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A modelling of the natural logarithm and Mercator series as 5^{th}, 6^{th}, 7^{th} order Bézier curve in plane

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ABSTRACT

In this study first, natural logarithm function $f(x) = \ln x$ with base e has been examined as polynomial function of 5^{th}, 6^{th}, 7^{th} order Bézier curve. By modelling matrix representation of 5^{th}, 6^{th}, 7^{th} order Bézier curve we have found the control points in plane. Further, Mercator series for the curves $\ln(1+x)$ and $\ln(1-x)$ have been written too as the polynomial functions as 5^{th}, 6^{th}, 7^{th} order Bézier curve in plane based on the control points with matrix form in $\mathbf{E}^2$. Finally, the curve $\ln(1-x^2)$ has been expressed as 5^{th}, 6^{th}, 7^{th} order Bézier curve, examined the control points and given matrix forms.

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1. Introduction

Bézier curves are named after Pierre Bézier, a French engineer who used them in the 1960s for designing automobiles at Renault. They are widely used in computer graphics software like Adobe Illustrator, Photoshop, and in programming libraries such as SVG (Scalable Vector Graphics) and OpenGL for creating smooth curves in digital designs and animations. In 3D animation, Bézier curves are commonly used to define paths that objects follow through space. This is often used for creating smooth and natural-looking motion. Overall, Bézier curves are a powerful tool in 3D animation for defining both the overall path of movement and the interpolation between keyframes, helping animators create lifelike and fluid animations.

Bézier curves have been the focus of attention of many researchers due to their properties. Some of the publications that attracted our attention and examined while preparing our study are as follows: In [3], Marsh showed geometric applications for computer graphics and CAD, and also emphasized the importance of Bézier and B-spline curves in this regard. In [6], H. Hagen investigated Bézier curves with curvature and torsion continuity. In [5] and [7], Bézier curves and Bézier surfaces were examined by H. Zhang, F. Jieqing and S.Michael. In [4], G. Farin used Bézier curves for CAD and also studied the equivalence conditions of control points and their application to planar Bézier curves. In [8], [9], [10] and [12] it has been examined cubic Bézier curves, their involutes, Bertrand and Mannheim mate of a cubic Bézier curve by using matrix representation in E^3, respectively. In [11] and [13], it has been researched matrix representation of Bézier curves and 5^{th} order Bézier curve and their derivatives, respectively. In [14] and [15], it has been investigated Bézier curves and 5^{th} order Bézier Curve in three dimensional Euclidean space. In [16],[17],[18] and [19], approaches to various curves (circular helix, sine wave, cosine curve and exponential curves, respectively) with various order Bézier curves were examined.

In simplest form, a Bézier curve is defined by a set of control points. A linear Bézier curve, for example, is defined by two points, while a quadratic Bézier curve is defined by three points, and a cubic Bézier curve is defined by four points.

Generally, it can be defined n^{th} *order* Bézier curve by $n+1$ control points $P_0, P_1, \ldots, P_n$ with the parametrization

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} [P_i].$$

For more detail see in [4], [5], [6], [7]. As is well known, Taylor series $f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}$ of a function is an infinite sum of the functions derivatives at a single point a, also a Maclaurin series $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$ is a Taylor series where $a = 0$.

In this study we will focus on the natural logarithm with base e, for 5^{th}, 6^{th}, 7^{th} order Bézier curves. For more detail see [2], [8]. We need to write the coefficients matrix of any 5^{th} order Bézier curve. It is clear that the coefficients matrix on matrix representation is 5^{th}, 6^{th}, 7^{th} $order$ Bézier curves as in the following (see [11]):

The coefficients matrix of any 5^{th} $order$ Bézier curve is

$$[B^5] = \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The coefficients matrix of any 6^{th} $order$ Bézier curve is

$$[B^6] = \begin{bmatrix} 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ -6 & 30 & -60 & 60 & -30 & 6 & 0 \\ 15 & -60 & 90 & -60 & 15 & 0 & 0 \\ -20 & 60 & -60 & 20 & 0 & 0 & 0 \\ 15 & -30 & 15 & 0 & 0 & 0 & 0 \\ -6 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The coefficients matrix of any 7^{th} $order$ Bézier curve is

$$[B^7] = \begin{bmatrix} -1 & 7 & -21 & 35 & -35 & 21 & -7 & 1 \\ 7 & -42 & 105 & -140 & 105 & -42 & 7 & 0 \\ -21 & 105 & -210 & 210 & -105 & 21 & 0 & 0 \\ 35 & -140 & 210 & -140 & 35 & 0 & 0 & 0 \\ -35 & 105 & -105 & 35 & 0 & 0 & 0 & 0 \\ 21 & -42 & 21 & 0 & 0 & 0 & 0 & 0 \\ -7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. The curve ln$x$ as a 5^{th}, 6^{th}, 7^{th} order Bézier curve

The natural logarithm is used in many areas of mathematics, science, and engineering, particularly in calculus, probability theory, and the natural sciences. It has applications in areas such as exponential growth and decay, compound interest, and solving differential equations. The natural logarithm of a number is its logarithm to the base of the mathematical constant e, which is an irrational and transcendental number. We cannot find the Maclaurin series for $\ln(x)$. Hence the natural logarithm of x is generally written as $f(x) = \ln x$ with Taylor series for centered at $x = 1$ is

$$\ln x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}, \quad if\ 0 < x \le 2$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7} - \cdots$$

Proposition 1: The matrix representation of the curve $f(x) = \ln x$ as a 7^{th} $order$ Bézier curve has the control points P_0, P_1, P_2, P_3, P_4, P_5, P_6, and P_7

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & \frac{363}{140} \\ \frac{1}{7} & \frac{503}{140} \\ \frac{2}{7} & \frac{573}{140} \\ \frac{3}{7} & \frac{1859}{420} \\ \frac{4}{7} & \frac{491}{105} \\ \frac{5}{7} & \frac{512}{105} \\ \frac{6}{7} & \frac{353}{70} \\ 1 & \frac{363}{70} \end{bmatrix}.$$

Proof. The function $f(x) = \ln x$ has 7^{th} degree Taylor series expansion centered $x = 1$

$$\ln x = \frac{1}{7}x^7 - \frac{7}{6}x^6 + \frac{21}{5}x^5 - \frac{35}{4}x^4 + \frac{35}{3}x^3 - \frac{21}{2}x^2 + 7x - \frac{363}{140}$$

It can be written as in parametric form and a 7^{th} degree polynomial function

$$(t, \ln t) = (t, \frac{1}{7}t^7 - \frac{7}{6}t^6 + \frac{21}{5}t^5 - \frac{35}{4}t^4 + \frac{35}{3}t^3 - \frac{21}{2}t^2 + 7t - \frac{363}{140}).$$

It has been already known that the matrix representation of $\alpha(t) = (t, a_7t^7 + a_6t^6 + \ldots + a_1t + a_0)$ is the following matrix equation,

$$\begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{7} \\ 0 & -\frac{7}{6} \\ 0 & \frac{21}{5} \\ 0 & -\frac{35}{4} \\ 0 & \frac{35}{3} \\ 0 & -\frac{21}{2} \\ 1 & 7 \\ 0 & \frac{363}{140} \end{bmatrix} = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^7] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix}.$$

Proposition 2: The matrix representation of the curve $f(x) = \ln x$ as a 6^{th} $order$ Bézier curve has the control points that P_0, P_1, P_2, P_3, P_4, P_5, P_6 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{49}{20} \\ \frac{1}{6} & -\frac{29}{20} \\ \frac{1}{3} & -\frac{19}{20} \\ \frac{1}{2} & -\frac{37}{60} \\ \frac{2}{3} & -\frac{11}{30} \\ \frac{5}{6} & -\frac{1}{6} \\ 1 & 0 \end{bmatrix}.$$

Proof. The function $f(x) = \ln x$ has 6^{th} degree Taylor series expansion centered $x = 1$.

$$\ln x = -\frac{1}{6}x^6 + \frac{6}{5}x^5 - \frac{15}{4}x^4 + \frac{20}{3}x^3 - \frac{15}{2}x^2 + 6x - \frac{49}{20}, \quad if \quad 0 < x \le 2$$

it can be written as in parametric form and a 6^{th} degree polynomial function

$$\alpha(t) = (t, \ln t) = (t, -\frac{1}{6}t^6 + \frac{6}{5}t^5 - \frac{15}{4}t^4 + \frac{20}{3}t^3 - \frac{15}{2}t^2 + 6t - \frac{49}{20}).$$

It has been already known that the matrix representation of $\alpha(t) = (t, a_6t^6 + \ldots + a_0)$ is as in the following equation

$$\begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{6} \\ 0 & \frac{6}{5} \\ 0 & -\frac{15}{4} \\ 0 & \frac{20}{3} \\ 0 & -\frac{15}{2} \\ 1 & 6 \\ 0 & -\frac{49}{20} \end{bmatrix} = \begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^6] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}.$$

Proposition 3: The matrix representation of the curve $f(x) = \ln x$ as a 5^{th} $order$ Bézier curve has the control points P_0, P_1, P_2, P_3, P_4, and P_5, where P_0, P_1, P_2, P_3, P_4, and P_5 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{49}{20} \\ \frac{1}{5} & -\frac{5}{4} \\ \frac{2}{5} & -\frac{4}{5} \\ \frac{3}{5} & -\frac{13}{30} \\ \frac{4}{5} & -\frac{7}{30} \\ 1 & \frac{1}{6} \end{bmatrix}.$$

Proof. The function $f(x) = \ln(x)$ has 5^{th} degree Taylor series expansion is

$$\ln x = \frac{6}{5}x^5 - \frac{15}{4}x^4 + \frac{20}{3}x^3 - \frac{15}{2}x^2 + 6x - \frac{49}{20}$$

it can be written as in parametric form and a 5^{th} degree polynomial function

$$(t, \ln t) = (t, \frac{6}{5}t^5 - \frac{15}{4}t^4 + \frac{20}{3}t^3 - \frac{15}{2}t^2 + 6t - \frac{49}{20})$$

Hence we get the following equation

$$\begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{6}{5} \\ 0 & -\frac{15}{4} \\ 0 & \frac{20}{3} \\ 0 & -\frac{15}{2} \\ 1 & 6 \\ 0 & -\frac{49}{20} \end{bmatrix} = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^5] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

3. Mercator Series as a 5^{th}, 6^{th}, 7^{th} order Bézier curve

The series known in mathematics as the Mercator series or Newton-Mercator series are actually the Taylor series for the natural logarithms with of $(1 + x)$ and $(1 - x)$ that are generally written as

$$\ln(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} \ldots, \quad if \; -1 < x \le 1$$

$$\ln(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} \ldots, \quad if \; -1 < x \le 1$$

Although the serias were first discovered by Johannes Hudde and Isaac Newton, they were independently published by Nicholas Mercator in his 1668 treatise named Logarithmotechnia [1].

3.1. The curve $ln(1+x)$ as a $5^{th}, 6^{th}, 7^{th}$ order Bézier curve

Proposition 4: The matrix representation of the curve $f(x) = \ln(1+x)$ as a 7^{th} *order* Bézier curve has the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7$ as follows

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{11}{42} \\ \frac{3}{7} & \frac{11}{30} \\ \frac{4}{7} & \frac{193}{420} \\ \frac{5}{7} & \frac{229}{420} \\ \frac{6}{7} & \frac{37}{60} \\ 1 & \frac{319}{420} \end{bmatrix}.$$

Proof. Function $f(x) = \ln(1+x)$ has 7^{th} degree Maclaurin series expansion is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7}, \quad (-1 < x \le 1)$$

it can be written as in parametric form and a 7^{th} degree polynomial function

$$\left(t, \ln(1+t)\right) = (t, t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \frac{t^6}{6} + \frac{t^7}{7}).$$

It has been already known that the matrix representation of $\alpha(t) = (t, a_7 t^7 + a_6 t^6 + \ldots + a_0)$ is as in the following equation

$$\begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{7} \\ 0 & -\frac{1}{6} \\ 0 & \frac{1}{5} \\ 0 & -\frac{1}{4} \\ 0 & \frac{1}{3} \\ 0 & -\frac{1}{2} \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^7] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix}.$$

Proposition 5: The matrix representation of the curve $f(x) = \ln(1+x)$ as a 6^{th} *order* Bézier curve has the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6$ as follows

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{3}{10} \\ \frac{1}{2} & \frac{5}{12} \\ \frac{2}{3} & \frac{31}{60} \\ \frac{5}{6} & \frac{37}{60} \\ 1 & \frac{37}{60} \end{bmatrix}.$$

Proof. Function $f(x) = \ln(1+x)$ has 6^{th} degree Maclaurin series expansion is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}, \quad (-1 < x \le 1)$$

it can be written as in parametric form and a 6^{th} degree polynomial function

$$\left(t, \ln(1+t)\right) = (t, t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \frac{t^6}{6}).$$

It has been already known that the matrix representation of $\alpha(t) = (t, a_6 t^6 + \ldots + a_0)$ is as in the following equation

$$\begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{6} \\ 0 & \frac{1}{5} \\ 0 & -\frac{1}{4} \\ 0 & \frac{1}{3} \\ 0 & -\frac{1}{2} \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^6] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}.$$

Proposition 6: The matrix representation of the curve $f(x) = \ln(1+x)$ as a 5^{th} *order* Bézier curve has the control points $P_0, P_1, P_2, P_3, P_4, P_5$ as follows

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{7}{20} \\ \frac{3}{5} & \frac{29}{60} \\ \frac{4}{5} & \frac{7}{12} \\ 1 & \frac{47}{60} \end{bmatrix}.$$

Proof. Function $f(x) = \ln(1+x)$ has 5^{th} degree Maclaurin series expansion is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}, \quad (-1 < x \le 1)$$

it can be written as in parametric form and a 5^{th} degree polynomial function

$$\left(t, \ln(1+t)\right) = (t, t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5}).$$

It has been already known that the matrix representation of $\alpha(t) = (t, a_5 t^5 + \ldots + a_0)$ is as in the following equation

$$\begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{5} \\ 0 & -\frac{1}{4} \\ 0 & \frac{1}{3} \\ 0 & -\frac{1}{2} \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^5] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

3.2. The curve $\ln(1-x)$ as a 5^{th}, 6^{th}, 7^{th} order Bézier curve

Proposition 7: The matrix representation of the curve $f(x) = \ln(1-x)$ as a 7^{th} $order$ Bézier curve has the control points P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7 as follows

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{7} & -\frac{1}{7} \\ \frac{2}{7} & -\frac{13}{42} \\ \frac{3}{7} & -\frac{107}{210} \\ \frac{4}{7} & -\frac{319}{420} \\ \frac{5}{7} & -\frac{153}{140} \\ \frac{6}{7} & -\frac{223}{140} \\ 1 & -\frac{363}{140} \end{bmatrix}.$$

Proof. Function $f(x) = \ln(1-x)$ has 7^{th} degree Maclaurin series expansion is

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7}, \quad (-1 < x \leq 1)$$

it can be written as in parametric form and a 7^{th} degree polynomial function

$$\left(t, \ln(1-t)\right) = (t, -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{6} - \frac{t^7}{7}).$$

Hence we get the following matrix equation

$$\begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{7} \\ 0 & -\frac{1}{6} \\ 0 & -\frac{1}{5} \\ 0 & -\frac{1}{4} \\ 0 & -\frac{1}{3} \\ 0 & -\frac{1}{2} \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^7] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix}.$$

Proposition 8: The matrix representation of the curve $f(x) = \ln(1-x)$ as a 6^{th} $order$ Bézier curve has the control points P_0, P_1, P_2, P_3, P_4, P_5, P_6 as follows

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{11}{30} \\ \frac{1}{2} & -\frac{37}{60} \\ \frac{2}{3} & -\frac{19}{20} \\ \frac{5}{6} & -\frac{29}{20} \\ 1 & -\frac{49}{20} \end{bmatrix}.$$

Proof. Function $f(x) = \ln(1-x)$ has 6^{th} degree Maclaurin series expansion is

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6}, \quad (-1 < x \leq 1)$$

it can be written as in parametric form and a 6^{th} degree polynomial function

$$\left(t, \ln(1-t)\right) = (t, -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{6}).$$

Hence we get the following matrix equation

$$\begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{6} \\ 0 & -\frac{1}{5} \\ 0 & -\frac{1}{4} \\ 0 & -\frac{1}{3} \\ 0 & -\frac{1}{2} \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^6] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}.$$

Proposition 9: The matrix representation of the curve $f(x)=\ln(1-x)$ as a 5^{th} $order$ Bézier curve has the control points $P_0, P_1, P_2, P_3, P_4, P_5$ as follows

$$\begin{bmatrix}P_0\\P_1\\P_2\\P_3\\P_4\\P_5\end{bmatrix}=\begin{bmatrix}0&0\\\frac{1}{5}&-\frac{1}{5}\\\frac{2}{5}&-\frac{9}{20}\\\frac{3}{5}&-\frac{47}{60}\\\frac{4}{5}&-\frac{77}{60}\\1&-\frac{137}{60}\end{bmatrix}.$$

Proof. Function $f(x)=\ln(1-x)$ has 5^{th} degree Maclaurin series expansion is

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5},\quad(-1<x\le 1)$$

it can be written as in parametric form and a 5^{th} degree polynomial function

$$\big(t,\ln(1-t)\big)=(t,-t-\frac{t^2}{2}-\frac{t^3}{3}-\frac{t^4}{4}-\frac{t^5}{5}).$$

Hence we get the following matrix equation

$$\begin{bmatrix}t^5\\t^4\\t^3\\t^2\\t\\1\end{bmatrix}^T\begin{bmatrix}0&-\frac{1}{5}\\0&-\frac{1}{4}\\0&-\frac{1}{3}\\0&-\frac{1}{2}\\1&-1\\0&0\end{bmatrix}=\begin{bmatrix}t^5\\t^4\\t^3\\t^2\\t\\1\end{bmatrix}^T[B^5]\begin{bmatrix}P_0\\P_1\\P_2\\P_3\\P_4\\P_5\end{bmatrix}.$$

3.3. The curve ln$(1-x^2)$ as a 4^{th} and 6^{th} order Bézier curve

Proposition 10: The matrix representation of the curve $f(x)=\ln(1-x^2)$ as a 6^{th} $order$ Bézier curve has the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6$ as follows

$$\begin{bmatrix}P_0\\P_1\\P_2\\P_3\\P_4\\P_5\\P_6\end{bmatrix}=\begin{bmatrix}0&0\\\frac{1}{6}&0\\\frac{1}{3}&-\frac{1}{15}\\\frac{1}{2}&-\frac{1}{5}\\\frac{2}{3}&-\frac{13}{30}\\\frac{5}{6}&-\frac{5}{6}\\1&-\frac{11}{6}\end{bmatrix}.$$

Proof. We have already known that $f(x)=\ln(1-x^2)=\ln[(1-x)(1+x)]$. Hence we get

$$\begin{aligned}\ln(1-x^2)&=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^7}{7}\right)\\&\quad+\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\frac{x^6}{6}-\frac{x^7}{7}\right)\\&=-\frac{1}{3}x^6-\frac{1}{2}x^4-x^2\end{aligned}$$

Also it can be written as in parametric form and a 6^{th} degree polynomial function $\big(t,\ln(1-t^2)\big)=\left(t,-\frac{1}{3}t^6-\frac{1}{2}t^4-t^2\right)$. Hence we get the following matrix equation

$$\begin{bmatrix}t^6\\t^5\\t^4\\t^3\\t^2\\t\\1\end{bmatrix}^T\begin{bmatrix}0&-\frac{1}{3}\\0&0\\0&-\frac{1}{2}\\0&0\\0&-1\\1&0\\0&0\end{bmatrix}=\begin{bmatrix}t^6\\t^5\\t^4\\t^3\\t^2\\t\\1\end{bmatrix}^T[B^6]\begin{bmatrix}P_0\\P_1\\P_2\\P_3\\P_4\\P_5\\P_6\end{bmatrix}.$$

Proposition 11: The matrix representation of the curve $f(x)=\ln(1-x^2)$ as a 4^{th} $order$ Bézier curve has control points P_0, P_1, P_2, P_3, P_4 as follows

$$\begin{bmatrix}P_0\\P_1\\P_2\\P_3\\P_4\end{bmatrix}=\begin{bmatrix}0&0\\\frac{1}{4}&0\\\frac{1}{2}&-\frac{1}{6}\\\frac{3}{4}&-\frac{1}{2}\\1&-\frac{3}{2}\end{bmatrix}$$

Proof. Since $f(x)=\ln(1-x^2)=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\right)+\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}\right)=-\frac{1}{2}x^4-x^2$,
it can be written as in parametric form and a 4^{th} degree polynomial function $\big(t,\ln(1-t^2)\big)=\left(t,-\frac{1}{2}t^4-t^2\right)$

$$\begin{bmatrix}t^4\\t^3\\t^2\\t\\1\end{bmatrix}^T\begin{bmatrix}0&-\frac{1}{2}\\0&0\\0&-1\\1&0\\0&0\end{bmatrix}=\begin{bmatrix}t^4\\t^3\\t^2\\t\\1\end{bmatrix}^T[B^4]\begin{bmatrix}P_0\\P_1\\P_2\\P_3\\P_4\end{bmatrix}$$

Solving the above equation give us the control points, where $[B^4]$ is the 4^{th} order Bézier curves matrix in $\mathbf{E}^2$.

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