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ON IDEAL BOUNDED SEQUENCES

Leila MILLER-VAN WIEREN

International University of Sarajevo, Faculty of Engineering and Natural Sciences,
71000 Sarajevo, BOSNIA-HERZEGOVINA

Abstract. In this paper, we study the notion of ideal bounded sequences, related to a given ideal, generalizing an earlier concept known as statistical boundedness of a sequence. We proceed to prove some results connecting ideal boundedness of a sequence to that of its subsequences. For this purpose, we use Lebesgue measure and Baire category to measure size.

1. Introduction

The convergence of sequences has undergone numerous generalizations, one of the first and most important being the concept of statistical convergence, introduced by Fast (1951), [8]. Later on, other types of summability including almost convergence, uniform statistical convergence and more generally ideal convergence of sequences were researched by many authors in different directions.

In classical and recent works the relationships between a given sequence and its subsequences regarding different kinds of summability have been studied using measure or category as gauges of size. It is well known that every $x \in (0,1]$ has a binary expansion $x = \sum_{n=1}^{\infty} 2^{-n} d_n(x)$ such that $d_n(x) = 1$ for infinitely many positive integers n, that is unique. Then for any $x \in (0,1]$ and any sequence $s = (s_n)$ we can construct a subsequence (sx) of s in such a way that: $(sx)_i = s_{n_i}$, where $n_1 < n_2 < ... < n_i < ...$ is the set of $n \in \mathbb{N}$ for which $d_n(x) = 1$.

Using this one-to-one correspondence, the sets of all almost convergent, statistically convergent, uniformly statistically convergent, ideal convergent subsequences of a sequence s have been studied in detail in several papers (see [3, 4, 12–15, 17–19, 21–23]).

The concept of statistical boundedness of a sequence first appeared in the work of Fridy and Orhan (1997), [10]. Theorems researching statistical boundedness

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lmiller@ius.edu.ba; 0000-0002-7621-9231.

and its relation to statistical convergence were proved by Tripathy (1997) [20], and Bhardwaj and Gupta (2014), [5]. Recently the authors, Miller-Van Wieren (2022), [16] studied statistical boundedness of a sequence and its relation to statistical boundedness of its subsequences using Lebesgue measure and category. In this paper we wish to generalize this concept to ideal boundedness, and to obtain results connecting ideal boundedness of a sequence to that of its subsequences, again with regards to measure and category.

We will first introduce some necessary notation. A family $I \subseteq P(\mathbb{N})$ of subsets of $\mathbb{N}$ is said to be an ideal on $\mathbb{N}$ if I is closed under subsets and finite unions, i.e. for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and $B \subset A$, we have $B \in I$. An ideal I is said to be proper if it does not contain $\mathbb{N}$. We say a proper ideal is admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$. Clearly any admissible ideal contains all finite subsets of $\mathbb{N}$. Throughout the paper, we will assume that the ideal I is admissible.

A sequence of real numbers s is said to be I-convergent to L if for every $\varepsilon > 0$ the set $K_\varepsilon = \{n \in \mathbb{N} : |s_n - L| > \varepsilon\}$ belongs to I, and we write $I - \lim s = L$ (see Kostyrko, Šalát and Wilczyński, 2000/01 , Balaz and Šalát , 2006) [2, 11]. It is easy to see that if $I = I_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then I_d-convergence is statistical convergence where $d(A)$ denotes the asymptotic density of A [8], and if $I = I_u = \{A \subset \mathbb{N} : u(A) = 0\}$, then I_u-convergence coincides with uniform statistical convergence where $u(A)$ denotes the uniform density of A (Yurdakadim and Miller-Van Wieren 2016, Yurdakadim and Miller-Van Wieren 2017) [21, 22]. Ideals on $\mathbb{N}$ can be observed as subsets of the Polish space $\{0,1\}^{\mathbb{N}}$. Therefore ideals can have the Baire property or can be Borel, analytic, coanalytic etc. (Farah, 2000) [7]. From now on, we will refer to sets of first Baire category as meager, and to sets whose complement is of first category as comeager.

Next we state a well known lemma that can be found in several sources, recently in (M. Balcerzak, S. Glab, A. Wachowicz , 2016) [3].

Lemma 1. *Suppose I is an ideal on $\mathbb{N}$. The following conditions are equivalent:*

- *I has the Baire property;*
- *I is meager;*
- *There exists a sequence $n_1 < n_2 < ... < n_k < ...$ of integers in $\mathbb{N}$ such that no member of I contains infinitely many intervals $[n_k, n_{k+1})$ in $\mathbb{N}$.*

It is simple to verify that I_d and I_u have the Baire property. Additionally any analytic or coanalytic ideal has the Baire property.

2. Main Results

First, we recall the definition of a statistically bounded sequence of reals.

Definition 1. *A sequence of reals $s = (s_n)$ is said to be statistically bounded if there exists $L > 0$ such that $d(\{n : |s_n| \geq L\}) = 0$.*

Statistical boundedness of sequences was studied by Tripathy (1997) [20], Bhardwaj and Gupta (2014) [5], Aytar and Pehlivan (2006) [1] and by the authors (Miller-Van Wieren, 2022) [16].

Now we present a generalization of Definition 1, for a given ideal I introduced by Demirci (2001) [6].

Definition 2. *A sequence of reals $s = (s_n)$ is said to be I-bounded if there exists $L > 0$ such that $\{n : |s_n| \geq L\} \in I$.*

Given a sequence $s = (s_n)$ and $n_1 < n_2 < ... < n_k < ...$ we say that $s = (s_{n_k})$ is I-dense in s if $\mathbb{N} \setminus \{n_k : k \in \mathbb{N}\} \in I$. It is clear that $s = (s_{n_k})$ is I-bounded if and only if it has an I-dense subsequence that is bounded.

We will study the relationship of sequences and their subsequences regarding their I-boundedness, using Lebesgue measure as gauge of size.

In (Miller-Van Wieren, 2022) [16], we have shown the following theorem.

Theorem 1. *Suppose s is a sequence of reals. Then s is statistically bounded if and only if the set $\{x \in (0,1] : (sx)\text{ is statistically bounded}\}$ has Lebesgue measure 1. Additionally, s is not statistically bounded if and only if the set $\{x \in (0,1] : (sx)\text{ is statistically bounded}\}$ has Lebesgue measure 0.*

Now we direct our attention to sequences and their subsequences with regard to their I-boundedness. The discussion in the theorems that follow is related to some results obtained in [17, 18].

Theorem 2. *Suppose s is a I-bounded sequence, I is an analytic or coanalytic ideal. Then the set $\{x \in (0,1] : (sx)\text{ is } I\text{-bounded}\}$ has Lebesgue measure 0 or 1. Both cases of measure 0 and 1 can occur.*

Proof. Let us first prove that $\{x \in (0,1] : (sx)\text{ is } I\text{-bounded}\}$ is measurable. We have

$$\{x \in (0,1] : (sx)\text{ is } I\text{-bounded}\} = \bigcup_{M \in \mathbb{N}} \{x : \{i : |\,(sx)_i\,| \geq M\} \in I\}.$$

We define the characteristic function

$$\chi_M : (0,1] \to \{0,1\}^{\mathbb{N}}$$

by setting $(\chi_M(x))_i = \begin{cases} 1 & , \quad |\,(sx)_i\,| \geq M \\ 0 & , \quad otherwise \end{cases}$

for $M \in \mathbb{N}$. We will verify that χ_M is continuous. For this purpose it is sufficient to check that the i-th component of χ_M, $(\chi_M)_i$ is continuous on $(0,1]$. We will check that the set $(\chi_M)_i^{-1}(\{1\})$ is open. Suppose that $x \in (\chi_M)_i^{-1}(\{1\})$ is arbitrarily fixed. Easily if $y \in (0,1]$ is such that $(sx)_j = (sy)_j$ for $1 \leq j \leq i$, then $y \in (\chi_M)_i^{-1}(\{1\})$. We can conclude that there exists a $k \geq i$ such that: if $y \in (0,1]$ satisfies $x_j = y_j$ for $1 \leq j \leq k$ (where x_j, y_j are the j-th coordinates of x, y respectively as $0-1$ sequences), then $y \in (\chi_M)_i^{-1}(\{1\})$. We obtain that $(\chi_M)_i^{-1}(\{1\})$ is open. In the same manner, we conclude that $(\chi_M)_i^{-1}(\{0\})$ is open.

Since I is analytic or coanalytic, we conclude that $\chi_M^{-1}(I)$ is analytic or coanalytic and hence measurable. Therefore
$\{x : \{i : |(sx)_i| \geq M\} \in I\} = \chi_M^{-1}(I)$ is measurable for $M \in \mathbb{N}$ and consequently $\{x \in (0,1] : (sx)\text{ is } I\text{-bounded}\}$ is measurable. Clearly $\{x \in (0,1] : (sx)\text{ is } I\text{-bounded}\}$ is a tail set. Since we proved it is measurable we conclude that $X = \{x \in (0,1] : (sx)\text{ is } I\text{-bounded}\}$ must have Lebesgue measure 0 or 1.

To see that both values can occur observe the following. If the sequence s is bounded (and consequently I-bounded), then for every $x \in (0,1]$, (sx) is bounded and consequently I-bounded, therefore $m(X) = 1$. Additionaly we can remark that in the case when $I = I_d$, the authors proved in [16] that the set X is of measure 1 for any I-bounded sequence s. Now we construct an example in which $m(X) = 0$ occurs.

In [13], Miller and Orhan (2001) constructed a sequence t of 0's and 1's,

$$t = 01001001...00010001....$$

that we made use of in (Yurdakadim and Miller-Van Wieren, 2016) [21] showing that t uniformly statistically converges to 0, $u(\{n : t_n = 1\}) = 0$, and $X^* = \{x \in (0,1] : \{n : (tx)_n = 1\}\text{is not in } I_u\}$ has measure 1.

Now we will construct a sequence s that is I_u-bounded but $m(X) = 0$. We define $s = (s_n)$ as follows:

$$s_n = \begin{cases} 0 & , \quad t_n = 0 \\ n & , \quad t_n = 1 \end{cases} \quad \text{for } n \in \mathbb{N}.$$

Now from this definition it follows that $u(\{n : s_n \neq 0\}) = 0$, so s is I_u-bounded. Suppose $x \in X^*$. From the definitions of s and X^* we conclude that there exists a subset of $\mathbb{N}$, $\{n_k : k \in \mathbb{N}\}$ not in I_u such that $(sx)_{n_k} \to \infty$ and therefore sx is not I_u-bounded. Since $m(X^*) = 1$, it follows that $m(X) = 0$. This completes the proof.

□

Now we will observe the case when I is an analytic or coanalytic ideal with property (G). We will use some notation from (M. Balcerzak, S. Glab, A. Wachowicz , 2016) [3].

We will denote by T the set of all 0-1 sequences that have an infinite number of ones. A mapping $f : \mathbb{N} \to \mathbb{N}$ is said to be bi-I-invariant if $E \in I$ if and only if $f[E] \in I$ whenever $E \subset \mathbb{N}$. Given a sequence $x \in T$ we can denote $\{n_1 < n_2 < ... < n_i < ...\} = \{k \in \mathbb{N} : x_k = 1\}$. Define $f_x : \mathbb{N} \longrightarrow \mathbb{N}$ by $f_x(k) = n_k$ and define $T_I = \{x \in T : f_x\text{is bi-}I\text{-invariant}\}$.

An ideal I is said to have property (G) if $\mu(T_I) = 1$. For instance, it is easy to check that I_d has property (G) while I_u does not. Now we have an analog of Theorem 1 for ideals with property (G).

Theorem 3. *Suppose s is a sequence, I is an analytic or coanalytic ideal with property (G). Then s is I-bounded if and only if the set $X = \{x \in (0,1] :$*

(sx) is I-bounded$\}$ has Lebesgue measure 1. Additionally, s is not I-bounded if and only if the set $\{x \in (0,1] : (sx)$ is I-bounded$\}$ has Lebesgue measure 0.

Proof. Suppose s is I-bounded. Suppose $M > 0$ is fixed so that $\{n : |s_n| \geq M\} \in I$. Let $x \in T_I$ be arbitrarily fixed (using the earlier mentioned definition of T_I). Then $(sx) = (s_{n_i})_i$ where $n_1 < n_2 < ... < n_i < ....$ Then,

$\{n_i : |s_{n_i}| \geq M\} \subseteq \{n : |s_n| \geq M\}$ and consequently from above $\{n_i : |s_{n_i}| \geq M\} \in I$. Now since $x \in T_I$, we have $\{n_i : |s_{n_i}| \geq M\} \in I \to f_x^{-1}(\{n_i : |s_{n_i}| \geq M\}) \in I \to \{i : |s_{n_i}| \geq M\} \in I$. Hence (sx) is I-bounded. We conclude that $T_I \subseteq X$. Since $m(T_I) = 1$, $m(X) = 1$.

Conversely suppose that $m(X) = 1$.

Let $T = X \cap (1 - X) \cap T_I \cap (1 - T_I)$ where $1 - X = \{x : 1 - x \in X\}$ and $1 - T_I$ is defined analogously. Then $m(T) = 1$ and $x \in T \to 1 - x \in T$. Suppose $x \in T$ is fixed. We will denote by $\{n_i\}$ the set of indices corresponding to x and by $\{n_j\}$ the set of indices corresponding to $1 - x$. Trivially $\{n_i\} \cap \{n_j\} = \varnothing$, $\{n_i\} \cup \{n_j\} = \mathbb{N}$. Then there exists $M > 0$ for which $\{i : |s_{n_i}| \geq M\} \in I$ and $\left\{j : |s_{n_j}| \geq M\right\} \in I$. From the above, $f_x(\{i : |s_{n_i}| \geq M\}) \in I$ and $f_{1-x}(\left\{j : |s_{n_j}| \geq M\right\}) \in I$. Therefore $\{n_i : |s_{n_i}| \geq M\} \in I$ and $\left\{n_j : |s_{n_j}| \geq M\right\} \in I$ and consequently

$$\{n : |s_n| \geq M\} = \{n_i : |s_{n_i}| \geq M\} \bigcup \left\{n_j : |s_{n_j}| \geq M\right\} \in I.$$

Therefore s is I-bounded. This completes the proof of the first statement.

To prove the second statement observe that in the proof of Theorem 2, we have shown that X is a measurable tail set with measure 0 or 1. Therefore the second statement follows immediately from the first one. The proof is complete.

□

Next we observe the relationship of the subsequences of a given sequence regarding I-boundedness, using Baire category as a gauge of size. In (Miller-Van Wieren, 2022) [16] we showed the following theorem .

Theorem 4. *Suppose $s = (s_n)$ is an unbounded sequence of reals, and let $X = \{x \in (0,1] : (sx)$ is statistically bounded$\}$. Then X is meager.*

We focus on I-boundedness with the assumption that I is an ideal with the Baire property. If s is a bounded sequence of reals, then all of its subsequences are likewise bounded, and hence I-bounded as well. If that is not the case we can show the following theorem.

Theorem 5. *Suppose $s = (s_n)$ is an unbounded sequence of reals, I an ideal with the Baire property and $X = \{x \in (0,1] : (sx)$ is I–bounded$\}$. Then X is meager.*

Proof. Since s is unbounded, it has ∞ or $-\infty$ as a limit point. Let us assume that ∞ is a limit point of s (the case of $-\infty$ is analogous) . Now since I has the Baire property, we can find a sequence $n_1 < n_2 < ... < n_k < ...$ of integers such that no member of I contains infinitely many intervals $[n_k, n_{k+1})$.

For arbitrary $m, j \in \mathbb{N}$, let

$$K_{m,j} = \{x \in (0,1] : there\ exists\ k \in \mathbb{N}, n_k > m : |(sx)_i| > j\ for\ i \in [n_k, n_{k+1})\}. \tag{1}$$

Let $m, j \in \mathbb{N}$ be arbitrarily fixed. We proceed to prove that $K_{m,j}$ is comeager.

Fix an arbitrary finite sequence of $0's$ and $1's$ denoted by $\overline{x} = (x_1, x_2, ..., x_d)$. It suffices to prove that we can find a finite extension x^* of $\overline{x}$ such that any $x \in (0,1]$ starting with x^* belongs to $K_{m,j}$. Suppose that $\overline{x}$ has t $1's$ where $t \geq m$ (we can assume this without loss of generality). Let $k = \min\{i : n_i > t\}$. We first extend $\overline{x}$ to a sequence $(x_1, x_2, ..., x_g)$, $g \geq d$ that has exactly $n_k - 1$ $1's$. Since ∞ is a limit point of s we can find $i_{n_k} < i_{n_k+1} < ... < i_{n_{k+1}-1}$ greater than g such that the terms of s corresponding to those indices are greater than j. Now define the following extension of $\overline{x}$

$$x^* = (x_1, x_2, ..., x_g, ..., x_{i_{n_k}}, ..., x_{i_{n_k+1}}, ...x_{i_{n_{k+1}-1}})$$

where for $i > g$: $x_i = 1\ for\ i \in \{i_{n_k}, i_{n_k+1}, ..., i_{n_{k+1}-1}\}$ and $x_i = 0,\ otherwise$. It is clear that any $x \in (0,1]$ that extends x^* belongs to $K_{m,j}$. We conclude $K_{m,j}$ is comeager. Consequently $K = \cap_m \cap_j K_{m,j}$ is also comeager. Now if $x \in K$, for every j the set $\{n : |(sx)_n| > j\}$ contains infinitely many $[n_k, n_{k+1})$. Consequently for $x \in K$, sx cannot be I-bounded, since if we assumed otherwise, there would exist j for which $\{n : |(sx)_n| > j\} \in I$, a contradiction. Since K is comeager, it follows that X is meager. □

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