Journal of Mathematical Sciences and Modelling, 7(3) (2024) 146-156
Research Article

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm
ISSN: 2636-8692
DOI: https://doi.org/10.33187/jmsm.1475211

Unveiling New Exact Solutions of the Complex-Coupled Kuralay System Using the Generalized Riccati Equation Mapping Method

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Article Info

Keywords: *Complex-coupled Kuralay system (CCKS), Soliton solutions, The generalized Riccati equation mapping (GREM) method*
2010 AMS: *35C08, 35C09, 35C07, 35A99*
Received: *29 April 2024*
Accepted: *21 December 2024*
Available online: *29 December 2024*

Abstract

This examination analyzes the integrable dynamics of induced curves by utilizing the complex-coupled Kuralay system (CCKS). The significance of the coupled complex Kuralay equation lies in its role as an essential model that contributes to the understanding of intricate physical and mathematical concepts, making it a valuable tool in scientific research and applications. The soliton solutions originating from the Kuralay equations are believed to encapsulate cutting-edge research in various essential domains such as optical fibers, nonlinear optics, and ferromagnetic materials. Analytical procedures are operated to derive traveling wave solutions for this model, given that the Cauchy problem cannot be resolved using the inverse scattering transform. This study uses the generalized Riccati equation mapping (GREM) method to search for analytical solutions. This method observes single and combined wave solutions in the shock, complex solitary shock, shock singular, and periodic singular forms. Rational solutions also emerged during the derivation. In addition to the analytical results, numerical simulations of the solutions are presented to enhance comprehension of the dynamic features of the solutions generated. The study's conclusions could provide insightful information about how to solve other nonlinear partial differential equations (NLPDEs). The soliton solutions found in this work provide valuable information on the complex nonlinear problem under investigation. These results provide a foundation for further investigation, making the solutions helpful, manageable, and trustworthy for the future development of intricate nonlinear issues. This study's methodology is reliable, robust, effective, and applicable to various NLPDEs. The Maple software application is used to verify the correctness of all obtained solutions.

1. Introduction

Partial differential equations (PDEs) play a crucial role in mathematical modeling by serving as a critical mechanism for analyzing and understanding the dynamics of intricate systems across various fields, including physics, engineering, economics, and other disciplines. PDEs offer a robust framework for predicting and interpreting the behavior of diverse phenomena, effectively capturing intricate relationships between variables and their rates of change. These equations elucidate the temporal and spatial evolution of functions with multiple variables. NLPDEs hold significance across various disciplines, such as mathematics, science, and engineering. Contemporary research has unveiled a diverse range of complex nonlinear models, prompting the development of sophisticated mathematical methodologies by numerous scholars to derive exact solutions. Consequently, the exploration of nonlinear phenomena has garnered considerable attention among academics. The construction of soliton structures is progressively gaining importance due to its relevance in multiple scientific domains [1–5].
Furthermore, an emerging application of NLPDEs lies in investigating soliton waves. Soliton waves, characterized by localized wave packets that maintain their shape and velocity as they propagate, are the subject of study in various nonlinear physical models researchers employ to elucidate and forecast their dynamics. Consequently, the significance of soliton waves is progressively growing across diverse

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Cite as "B. Kopçasız, *Unveiling new exact solutions of the complex-coupled Kuralay system using the generalized Riccati equation mapping method*, J. Math. Sci. Model., 7(3) (2024), 146-156"

disciplines, including nonlinear optics, optical fibers, and ferromagnetic materials. Recent advancements in understanding soliton waves are comprehensively examined in the literature reference [6–10]. Researchers can advance their knowledge and explore novel applications by enhancing their comprehension of soliton waves. Alongside the advantages of utilizing NLPDEs, numerous methodologies have been devised to address the difficulty of accurately determining analytical solutions for NLPDEs. Recently, there has been a growing interest among scholars in investigating the exact solutions of NLPDEs, particularly concerning nonlinear physical phenomena. Nonlinear critical approaches have been introduced and employed by many physicists and mathematicians, such as the Hirota bilinear method [11], the Painlevé analysis [12], the Bäcklund transformation [13], Lie symmetry analysis and conservation laws [14]. Some more techniques can be found in the literature [15–20].
The Kuralay equation is a mathematical model that has been the subject of a thorough examination by scholars in integrable systems and nonlinear dynamics. Its origins are not attributed to a singular discoverer but have emerged through research endeavors and mathematical inquiries, solidifying its significance as a foundational model in scientific exploration. The Kuralay equation is utilized across diverse disciplines in contemporary scientific studies, notably in analyzing ferromagnetic materials and wave propagation phenomena. This mathematical framework investigates the generation and movement of solitary wave solutions, enhancing comprehension of intricate physical phenomena. Moreover, the Kuralay equation is crucial in examining the integrable dynamics of spatial curves and geometric flows, offering valuable insights into the interactions among curves, multilayer spin systems, and the vector nonlinear Schrödinger equation. The coupled complex Kuralay equation is fundamental in contemporary scientific disciplines because of its integrability and practical implications. This mathematical framework plays a crucial role in examining diverse phenomena and has been subject to thorough investigation for its capacity to depict intricate systems precisely. Scholars have concentrated on deducing exact solutions, examining optical solitons, and delving into innovative soliton solutions through computational modeling techniques centered on the Kuralay equation [21–27].
The primary aim of this study is to utilize the GREM technique on CCKS to investigate novel soliton solutions. For this purpose, we discuss some previous work and our newly obtained results. Researchers have explored various solutions to the complex nonlinear Kuralay-IIA model operating methods, such as the Hirota bilinear procedure [22], simple equation and Paul-Painlevé approaches [23], generalized Kudryashov scheme, extended Sinh Gordon equation expansion scheme, $\exp_{\alpha}$-function scheme [24], new auxiliary equation scheme [25], modified F-expansion, and new extended auxiliary equation techniques [26]. In this study, we have considered the GREM method. The GREM method is a valuable tool to get the exact solitary wave solutions and control theory, particularly for linear time-invariant systems. Compared to other methods, this method is highly convenient and easy to implement, offers a wide range of solutions, and works efficiently.
The paper is structured as follows: Section 2 presents the mathematical analysis of the model under consideration. Section 3 describes the offered method. Analytical solutions for the model are discussed in Section 4. Section 5 presents a graphical discussion and Section 6 explains the conclusions.

2. Analytical Analysis of the Complex-Coupled Kuralay System (CCKS)

Consider the CCKS as [27]:

$$\begin{cases} i\Phi_t - \Phi_{xt} - \tau\Phi = 0, \\ i\Omega_t + \Omega_{xt} + \tau\Phi = 0, \\ \tau_x + 2R^2(\Omega\Phi)_t = 0, \end{cases}$$

in which $\Phi(x,t)$ is complex function with a complex conjugate as $\Phi^*(x,t)$. Subsequent, τ represents the potential real function, and they depend on the independent spatial and temporal variables x and t, respectively.
By assuming that $R = 1$, $\Omega = \varepsilon\Phi^*$, where $\varepsilon = \pm 1$, the above system of differential equation will become:

$$\begin{cases} i\Phi_t - \Phi_{xt} - \tau\Phi = 0, \\ \tau_x - 2\varepsilon(|\Phi|^2)_t = 0. \end{cases} \tag{2.1}$$

Now, the following complex traveling wave transformation is applied to the complex system of PDEs Eq. (2.1)

$$\Phi(x,t) = \Upsilon(\xi) \times \exp(i(\alpha x + \beta t + \gamma)), \quad \tau(x,t) = \Psi(\xi) \times \exp(i(\alpha x + \beta t + \gamma)), \quad \xi = sx + vt, \tag{2.2}$$

where,

$$\begin{cases} \Phi_t = (v\Upsilon' + \imath\beta\Upsilon) \times \exp(i(\alpha x + \beta t + \gamma)), \\ \Phi_x = (s\Upsilon' + \imath\alpha\Upsilon) \times \exp(i(\alpha x + \beta t + \gamma)), \\ \Phi_{xt} = (vs\Upsilon'' + \imath\beta s\Upsilon' + iv\alpha\Upsilon' + \imath\alpha\beta\Upsilon) \times \exp(i(\alpha x + \beta t + \gamma)), \end{cases} \tag{2.3}$$

and α, β, γ, s, and v are real numbers. The Eq. (2.2) along with Eq. (2.3) is plugging into Eq. (2.1) and gets,

$$\begin{cases} i(v\Upsilon' + i\beta\Upsilon) - (vs\Upsilon'' + i\beta s\Upsilon' + iv\alpha\Upsilon' - \alpha\beta\Upsilon) - \Psi\Upsilon = 0, \\ s\Psi' - 4v\varepsilon\Upsilon\Upsilon' = 0, \end{cases} \tag{2.4}$$

integrating the second part of Eq. (2.4), we attain

$$\Psi = \frac{2\varepsilon v\Upsilon^2}{s} - \frac{v_1}{s}. \tag{2.5}$$

The Eq. (2.5) is putting into the first part of Eq. (2.4):

$$i(v\Upsilon' + i\beta\Upsilon) - (vs\Upsilon'' + i\beta s\Upsilon' + iv\alpha\Upsilon' - \alpha\beta\Upsilon) - \left(\frac{2\varepsilon v\Upsilon^2}{s} - \sigma\right) = 0, \tag{2.6}$$

where $\sigma = \frac{v_1}{s}$.
The real part and imaginary part of Eq. (2.6) are give as, respectively,

$$\Upsilon'' + \frac{(\beta(1-\alpha)-\sigma)}{vs}\Upsilon + \frac{2\varepsilon}{s^2}\Upsilon^3 = 0, \tag{2.7}$$

$$(v - \beta s - v\alpha)\Upsilon' = 0. \tag{2.8}$$

The imaginary part Eq. (2.8) implies

$$s = \frac{v(\alpha-1)}{\beta}. \tag{2.9}$$

The value of s Eq. (2.9) is substituting into Eq. (2.7), and get a nonlinear ordinary differential equation (NLODE)

$$\Upsilon'' + \frac{\beta(\beta(1-\alpha)-\sigma)}{v^2(\alpha-1)}\Upsilon + \frac{2\beta^2\varepsilon}{v^2(\alpha-1)^2}\Upsilon^3 = 0. \tag{2.10}$$

3. GREM Method

This section will present the GREM method [28–30].
Consider the following NLPDE:

$$Z(\Phi(x,t), \Phi_x(x,t), \Phi_t(x,t), \Phi_{xx}(x,t), \Phi_{xt}(x,t), \Phi_{tt}(x,t), \ldots) = 0, \tag{3.1}$$

in which Z is generally a polynomial function of its argument, and the subscripts of the dependent variable denote the partial derivatives. By using the Eq. (2.2), Eq. (3.1) can be transformed to an NLODE:

$$W(\Upsilon, \Upsilon', \Upsilon'', \ldots) = 0. \tag{3.2}$$

Suppose that the solution of Eq. (3.2) is in the polynomial form

$$\Upsilon = \sum_{j=0}^{p} A_j \Lambda^j(\xi), \qquad A_p \neq 0, \tag{3.3}$$

in which $A_j (0 \leq j \leq p)$ are arbitrary constants that are determine later, and p is a positive integer that is obtained by the help of the balancing principle in Eq. (3.2). In Eq. (3.3), the function $\Lambda(\xi)$ satisfies the generalized Riccati equation is provided as follows:

$$\Lambda(\xi)' = \theta_0 + \theta_1\Lambda(\xi) + \theta_2\Lambda(\xi)^2, \quad \theta_2 \neq 0, \tag{3.4}$$

in which θ_0, θ_1, and θ_2 are all real constants. Substituting the Eq. (3.3) with Eq. (3.4) into the regarding NLODE and removing all the coefficients of $\Lambda^j(\xi)$ will obtain a system of algebraic equations. Solving the algebraic equations, with the known solutions of Eq. (3.3), one can easily obtain the solutions to the Eq. (3.1). We can obtain the following twenty seven solutions to Eq. (3.2) such as:
Set 1: For $\Delta = \theta_1^2 - 4\theta_0\theta_2 > 0$, and $\theta_1\theta_2 \neq 0$ (or $\theta_0\theta_2 \neq 0$), the solutions of Eq. (3.4) are

$$\Lambda_1(\xi) = -\frac{1}{2\theta_2}\left(\theta_1 + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right), \tag{3.5}$$

$$\Lambda_2(\xi) = -\frac{1}{2\theta_2}\left(\theta_1 + \sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right), \tag{3.6}$$

$$\Lambda_3(\xi) = -\frac{1}{2\theta_2}\left(\theta_1 + \sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right) \pm i\,\mathrm{sec}h\left(\sqrt{\Delta}\xi\right)\right)\right), \tag{3.7}$$

$$\Lambda_4(\xi) = -\frac{1}{2\theta_2}\left(\theta_1 + \sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\xi\right) \pm \mathrm{csc}h\left(\sqrt{\Delta}\xi\right)\right)\right), \tag{3.8}$$

$$\Lambda_5(\xi) = -\frac{1}{4\theta_2}\left(2\theta_1 + \sqrt{\Delta}\left(\tanh\left(\frac{\sqrt{\Delta}}{4}\xi\right) + \mathrm{cot}h\left(\frac{\sqrt{\Delta}}{4}\xi\right)\right)\right), \tag{3.9}$$

$$\Lambda_6(\xi) = \frac{1}{2\theta_2}\left(-\theta_1 + \frac{\sqrt{\Delta(P^2+Q^2)} - P\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{P\sinh\left(\sqrt{\Delta}\xi\right) + Q}\right), \tag{3.10}$$

$$\Lambda_7(\xi) = \frac{1}{2\theta_2}\left(-\theta_1 - \frac{\sqrt{\Delta(P^2+Q^2)} + P\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{P\sinh\left(\sqrt{\Delta}\xi\right) + Q}\right), \tag{3.11}$$

in which P and Q are two non-zero real constants and satisfies $P^2 - Q^2 > 0$.

$$\Lambda_8(\xi) = \frac{2\theta_0 \cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - \theta_1 \cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}, \tag{3.12}$$

$$\Lambda_9(\xi) = -\frac{2\theta_0 \sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{\theta_1 \sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - \sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}, \tag{3.13}$$

$$\Lambda_{10}(\xi) = \frac{2\theta_0 \cosh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right) - \theta_1 \cosh\left(\sqrt{\Delta}\xi\right) \pm i\sqrt{\Delta}}, \tag{3.14}$$

$$\Lambda_{11}(\xi) = \frac{2\theta_0 \sinh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right) - \theta_1 \sinh\left(\sqrt{\Delta}\xi\right) \pm \sqrt{\Delta}}, \tag{3.15}$$

$$\Lambda_{12}(\xi) = \frac{4\theta_0 \sinh\left(\frac{\sqrt{\Delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\Delta}}{4}\xi\right)}{2\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{4}\xi\right)^2 - 2\theta_1 \sinh\left(\frac{\sqrt{\Delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\Delta}}{4}\xi\right) - \sqrt{\Delta}}. \tag{3.16}$$

Set 2: For $\Delta = \theta_1^2 - 4\theta_0\theta_2 < 0$, and $\theta_1\theta_2 \neq 0$ (or $\theta_0\theta_2 \neq 0$), the solutions of Eq. (3.4) are

$$\Lambda_{13}(\xi) = \frac{1}{2\theta_2}\left(-\theta_1 + \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right), \tag{3.17}$$

$$\Lambda_{14}(\xi) = -\frac{1}{2\theta_2}\left(\theta_1 + \sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right), \tag{3.18}$$

$$\Lambda_{15}(\xi) = \frac{1}{2\theta_2}\left(-\theta_1 + \sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right)\right), \tag{3.19}$$

$$\Lambda_{16}(\xi) = -\frac{1}{2\theta_2}\left(\theta_1 + \sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\xi\right) \pm \csc\left(\sqrt{-\Delta}\xi\right)\right)\right), \tag{3.20}$$

$$\Lambda_{17}(\xi) = \frac{1}{4\theta_2}\left(-2\theta_1 + \sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}}{4}\xi\right) - \cot\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\right)\right), \tag{3.21}$$

$$\Lambda_{18}(\xi) = \frac{1}{2\theta_2}\left(-\theta_1 + \frac{\pm\sqrt{\Delta(-P^2+Q^2)} \mp P\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{P\sin\left(\sqrt{-\Delta}\xi\right) + Q}\right), \tag{3.22}$$

in which P and Q are two non-zero real constants and satisfies $P^2 - Q^2 > 0$.

$$\Lambda_{19}(\xi) = -\frac{2\theta_0 \cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + \theta_1 \cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}, \tag{3.23}$$

$$\Lambda_{20}(\xi) = \frac{2\theta_0 \sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - \theta_1 \sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}, \tag{3.24}$$

$$\Lambda_{21}(\xi) = -\frac{2\theta_0 \cos\left(\sqrt{-\Delta}\xi\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right) + \theta_1 \cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}}, \tag{3.25}$$

$$\Lambda_{22}(\xi) = -\frac{2\theta_0 \sin\left(\sqrt{-\Delta}\xi\right)}{\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right) + \theta_1 \sin\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}}, \tag{3.26}$$

$$\Lambda_{23}(\xi) = \frac{4\theta_0 \sin\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\cos\left(\frac{\sqrt{-\Delta}}{4}\xi\right)}{2\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}}{4}\xi\right)^2 - 2\theta_1 \sin\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\cos\left(\frac{\sqrt{-\Delta}}{4}\xi\right) - \sqrt{-\Delta}}. \tag{3.27}$$

Set 3: For $\theta_0 = 0$, and $\theta_1\theta_2 \neq 0$, the solutions of Eq. (3.4) are

$$\Lambda_{24}(\xi) = -\frac{\theta_1\zeta_0}{\theta_2\left(\zeta_0 + \cosh(\theta_1\xi) - \sinh(\theta_1\xi)\right)}, \tag{3.28}$$

$$\Lambda_{25}(\xi) = -\frac{\theta_1\left(\cosh(\theta_1\xi) - \sinh(\theta_1\xi)\right)}{\theta_2\left(\zeta_0 + \cosh(\theta_1\xi) - \sinh(\theta_1\xi)\right)}, \tag{3.29}$$

$$\Lambda_{26}(\xi) = -\frac{\theta_1\left(\cosh(\theta_1\xi) + \sinh(\theta_1\xi)\right)}{\theta_2\left(\zeta_0 + \cosh(\theta_1\xi) + \sinh(\theta_1\xi)\right)}, \tag{3.30}$$

in which ζ_0 is any arbitrary constant.
Set 4: For $\theta_0 \neq 0$, and $\theta_1 = \theta_2 = 0$, the solutions of Eq. (3.4) are

$$\Lambda_{27}(\xi) = -\frac{1}{\theta_0 \xi + c_1}, \tag{3.31}$$

in which c_1 is an arbitrary constant.

4. Solving the CCKS Using the Recommended Method

In this part, we explore the analytical solutions of the CCKS by applying the GREM technique. Balancing Φ'' with Φ^3, we get $p = 1$. From Eq. (3.3), the solution is presumed as

$$\Upsilon = K_0 + K_1 \Lambda, \qquad K_1 \neq 0. \tag{4.1}$$

Putting Eq. (4.1) along with Eq. (3.4) into Eq. (2.10), and setting the coefficient of $\Lambda^j(\xi)$ to be zero, which gives a system of algebraic equations. After solving the system of equations, we get the solutions are as follows:

$$K_0 = \mp \frac{\theta_1 v \sigma}{\varepsilon(-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2) \times \sqrt{-\frac{1}{\varepsilon}}}, \quad K_1 = \pm \frac{2\theta_2 v \sigma \sqrt{-\frac{1}{\varepsilon}}}{(-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2)},$$

$$\alpha = \frac{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2 - 2\beta\sigma}{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2}. \tag{4.2}$$

By putting constant values Eq. (4.2) along with Eq. (3.5)-Eq. (3.16) in the Eq. (4.1), and by using the wave transformation Eq. (2.2), we get the solutions as follows:

$$\begin{aligned} \Phi_1(x,t) &= -\frac{\sigma v \sqrt{\Delta} \tanh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2) \times \sqrt{-\frac{1}{\varepsilon}}} \\ &\quad \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2 - 2\beta\sigma}{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2}\right)x + \beta t + \gamma\right)\right), \end{aligned}$$

$$\begin{aligned} \Phi_2(x,t) &= -\frac{\sigma v \sqrt{\Delta} \coth\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2) \times \sqrt{-\frac{1}{\varepsilon}}} \\ &\quad \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2 - 2\beta\sigma}{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2}\right)x + \beta t + \gamma\right)\right), \end{aligned}$$

$$\begin{aligned} \Phi_3(x,t) &= -\frac{\sigma v \sqrt{\Delta} \left(\tanh\left(\sqrt{\Delta}(sx+vt)\right) + i\,\mathrm{sec}h\left(\sqrt{\Delta}(sx+vt)\right)\right)}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2) \times \sqrt{-\frac{1}{\varepsilon}}} \\ &\quad \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2 - 2\beta\sigma}{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2}\right)x + \beta t + \gamma\right)\right), \end{aligned}$$

$$\begin{aligned} \Phi_4(x,t) &= -\frac{\sigma v \sqrt{\Delta} \left(\coth\left(\sqrt{\Delta}(sx+vt)\right) + \mathrm{csc}h\left(\sqrt{\Delta}(sx+vt)\right)\right)}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2) \times \sqrt{-\frac{1}{\varepsilon}}} \\ &\quad \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2 - 2\beta\sigma}{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2}\right)x + \beta t + \gamma\right)\right), \end{aligned}$$

$$\begin{aligned} \Phi_5(x,t) &= -\frac{\sigma v \sqrt{\Delta} \left(\tanh\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right) + \coth\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right)\right)}{2\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2) \times \sqrt{-\frac{1}{\varepsilon}}} \\ &\quad \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2 - 2\beta\sigma}{-4v^2\theta_0\theta_2 + v^2\theta_1^2 + 2\beta^2}\right)x + \beta t + \gamma\right)\right), \end{aligned}$$

$$\Phi_6(x,t) = -\frac{\sigma v\left(P\sqrt{\Delta}\cosh\left(\sqrt{\Delta}(sx+vt)\right)-\sqrt{-(-\Delta)(P^2+Q^2)}\right)}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)(P\sinh\left(\sqrt{\Delta}(sx+vt)\right)+Q)\times\sqrt{-\frac{1}{\varepsilon}}}$$
$$\times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_7(x,t) = -\frac{\sigma v\sqrt{-(-\Delta)(P^2+Q^2)}+P\sqrt{\Delta}\cosh\left(\sqrt{\Delta}(sx+vt)\right)}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)(P\sinh\left(\sqrt{\Delta}(sx+vt)\right)+Q)\times\sqrt{-\frac{1}{\varepsilon}}}$$
$$\times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_8(x,t) = -\frac{\sigma v\begin{pmatrix}4\theta_2\theta_0\cosh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)\\-\theta_1^2\cosh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)+\theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)\sqrt{\Delta}\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}\theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)\\-\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}}$$
$$\times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_9(x,t) = -\frac{\sigma v\begin{pmatrix}4\theta_2\theta_0\sinh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)-\theta_1^2\sinh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)\\+\theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)\sqrt{\Delta}\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}\theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)\\-\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}}$$
$$\times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{10}(x,t) = \frac{\sigma v\begin{pmatrix}4\theta_2\theta_0\cosh\left(\sqrt{\Delta}(sx+vt)\right)+\sqrt{\Delta}\theta_1\sinh\left(\sqrt{\Delta}(sx+vt)\right)\\+i\sqrt{\Delta}\theta_1-\cosh\left(\sqrt{\Delta}(sx+vt)\right)\theta_1^2\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}\sqrt{\Delta}\sinh\left(\sqrt{\Delta}(sx+vt)\right)\\-\theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}(sx+vt)\right)+i\sqrt{\Delta}\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}}$$
$$\times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{11}(x,t) = -\frac{\sigma v\begin{pmatrix}4\theta_2\theta_0\sinh\left(\sqrt{\Delta}(sx+vt)\right)-\theta_1^2\sinh\left(\sqrt{\Delta}(sx+vt)\right)\\+\sqrt{\Delta}\theta_1\cosh\left(\sqrt{\Delta}(sx+vt)\right)+\sqrt{\Delta}\theta_1\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}\theta_1\sinh\left(\sqrt{\Delta}(sx+vt)\right)\\-\sqrt{\Delta}\cosh\left(\sqrt{\Delta}(sx+vt)\xi\right)-\sqrt{\Delta}\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}}$$
$$\times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{12}(x,t) = -\frac{\sigma v\begin{pmatrix}8\theta_2\theta_0\sinh\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right)\cosh\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right)\\+2\theta_1\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right)^2\\-2\theta_1^2\sinh\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right)\cosh\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right)-\sqrt{\Delta}\theta_1\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}2\theta_1\sinh\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right)\cosh\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right)\\-2\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{4}(sx+vt)\right)^2+\sqrt{\Delta}\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}}$$
$$\times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right).$$

By putting constant values Eq. (4.2) along with Eq. (3.17)- Eq. (3.27) in the Eq. (4.1), and by using the wave transformation Eq. (2.2), we obtain the solutions as follows:

$$\Phi_{13}(x,t) = \frac{\sigma v\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right)}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2)\times\sqrt{-\frac{1}{\varepsilon}}} \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{14}(x,t) = -\frac{\sigma v\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right)}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2)\times\sqrt{-\frac{1}{\varepsilon}}} \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{15}(x,t) = \frac{\sigma v\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}(sx+vt)\right)+\sec\left(\sqrt{-\Delta}(sx+vt)\right)\right)}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2)\times\sqrt{-\frac{1}{\varepsilon}}} \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{16}(x,t) = -\frac{\sigma v\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}(sx+vt)\right)+\csc\left(\sqrt{-\Delta}(sx+vt)\right)\right)}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2)\times\sqrt{-\frac{1}{\varepsilon}}} \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{17}(x,t) = \frac{\sigma v\sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right)-\cot\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right)\right)}{2\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2)\times\sqrt{-\frac{1}{\varepsilon}}} \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{18}(x,t) = \frac{\sigma v\left(\sqrt{(-\Delta)(P-Q)(P+Q)}-P\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}(sx+vt)\right)\right)}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2)(P\sin\left(\sqrt{-\Delta}(sx+vt)\right)+Q)\times\sqrt{-\frac{1}{\varepsilon}}} \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{19}(x,t) = -\frac{\sigma v\begin{pmatrix} 4\theta_2\theta_0\cos\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right)-\theta_1^2\cos\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right) \\ -\theta_1\sin\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right)\sqrt{-\Delta}\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2)\begin{pmatrix}\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right) \\ +\theta_1\cos\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right)\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}} \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{20}(x,t) = -\frac{\sigma v\begin{pmatrix} 4\theta_2\theta_0\sin\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right) \\ -\theta_1^2\sin\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right)+\theta_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\sqrt{-\Delta}\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2 - v^2\theta_1^2 - 2\beta^2)\begin{pmatrix}\theta_1\sin\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right) \\ -\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}}{2}(sx+vt)\right)\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}} \times \exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{21}(x,t) = -\frac{\sigma v\begin{pmatrix} 4\theta_2\theta_0\cos\left(\sqrt{-\Delta}(sx+vt)\right)-\theta_1^2\cos\left(\sqrt{-\Delta}(sx+vt)\right) \\ -\theta_1\sin\left(\sqrt{-\Delta}(sx+vt)\right)\sqrt{-\Delta}-\theta_1\sqrt{-\Delta}\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}(sx+vt)\right) \\ +\theta_1\cos\left(\sqrt{-\Delta}(sx+vt)\right)+\sqrt{-\Delta}\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}} \times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{22}(x,t) = -\frac{\sigma v\begin{pmatrix} 4\theta_2\theta_0\sin\left(\sqrt{-\Delta}(sx+vt)\right)-\theta_1^2\sin\left(\sqrt{-\Delta}(sx+vt)\right) \\ -\theta_1\cos\left(\sqrt{-\Delta}(sx+vt)\right)\sqrt{-\Delta}-\theta_1\sqrt{-\Delta}\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}(sx+vt)\right) \\ +\theta_1\sin\left(\sqrt{-\Delta}(sx+vt)\right)+\sqrt{-\Delta}\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}} \times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{23}(x,t) = -\frac{\sigma v\begin{pmatrix} 8\theta_2\theta_0\sin\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right)\cos\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right)+2\theta_1\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right)^2 \\ -2\theta_1^2\sin\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right)\cos\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right)-\theta_1\sqrt{-\Delta}\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}2\theta_1\sin\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right)\cos\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right) \\ -2\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}}{4}(sx+vt)\right)^2+\sqrt{-\Delta}\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}} \times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right).$$

By putting constant values Eq. (4.2) along with Eq. (3.28)- Eq. (3.30) in the Eq. (4.1), and by using the wave transformation Eq. (2.2), we reach the solutions as follows:

$$\Phi_{24}(x,t) = \frac{\sigma v\theta_1\begin{pmatrix}-\zeta_0+\cosh(\theta_1(sx+vt)) \\ -\sinh(\theta_1(sx+vt))\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}\zeta_0+\cosh(\theta_1(sx+vt)) \\ -\sinh(\theta_1(sx+vt))\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}} \times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{25}(x,t) = -\frac{\sigma v\theta_1\begin{pmatrix}\cosh(\theta_1(sx+vt)) \\ -\sinh(\theta_1(sx+vt))-\zeta_0\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}\zeta_0+\cosh(\theta_1(sx+vt)) \\ -\sinh(\theta_1(sx+vt))\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}} \times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right),$$

$$\Phi_{26}(x,t) = -\frac{\sigma v\theta_1\begin{pmatrix}\cosh(\theta_1(sx+vt)) \\ +\sinh(\theta_1(sx+vt))-\zeta_0\end{pmatrix}}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)\begin{pmatrix}\zeta_0+\cosh(\theta_1(sx+vt)) \\ +\sinh(\theta_1(sx+vt))\end{pmatrix}\times\sqrt{-\frac{1}{\varepsilon}}} \times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right).$$

By putting constant values Eq. (4.2) along with Eq. (3.31) in the Eq. (4.1), and by using the wave transformation Eq. (2.2), we attain the solution as follows:

$$\Phi_{27}(x,t) = \frac{\sigma v(\theta_0\theta_1(sx+vt)+c_1\theta_1-2\theta_2)}{\varepsilon(4v^2\theta_0\theta_2-v^2\theta_1^2-2\beta^2)((sx+vt)\theta_0+c_1)\times\sqrt{-\frac{1}{\varepsilon}}} \times\exp\left(i\left(\left(\frac{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2-2\beta\sigma}{-4v^2\theta_0\theta_2+v^2\theta_1^2+2\beta^2}\right)x+\beta t+\gamma\right)\right).$$

5. Graphical Discussion

In this section, we discuss the graphical behavior of the solutions successfully obtained using the GREM method for the CCKS.

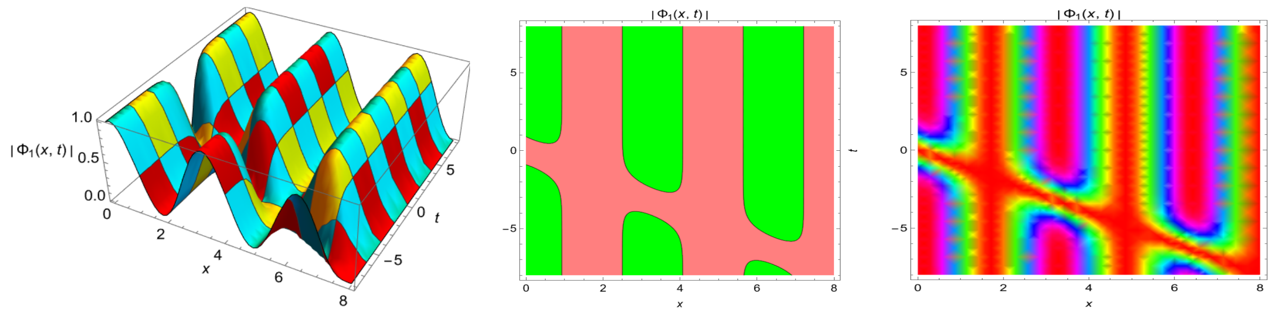

Figure 1. The 3d, contour, and density plots for the solution $|\Phi_1(x,t)|$.

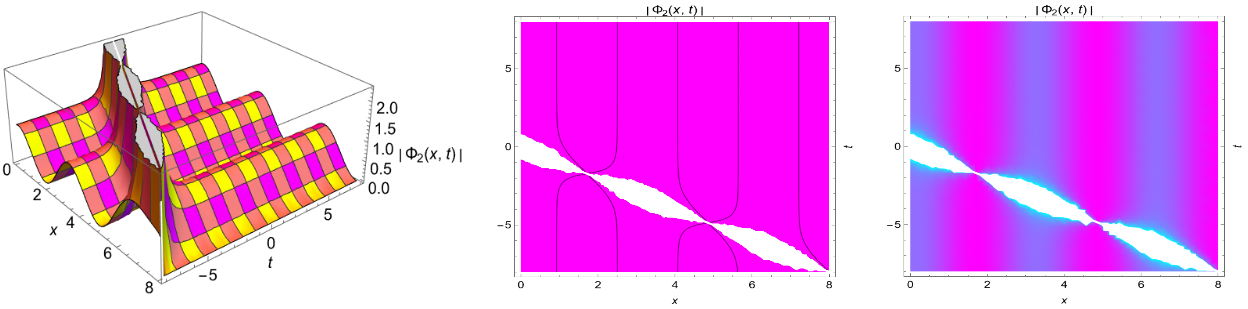

Figure 2. The 3d, contour, and density plots for the solution $|\Phi_2(x,t)|$.

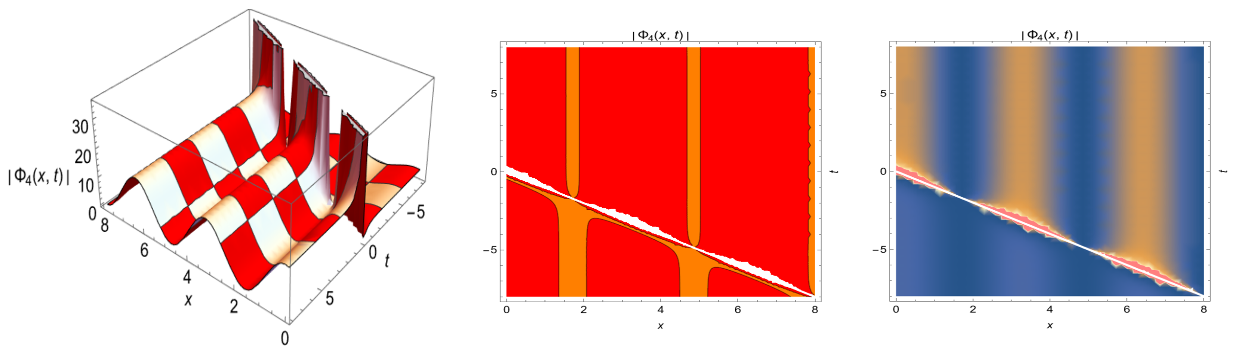

Figure 3. The 3d, contour, and density plots for the solution $|\Phi_4(x,t)|$.

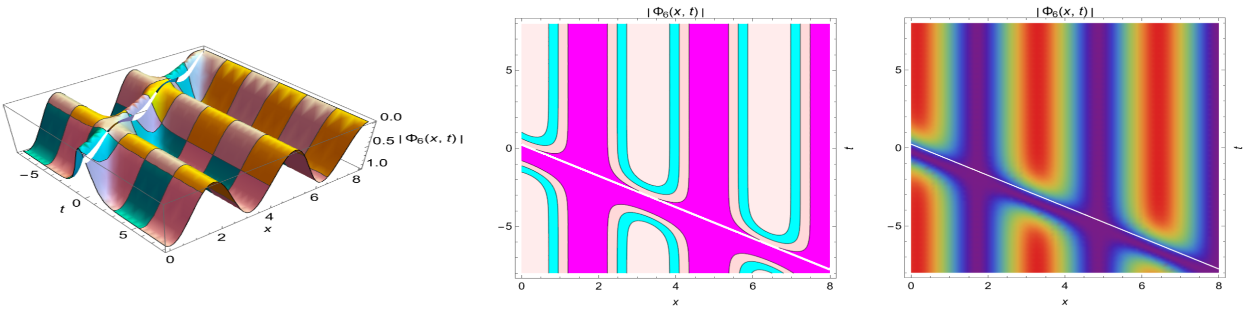

Figure 4. The 3d, contour, and density plots for the solution $|\Phi_6(x,t)|$.

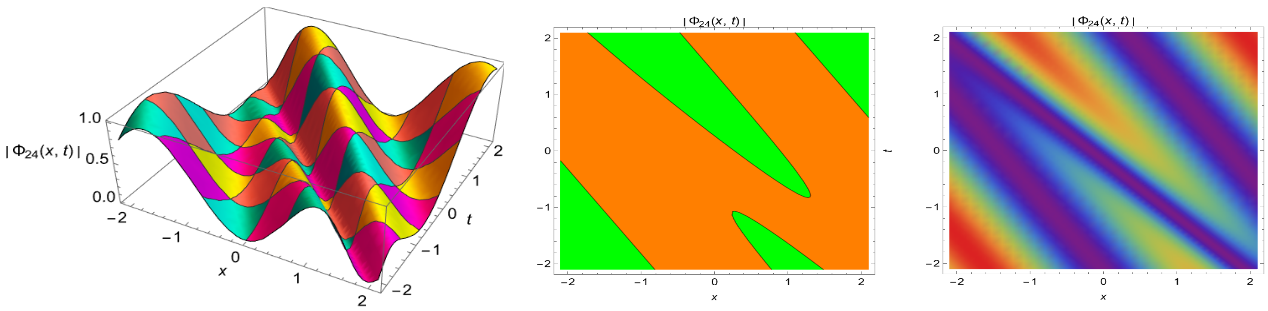

Figure 5. The 3d, contour, and density plots for the solution $|\Phi_{24}(x,t)|$.

Fig.1. The diagram of $|\Phi_1(x,t)|$ when $v=2$, $\beta=1$, $s=2$, $\sigma=3$, $\varepsilon=1$, $\theta_0=1$, $\theta_1=3$, $\theta_2=2$, $\gamma=3$, $\xi=sx+vt$.

Fig.2. The diagram of $|\Phi_2(x,t)|$ when $v=2$, $\beta=1$, $s=2$, $\sigma=3$, $\varepsilon=1$, $\theta_0=1$, $\theta_1=3$, $\theta_2=2$, $\gamma=3$, $\xi=sx+vt$.

Fig.3. The diagram of $|\Phi_4(x,t)|$ when $v=2$, $\beta=1$, $s=2$, $\sigma=3$, $\varepsilon=1$, $\theta_0=1$, $\theta_1=3$, $\theta_2=2$, $\gamma=3$, $\xi=sx+vt$.

Fig.4. The diagram of $|\Phi_6(x,t)|$ when $v=2$, $\beta=1$, $s=2$, $\sigma=3$, $\varepsilon=1$, $\theta_0=1$, $\theta_1=3$, $\theta_2=2$, $\gamma=3$, $P=2$, $Q=-1$, $\xi=sx+vt$.

Fig.5. The diagram of $|\Phi_{24}(x,t)|$ when $v=2$, $\beta=1$, $s=2$, $\sigma=3$, $\varepsilon=1$, $\theta_0=0$, $\theta_1=2$, $\theta_2=3$, $\gamma=3$, $\zeta_0=2$, $\xi=sx+vt$.

6. Conclusion

This examination focused on analyzing the CCKS, which is employed in diverse areas like ferromagnetic materials, nonlinear optics, and optical fibers. The CCKS is significant as it is a crucial model that enhances the understanding of complex physical and mathematical concepts, making it a valuable tool for scientific research and applications. The GREM approach was operated to scrutinize analytical solutions to the considered equation. It is a powerful analytical technique for solving various differential equations, particularly nonlinear ones. Following the application of the method, shock, complex solitary shock, shock singular, and periodic singular wave solutions were seen for both single and mixed wave solutions. The derivation also leads to reasonable solutions. 3D, contour and density graphs were plotted by choosing suitable parameter values via Mathematica to visualize the graphical representation of the acquired soliton solutions. These solutions we obtained using the GREM method can be extended to the analytical examination of various other types of NLPDEs in fields such as mathematical physics, plasma physics, applied sciences, nonlinear dynamics, and engineering. The accuracy of the results was verified by utilizing Maple to substitute the solutions back into the initial equation.

Article Information

Acknowledgements: The author would like to express his sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Supporting/Supporting organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical approval and participant consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

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