = 1$ for some $i > 0$.*

1. *Then $|p_{ij}| = k_j$ for all j.*
2. *If s is a real matrix then $p_{ij} = \pm k_j$ for all j.*

Proof. (i). Since $\delta(b_i) = 1$, $d_i = d_0$. Therefore, the row 1 and row i of the s-matrix are $[1, 1, \sqrt{k_2}, \ldots, \sqrt{k_{r-1}}]$ and $[1, p_{i1}, p_{i2}/\sqrt{k_2}, \ldots, p_{i,r-1}/\sqrt{k_{r-1}}]$, respectively. Since $d_0 = d_i$, $|p_{ij}|/\sqrt{k_j} = \sqrt{k_j}$, for all j. Hence $|p_{ij}| = k_j$, for all j.

(ii). By Part (i), $|p_{ij}| = k_j$ for all j. Since s is a real matrix, $p_{ij} = \pm k_j$ for all j. □

The next proposition classify the non-homogeneous Fourier matrices of rank 4 with one nontrivial degree equal to 1.

Proposition 5.2. *Let $(A, \mathbf{B}, \delta)$ be a non-homogeneous rational C-algebra arising from a Fourier matrix S of rank 4. Let $|s_{ij}| \leq s_{0j}$, for all j. Let $\delta(b_j) = k_j$, for all j, and $k_i = 1$, for some $i > 0$. Then the associated P-matrix is*

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 1 & 1 & 4 & -6 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

Proof. Since A is a rational C-algebra, the degrees of A are integers, see [11, Proposition 5]. Let $\delta(b_i) = k_i$, for all i. Without loss of generality, suppose $k_1 = 1$. Therefore, we have

$$P = \begin{bmatrix} 1 & 1 & k_2 & k_3 \\ 1 & p_{11} & p_{12} & p_{13} \\ 1 & p_{21} & p_{22} & p_{23} \\ 1 & p_{31} & p_{32} & p_{33} \end{bmatrix}.$$

Case 1. If p_{11}, p_{12} and p_{13} are not rational integers.

Since $d_0 = d_1$, $|p_{11}| = 1, |p_{12}| = k_2$ and $|p_{13}| = k_3$. Thus p_{11}, p_{12} and p_{13} cannot be irrational real numbers. Therefore, they can be non-real algebraic integers. Since the structure constants are rational numbers, a complex conjugate of an irreducible character of A is an irreducible character of A. Without loss of generality, assume that the third irreducible character is a complex conjugate of the second character. Thus $k_2 = 1$. But A is non-homogenous and $d_0 = d_i\delta(b_i)$, therefore, $k_3 = 3$ and $d_3 = 2$. Since S is a symmetric matrix, $|p_{31}| = |p_{32}| = 1$, thus $d_3 > 3$, a contradiction.

Case 2. If p_{11}, p_{12} and p_{13} are rational integers.

By Lemma 5.1, $p_{11} = \pm 1, p_{12} = \pm k_2$ and $p_{13} = \pm k_3$. Since the row sum of P-matrix is zero, the second row of P-matrix is either $[1, -1, k_2, -k_2]$ or $[1, 1, k_2, -(k_2 + 2)]$.

Subcase 1. Let the second row of P be $[1, -1, k_2, -k_2]$.

Then, the first row of P-matrix is $[1, 1, k_2, k_2]$. Since k_2 divides d_0 and A is non-homogenous, $k_2 = 2$. Thus, by the symmetry of Fourier matrix S and orthogonality of characters, we have

$$P = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

But the s-matrix associated to the above P-matrix does not have integral structure constants.

Subcase 2. Let the second row of P be $[1, 1, k_2, -(k_2 + 2)]$.

Then, the first row of P-matrix is $[1, 1, k_2, k_2 + 2]$. Since k_2 divides d_0 and A is non-homogenous, $k_2 = 2$ or 4. Thus, by the symmetry of Fourier matrix S and orthogonality of characters, we have

$$P = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & -4 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \text{ or } P = \begin{bmatrix} 1 & 1 & 4 & 6 \\ 1 & 1 & 4 & -6 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

But the s-matrix associated to the first P-matrix does not have integral structure constants. □

Remark 5.3. The above P-matrix is the first eigenmatrix of the adjacency algebra of an association scheme `as12(8)`, see [6].

Cuntz, with a computer, shows that there is no non-homogenous rational Fourier matrix of rank 4, see [3]. In the next theorem, we use C-algebra perspective to show that there is no non-homogeneous rational Fourier matrix of rank 4, that is, there is no non-homogeneous s-matrix with integral entries. Unlike the above proposition, we do not assume any nontrivial degree equal to 1.

Theorem 5.4. *There is no non-homogenous rational Fourier matrix S of rank 4.*

Proof. Let $(A, \mathbf{B}, \delta)$ be a C-algebra arising from a rational Fourier matrix S of rank 4. Let $\delta(b_i) = k_i$, for all $i > 0$. Since s-matrix is integral, k_1, k_2 and k_3 are square integers, see [11, Proposition 5 (iii)]. As $d_0 = 1+k_1+k_2+k_3$ and k_1, k_2 and k_3 divide d_0, therefore, $k_2+k_3 \equiv -1 \mod k_1$, $k_1+k_3 \equiv -1 \mod k_2$ and $k_1 + k_2 \equiv -1 \mod k_3$.

Claim: $k_1 = k_2 = k_3 = 1$. Without loss of generality, suppose k_1 is an even integer. Since k_1, k_2 and k_3 are squares, $k_1 \equiv 0 \mod 4$ and $d_0 \not\equiv 0 \mod 4$, a contradiction to the fact that k_1 divides d_0. Therefore k_1, k_2 and k_3 are odd integer. Suppose $k_1 \geq k_2, k_3$ and $k_1 > 1$. Now, if all k_1, k_2 and k_3 are odd integers then $k_1, k_2, k_3 \equiv 1 \mod 4$. But $d_0 \equiv 0 \mod 4$ implies $d_0 = k_1 a$, $a \geq 4$. Therefore, $k_1(a - 1) = 1 + k_2 + k_3$ implies $3k_1 \leq 1 + k_2 + k_3$. Thus k_2 or $k_3 > k_1$, again a contradiction. □

6. Fourier matrices of rank 5

In this section we prove that there is no non-homogenous s-matrix with integral entries (integral Fourier matrix) of rank 5. But the following proposition shows, under certain conditions, that there are three s-matrices of rank 5 with algebraic integer entries. Recall that, a Fourier matrix S with rational entries has associated integral s-matrix, and a complex Fourier matrix S has associated s-matrix with algebraic integer entries.

Proposition 6.1. *Let $(A, \mathbf{B}, \delta)$ be a non-homogeneous rational C-algebra arising from a Fourier matrix of rank 5. If $|s_{ij}| \leq s_{0j}$ for all j. If $\delta(b_i) = 1$ for one $i > 0$ and $\delta(b_j) = k_j$ for all $j \neq i$. Then up to simultaneous row and column permutations the P-matrices are as follows,*

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & -2 & -2 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & -\sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} & \sqrt{2} \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 & 4 & 8 \\ 1 & 1 & 2 & 4 & -8 \\ 1 & 1 & 2 & -4 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix} \textit{ and } \begin{bmatrix} 1 & 1 & 4 & 3 & 3 \\ 1 & 1 & 4 & -3 & -3 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{3} & -\sqrt{3} \\ 1 & -1 & 0 & -\sqrt{3} & \sqrt{3} \end{bmatrix}.$$

Proof. Since A is a rational C-algebra, the degrees of A are integers. Let $\delta(b_i) = k_i$, for all i. Without loss of generality, let $k_1 = 1$. Therefore, first eigenmatrix of A is given by

$$P = \begin{bmatrix} 1 & 1 & k_2 & k_3 & k_4 \\ 1 & p_{11} & p_{12} & p_{13} & p_{14} \\ 1 & p_{21} & p_{22} & p_{23} & -(1 + p_{21} + p_{22} + p_{23}) \\ 1 & p_{31} & p_{32} & p_{33} & -(1 + p_{31} + p_{32} + p_{33}) \\ 1 & p_{41} & p_{42} & p_{43} & -(1 + p_{41} + p_{42} + p_{43}) \end{bmatrix}.$$

Case 1. If p_{11}, p_{12}, p_{13} and p_{14} are not all rational integers.

Since $d_0 = d_1$, $|p_{11}| = 1, |p_{12}| = k_2, |p_{13}| = k_3$ and $|p_{14}| = k_4$. Since the row sum is zero, at least two of these p_{11}, p_{12}, p_{13} and p_{14} can be non-real algebraic integers. Since the structure constants are rational numbers, a complex conjugate of an irreducible character is an irreducible character. Without loss of generality, we assume that the third irreducible character is a complex conjugate of the second character, thus $k_2 = 1$. Without loss of generality, let $k_3 \leq k_4$. Therefore, $d_0 = d_i m_i$ implies $(k_3, k_4) \in \{(1,2),(1,4),(2,5),(3,6),(6,9)\}$.

If $k_3 = 1$ then $d_3 = d_0 = d_1 = d_2$ and $|s_{ij}| \leq s_{0j}$ imply all the entries of the rows $1,2,3$ and 4 are nonzero. Since $k_3 \neq k_4$, the entries of the fifth row are rational integers because Galois conjugate of an irreducible character is an irreducible character, and rows of s-matrix corresponding to the conjugate characters have equal norm. But $k_3 = 1$ implies $d_4 \leq 3$. Thus there might be at least two zero entries in the fifth row of P-matrix. But S is a symmetric matrix, we get a contradiction.

For $(k_3, k_4) \in \{(2,5),(3,6),(6,9)\}$, $k_3 \neq k_4$, thus the entries of the row 5 are rational integers because $k_2 = k_1 = k_0 = 1$ and the Galois conjugate of an irreducible character is an irreducible character. Each entry of the row $1,2$ and 3 of P-matrix is non-zero. But for each of the above pair there are exactly 3 zeros in the fifth row. Since S is a symmetric matrix, we get a contradiction.

Case 2. If p_{11}, p_{12}, p_{13} and p_{14} are all rational integers.

By Lemma 5.1, the only possible degree patterns are:

$[1,-1,k_2,k_3,-(k_2+k_3)]$, $[1,1,k_2,k_3,-(k_2+k_3+2)]$, $[1,\ 1,\ k_2,\ -k_3,\ -(k_2-k_3+2)]$,

$[1,\ -1,\ k_2,\ -k_3,\ -(k_2-k_3)]$, $[1,\ 1,\ -k_2,\ -k_3,\ k_2+k_3-2]$, $[1,\ -1,\ -k_2,\ -k_3,\ k_2+k_3]$.

Subcase 1. Let the second row of P be $[1,-1,k_2,k_3,-(k_2+k_3)]$.

Then the first row of the character table is $[1,1,k_2,k_3,k_2+k_3]$ and $(k_2+k_3)\big|(2+k_2+k_3)$. Thus $(k_2,k_3) = (1,1)$. Therefore, by the orthogonality of characters, we have

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 1 & -2 \\ 1 & 1 & p_{22} & p_{23} & -1 \\ 1 & 1 & p_{32} & p_{33} & -1 \\ 1 & -1 & p_{42} & p_{43} & 1 \end{bmatrix}.$$

Since $k_2 = k_3 = 1$, $d_2 = d_3 = 6$. But each of $|p_{22}|$, $|p_{23}|$, $|p_{32}|$ and $|p_{33}|$ can be at most 1. Thus both d_2 and d_3 are strictly less than 6, a contradiction. Hence this case is not possible.

Subcase 2. Let the second row of P be $[1,1,k_2,k_3,-(k_2+k_3+2)]$.

Then the first row of the character table is $[1,1,k_2,k_3,k_2+k_3+2]$. Therefore, by the orthogonality of the irreducible characters and symmetry of the matrix S, we have

$$P = \begin{bmatrix} 1 & 1 & k_2 & k_3 & k_2+k_3+2 \\ 1 & 1 & k_2 & k_3 & -(k_2+k_3+2) \\ 1 & 1 & p_{22} & -2-p_{22} & 0 \\ 1 & 1 & p_{32} & -2-p_{32} & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Without loss of generality, let $k_2 \leq k_3$. Since $k_2\big|(2k_3+4)$ and $k_3\big|(2k_2+4)$, we have

$$(k_2,k_3) \in \{(1,2),(1,3),(1,6),(2,4),(2,8),(3,10),(4,6),(4,12),(6,16),(8,10),(12,28)\}.$$

Note that $k_2 \neq k_3$, thus $d_3 \neq d_4$. The structure constants are rational. Therefore, if p_{22} or p_{33} is not rational then row 3 and 4 of P-matrix should be Galois conjugates. But the rows of s-matrix corresponding to conjugate irreducible characters should have equal norm. Thus p_{22} and p_{32} are rational integers. Therefore, $\det(P) \in \mathbb{Z}$ and $(\det P)^2 = n^5$. Thus $n = 2(k_2+k_3+2)$ need to be a square. But the only two pairs $(2,4),(4,12)$ do not fail this test. For $(k_2,k_3) = (2,4)$, $d_2 = 8 = 1+1+(\frac{p_{22}}{\sqrt{2}})^2 + (\frac{-2-p_{22}}{\sqrt{4}})^2$.

Since the entries of P-matrix are algebraic integers, we have $p_{22} = 2$. Similarly, $d_3 = 4$ implies $p_{32} = -2$. Therefore,

$$P = \begin{bmatrix} 1 & 1 & 2 & 4 & 8 \\ 1 & 1 & 2 & 4 & -8 \\ 1 & 1 & 2 & -4 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

For $(k_2, k_3) = (4, 12)$, $d_2 = 9$ and $d_3 = 3$. Therefore, we have $p_{22} = 4, -5$ and $p_{32} = 1, -2$. Since S is a symmetric matrix, $p_{22} = 4$ and $p_{32} = -2$ is the only possibility. But for $p_{22} = 4$ and $p_{32} = -2$, the associated s-matrix does not have integral structure constants.

Subcase 3. Let the second row of P be $[1,\ 1,\ k_2,\ -k_3,\ -(k_2 - k_3 + 2)]$.

Then, the first row of the character table is $[1, 1, k_2, k_3, k_2 - k_3 + 2]$. Therefore, by the orthogonality of characters, symmetry of the Fourier matrix S and $P\bar{P} = nI$, we have

$$P = \begin{bmatrix} 1 & 1 & k_2 & k_3 & k_2 - k_3 + 2 \\ 1 & 1 & k_2 & -k_3 & -(k_2 - k_3 + 2) \\ 1 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & p_{33} & -p_{33} \\ 1 & -1 & 0 & p_{43} & -p_{43} \end{bmatrix}.$$

Therefore, $k_2 \big| 4$, $k_3 \big| 2k_2 + 4$, $(k_2 - k_3 + 2)\big|(k_2 + k_3 + 2)$ and $k_2 - k_3 + 2 > 0$. Hence $(k_2, k_3) \in \{(1,1), (1,2), (2,2), (4,2), (4,3), (4,4)\}$. Since $|s_{ij}| \le s_{0j}$, $(k_2, k_3) \notin \{(1,1), (1,2)\}$. For $(k_2, k_3) = (2, 2)$, $d_3 = d_4 = 4$. Thus $p_{33}\bar{p}_{33} = 2$, $p_{43}\bar{p}_{43} = 2$. But the integrality of the structure constants of s-matrix and orthogonality of characters forces $p_{33} = \pm\sqrt{2}$ and $p_{43} = \mp\sqrt{2}$. Therefore, up to simultaneous permutation of row 4 and row 5, and column 4 and column 5, we have

$$P = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & -2 & -2 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & -\sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} & \sqrt{2} \end{bmatrix}.$$

For $(k_2, k_3) = (4, 2)$, $d_3 = 6$. Thus $|p_{33}| = \dfrac{4}{\sqrt{3}} > 2$, a contradiction. For $(k_2, k_3) = (4, 3)$, $k_4 = 3$, $d_3 = 4$ and $d_4 = 4$. Thus $|p_{33}| = \sqrt{3}$ and $|p_{43}| = \sqrt{3}$. But the integrality of structure constants and orthogonality of characters forces $p_{33} = \pm\sqrt{3}$ and $p_{43} = \mp\sqrt{3}$. Therefore, up to simultaneous permutation of row 4 and row 5, and column 4 and column 5, we have

$$P = \begin{bmatrix} 1 & 1 & 4 & 3 & 3 \\ 1 & 1 & 4 & -3 & -3 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{3} & -\sqrt{3} \\ 1 & -1 & 0 & -\sqrt{3} & \sqrt{3} \end{bmatrix}.$$

Although the structure constants are not all integers, for example $\lambda_{342} = 3/2$, but the associated s-matrix has integral structure constants.

Subcase 4. Let the second row of P be $[1,\ -1,\ k_2,\ -k_3,\ -(k_2 - k_3)]$.

Then, the first row of the character table is $[1,\ 1,\ k_2,\ k_3,\ k_2 - k_3]$. Therefore, by the orthogonality of the characters and symmetry of the matrix S, we have

$$P = \begin{bmatrix} 1 & 1 & k_2 & k_3 & k_2 - k_3 \\ 1 & -1 & k_2 & -k_3 & -(k_2 - k_3) \\ 1 & 1 & 0 & p_{23} & -(2 + p_{23}) \\ 1 & -1 & p_{32} & p_{33} & -(p_{32} + p_{33}) \\ 1 & -1 & p_{42} & p_{43} & -(p_{42} + p_{43}) \end{bmatrix}.$$

Since $P\bar{P} = nI$, from row 1, 2 and column 3, we get $k_2 = 0$, a contradiction.

Subcase 5. Let the second row of P be $[1,\ 1,\ -k_2,\ -k_3,\ k_2+k_3-2]$.

Then the first row of the character table is $[1,\ 1,\ k_2,\ k_3, k_2+k_3-2]$. Therefore, by the symmetry of the matrix S and orthogonality of characters, we have

$$P = \begin{bmatrix} 1 & 1 & k_2 & k_3 & k_2+k_3-2 \\ 1 & 1 & -k_2 & -k_3 & k_2+k_3-2 \\ 1 & -1 & p_{22} & -p_{22} & 0 \\ 1 & -1 & p_{32} & -p_{32} & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}.$$

Therefore $k_2 \big| 2k_3,\ k_3 \big| 2k_2,\ (k_2+k_3-2)\big|(k_2+k_3+2)$ and $k_2+k_3-2 > 0$. Without loss of generality, let $k_2 \leq k_3$. Hence $(k_2, k_3) \in \{(1,2),(2,2)\}$. Since $|s_{ij}| \leq s_{0j}$, $(k_2,k_3) \neq (1,2)$. For $(k_2,k_3) = (2,2)$, $d_2 = d_3 = 4$. Thus $p_{22}\bar{p}_{22} = 2$, $p_{32}\bar{p}_{32} = 2$. But the integrality of structure constants and orthogonality of characters forces $p_{22} = \pm\sqrt{2}$ and $p_{32} = \mp\sqrt{2}$. Therefore, up to simultaneous permutation of rows and columns, we have

$$P = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & -2 & -2 & 2 \\ 1 & -1 & \sqrt{2} & -\sqrt{2} & 0 \\ 1 & -1 & -\sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}.$$

Subcase 6. Let the second row of P be $[1,\ -1,\ -k_2,\ -k_3,\ k_2+k_3]$.

Then the first row of the character table is $[1,\ 1,\ k_2,\ k_3,\ k_2+k_3]$. Therefore, by the symmetry of the matrix S and orthogonality of the characters, we have

$$P = \begin{bmatrix} 1 & 1 & k_2 & k_3 & k_2+k_3 \\ 1 & -1 & -k_2 & -k_3 & k_2+k_3 \\ 1 & -1 & p_{22} & p_{23} & -1 \\ 1 & -1 & p_{32} & p_{33} & -1 \\ 1 & 1 & p_{42} & p_{43} & 0 \end{bmatrix}.$$

Similar to the Subcase 1, $k_2 = k_3 = 1$ implies $d_2 = d_3 = 6$, and we get a contradiction. □

Remark 6.2. We note that the associated s-matrices to the P-matrices in the above proposition are not integral Fourier matrices. The first two matrices are the character tables of `as08(10), as16(24)`, respectively, see [6]. The third matrix is not a first eigenmatrix of an adjacency algebra of any association scheme because the structure constants generated by the columns of P-matrix are not all nonnegative integers.

In the next theorem, by using the properties of C-algebras, we show that there is no non-homogeneous rational Fourier matrix of rank 5.

Theorem 6.3. *There is no non-homogeneous rational Fourier matrix S of rank 5.*

Proof. Let $(A, \mathbf{B}, \delta)$ be a non-homogenous C-algebra arising from a rational Fourier matrix S of rank 5. Let $\delta(b_i) = k_i$, for all i. Since s-matrix is integral, k_i are perfect square integers, see [11, Proposition 5]. By [3, Lemma 3.7], d_0 is a square. Therefore, $d_0 \equiv 0, 1 \mod 4$. Let $d_0 = k_4 a, d_0 = k_3 b, d_0 = k_2 c, d_0 = k_1 d$. Then each of a, b, c and d is greater than 1 and a square integer because A is non-homogeneous.

Case 1. If each of k_1, k_2, k_3, k_4 is an odd integer, then d_0 is odd. Also, the fact that k_1, k_2, k_3, k_4 are odd implies a, b, c, d are odd and greater than or equal to 9. Without loss of generality, let $k_4 \geq k_1, k_2, k_3$. Therefore, $d_0 \geq 9k_4, d_0 = 1 + k_1 + k_2 + k_3 + k_4 \leq 1 + 4k_4$, a contradiction.

Case 2. If three of k_1, k_2, k_3, k_4 are odd and one is even, then d_0 is even. Without loss of generality, suppose k_4 is even. Thus $d_0 = k_4 a$ implies $a \geq 4$.

Subcase 1. If $k_4 > k_1, k_2, k_3$, then $d_0 \geq 4k_4$, $d_0 \leq 1 + (k_4 - 1) + (k_4 - 1) + (k_4 - 1) + k_4 = 4k_4 - 2$, a contradiction.

Subcase 2. If $k_4 <$ one of k_1, k_2, k_3, say k_3, so $k_1, k_2 \leq k_3$, then $d_0 \geq 4k_3$ because b is an even square. Thus $d_0 \leq 1 + k_3 + k_3 + k_3 + (k_3 - 1) = 4k_3$ implies $k_1 = k_2 = k_3$ and $k_4 = k_3 - 1$, $d_0 = 4k_3$. Now $d_0 = 4x^2$ because d_0 is a square and an even integer. Hence $k_1 = k_2 = k_3 = x^2$ and $k_4 = x^2 - 1$. Since $x^2 - 1$ divides $4x^2$ and x is an odd integer, $x^2 - 1$ divides 4, we get a contradiction.

Case 3. If two of k_1, k_2, k_3, k_4 are odd and two are even, then $d_0 \equiv 3 \mod 4$, a contradiction.

Case 4. If one of k_1, k_2, k_3, k_4 is odd and three are even, then $d_0 \equiv 2 \mod 4$, a contradiction. □

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