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NEW APPLICATIONS IN THIRD-ORDER STRONG DIFFERENTIAL SUBORDINATION THEORY

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Abstract. The research conducted in this investigation focuses on extending known results from the second-order differential subordination theory for the special case of third-order strong differential subordination. This paper intends to facilitate the development of new results in this theory by showing how specific lemmas used as tools in classical second-order differential subordination theory are adapted for the context of third-order strong differential subordination. Two theorems proved in this study extend two familiar lemmas due to D.J. Hallenbeck and S. Ruscheweyh, and G.M. Goluzin, respectively. A numerical example illustrates applications of the new results but the theorems are hoped to become helpful tools in generating new outcome for this very recently initiated line of research concerning third-order strong differential subordination.

1. Introduction and Preliminaries

For the special case of third-order differential subordinations, J.A. Antonino and S.S. Miller [1] extended differential subordination theory first proposed by S.S. Miller and P.T. Mocanu [2, 3], setting a new direction for further research into this topic. Applications of the outcomes discussed in [1] rapidly followed, and this topic of research is currently progressing successfully. By applying fundamental results regarding the third-order differential subordination, a direction of study deals with defining appropriate classes of admissible functions. Specific devellopments of third-order differential subordination continue to be obtained nowadays in view of this approach. For example, p-valent functions connected to a generalized fractional

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differintegral operator are analyzed in [4]. The same approach delivers interesting conclusions for special functions in [5] and [6] as well as for a generalized operator in [7].

Recent studies have started the development of an alternative approach in third-order differential subordination theory concerning another essential concept, that of the the best dominant. New ways of identifying the best dominant of a third-order differential subordination are provided in [8, 9], along with techniques for finding the dominants for any third-order differential subordination.

The study presented in this paper intends to show how the classical results concerning third-order differential subordination are extended for the particular context of strong differential subordination theory in general and for the third-order strong differential subordination in particular. The first results in this directions are proposed in the very recent paper [10]. In their work, the authors extend the definitions specific to second-order strong differential subordination adapting them for the third-order strong differential subordination and develop some new results using the approach consisting in choosing appropriate classes of admissible functions. In this research, we propose other extensions form the classical theory of differential subordination to strong differential subordination and we obtain particular third-order strong differential subordination results.

Certain basic aspects concerning strong differential subordination theory were first presented in a published study from 2009 [11], following certain ideas set by J.A. Antonino and S. Romaguera through their work from 1994, [12], where the notion of strong differential subordination was first mentioned in the context of the special case of Briot-Bouquet differential subordination. The paper [11] defined the fundamental concepts of dominant of the solutions of the strong differential subordination and of solution of a strong differential subordination, as well as the three problems that form the basis of the theory and the fundamental tool in the analysis of strong differential subordination that is the class of admissible functions. The theory was further improved by the introduction of certain classes of analytic functions particularly applied in strong differential subordination studies in 2012 [13]. Latest results applying the results presented in [13] include strong differential results involving different operators [14, 15], multiplier transformation and Ruscheweyh derivative applications in strong differential subordination theory [16], first order strong differential subordinations [17], and q-calculus aspects included in strong differential subordination studies alongside particular operators [18].

Those classes [13], used also in the present investigation, are:

Analytic functions in $U \times \overline{U}$ represented by $H(U \times \overline{U})$;

$$H\zeta[a,n] = \left\{ f \in H(U \times \overline{U}) : f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots \right\},$$

considering $a_k(\zeta)$ holomorphic in $\overline{U}$, $k \geq n$, $a \in \mathbb{C}$, $n \in \mathbb{N}$, the class derived from the classical:

$$H[a,n] = \{ f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,\ z \in U \};$$

$$H\zeta_U(U) = \{f \in H_\zeta[a,n] : f(\cdot,\zeta) \text{ univalent in } U \text{ for all } \zeta \in \overline{U}\};$$

$$A\zeta_n = \{f \in H(U \times \overline{U}) : f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \cdots, \quad z \in U, \ \zeta \in \overline{U}\},$$

with $A\zeta_1 = A\zeta$ and $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n+1$, $n \in \mathbb{N}$, the class derived from the classical:

$$A_n = \{f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + \cdots, \quad z \in U\}, \quad \text{with } A_1 = A;$$

$$S^*\zeta = \{f \in A\zeta : \mathrm{Re}\frac{zf'_z(z,\zeta)}{f(z,\zeta)} > 0, \quad z \in U, \ \zeta \in \overline{U}\},$$

the class of starlike functions in $U \times \overline{U}$ derived from the classical class of starlike functions:

$$S^* = \{f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0\};$$

$$K\zeta = \{f \in A\zeta : \mathrm{Re}\left(\frac{zf''_{z^2}(z,\zeta)}{f'_z(z,\zeta)} + 1\right) > 0, \quad z \in U, \ \zeta \in \overline{U}\},$$

the class of convex functions in $U \times \overline{U}$, derived from the classical class of convex functions:

$$K = \{f \in A : \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, \ f(0) = 0, \ f'(0) \neq 0, \quad z \in U\}.$$

The notions of strong differential subordination necessary for this research are listed as follows.

Definition 1. *[13] Let $h(z,\zeta)$ and $f(z,\zeta)$ be analytic functions in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $h(z,\zeta)$, or $h(z,\zeta)$ is said to be strongly superordinate to $f(z,\zeta)$ if there exists a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$ such that $f(z,\zeta) = h(w(z),\zeta)$, for all $\zeta \in \overline{U}$, $z \in U$. In such a case, we write*

$$f(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

Remark 1. *[13] a) If $f(z,\zeta)$ is analytic in $U \times \overline{U}$ and univalent in U for $\zeta \in \overline{U}$, then Definition 1 is equivalent to:*

$$f(0,\zeta) = h(0,\zeta), \ \text{for all } \zeta \in \overline{U} \text{ and } f(U \times \overline{U}) \subset h(U \times \overline{U}).$$

b) If $f(z,\zeta) = f(z)$, $h(z,\zeta) = h(z)$, then the strong superordination becomes the usual superordination.

Definition 2. *[13] We denote by Q_ζ the set of functions $q(\cdot,\zeta)$ that are analytic and injective, as function of z, on $\overline{U} \backslash E(q(z,\zeta))$ where*

$$E(q(z,\zeta)) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z,\zeta = \infty\}$$

and are such that $q'_z(z,\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q(z,\zeta))$, $\zeta \in \overline{U}$.

The subclass of Q_ζ for which $q(0,\zeta) = a$ is denoted by $Q_\zeta(a)$.

Definition 3. *[13] Let Ω_ζ be a set in $\mathbb{C}$, $q(\cdot,\zeta) \in \Omega_\zeta$ and n a positive integer. The class of admissible functions $\phi_n[\Omega_\zeta, q(\cdot,\zeta)]$ consists of those functions $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(r,s,t;\xi,\zeta) \notin \Omega_\zeta$$

whenever

$$r = q(z,\zeta),\ \ s = nq'_z(z,\zeta),\ \ Re\left(\frac{t}{s}+1\right) \geq nRe\left[\frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}+1\right],$$

$z \in U$, $\zeta \in \partial U \backslash E(q(\cdot,\zeta))$ and $n \geq 1$. When $n = 1$, we write $\phi_1[\Omega_\zeta, q(\cdot,\zeta)]$ as $\phi[\Omega_\zeta, q(\cdot,\zeta)]$.

In the special case when $h(\cdot,\zeta)$ is an analytic mapping of $U \times \overline{U}$ onto $\Omega_\zeta \neq \mathbb{C}$ we denote the class $\phi_n[h(U \times \overline{U}), q(z,\zeta)]$ by $\phi_n[h(z,\zeta), q(z,\zeta)]$.

The class of admissible functions has been extended in [10] for the case of third-order strong differential subordination as it shows the next definition and will be used as such in the present investigation.

Definition 4. *[10] Let Ω_ζ be a set in $\mathbb{C}$, $q(\cdot,\zeta) \in \Omega_\zeta$ and $n \geq 2$. The class of admissible functions $\phi_n[\Omega_\zeta, q(\cdot,\zeta)]$ consists of those functions $\psi : \mathbb{C}^4 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(r,s,t,u;\xi,\zeta) \notin \Omega_\zeta \tag{1}$$

whenever

$$r = q(z,\zeta),\ \ s = nq'_z(z,\zeta),\ \ Re\left(\frac{t}{s}+1\right) \geq nRe\left[\frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}+1\right],$$

$$Re\frac{u}{s} \geq n^2 Re\frac{z^2 q'''_{z^3}(z,\zeta)}{q'_z(z,\zeta)},$$

$z \in U$, $\zeta \in \partial U \backslash E(q(\cdot,\zeta))$ and $n \geq 2$.

An important known result that will be applied for the proofs of the new results is the following lemma used in third-order differential subordination theory and given here having a particular form required by the theory of strong differential subordination:

Lemma 1. *([1], [19]) Let $q(z,\zeta) \in Q_\zeta(a)$ and let $p(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + ... \in H\left(U \times \overline{U}\right)$, with $p(z,\zeta) \neq \zeta$, and $n \geq 2$. If $p(\cdot,\zeta)$ is not subordinate to $q(\cdot,\zeta)$, then there exist points $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$ and $\xi_0 \in \partial U \backslash E(q(\cdot,\zeta))$ for which $p\left(U \times \overline{U_{r_0}}\right) \subset q\left(U \times \overline{U}\right)$ and $p(z_0,\zeta) = q(\xi_0,\zeta)$, and an $m \geq n$, such that the following conditions are satisfied:*

(i) $z_0 p'_z(z,\zeta) = q(\xi_0,\zeta)$;

(ii) $Re\frac{\xi_0 q''_{z^2}(\xi_0,\zeta)}{q'_z(\xi_0,\zeta)} \geq 0$ *and* $\left|\frac{z_0 p'_z(z_0,\zeta)}{q'_z(\xi_0,\zeta)}\right| \leq m$;

(iii) $z_0 p'_z(z_0,\zeta) = m\xi_0 q'_z(\xi_0,\zeta)$;

(iv) $Re\left(\frac{z_0 p''_{z^2}(z_0,\zeta)}{p'_z(z_0,\zeta)}+1\right) \geq mRe\left(\frac{\xi_0 q''_{z^2}(\xi_0,\zeta)}{q'_z(\xi_0,\zeta)}+1\right)$;

(v) $Re\frac{z_0^2 p'''_{z^3}(z_0,\zeta)}{p'_z(z_0,\zeta)} \geq m^2 Re\frac{\xi_0^2 q'''_{z^3}(\xi_0,\zeta)}{q'_z(\xi_0,\zeta)}$.

The first part of this lemma was used in [10] for developing a theorem. In the next section, the form of this lemma is adapted for strong differential subordination theory and will be applied for proving the original results contained in the Main results section of this paper.

The main concern of the present investigation is to present applications in third-order strong differential subordination studies of the known results due to D.J. Hallenbeck and S. Ruscheweyh [20] and G.M. Goluzin [21], respectively. The following two lemmas are used in the next section for developing two new theorems.

Lemma 2. *([20]) Let $h \in K$, with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$, $Re\gamma \geq 0$. If $p \in H[a,n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)\, t^{\frac{\gamma}{n}-1} dt.$$

Function $q \in K$ and is the best (a,n)-dominant.

Lemma 3. *([21]) Let $h \in K$. If the following differential subordination is satisfied:*

$$zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z) = \int_0^z \frac{h(t)}{t} dt,$$

and q is the best dominant.

Remark 2. *In 1970, T.J. Suffridge [22] proved that Goluzin's result remains true even if $h \in S^*$.*

Lemma 2 and Lemma 3 have facilitated major developments in second-order differential subordination theory, hence, the new theorems presented in the next section based on those popular results should help for the development of the newly initiated line of research concerning third-order strong differential subordinations.

2. Main Results

The first original outcome of the study extends the results obtained by Hallenbeck and Ruscheweyh [20] shown in Lemma 2. The theorem proved here also provides techniques of finding the best dominant of a third-order strong differential subordination.

Theorem 1. *Take $h(z,\zeta)\in K\zeta$, satisfying $h(0,\zeta)=a\in\mathbb{C}$ for all $\zeta\in\overline{U}$. Consider the functions $p(z,\zeta)\in H[a,n]$, $n\geq 2$, $p(z,\zeta)\neq a$ and $q(z,\zeta)\in H[a,n]$, $q(z,\zeta)\in Q_\zeta(a)$ satisfying:*

(i) $Re\frac{\zeta q''_{z^2}(\xi,\zeta)}{q'_z(\xi,\zeta)}\geq 0$ and $\left|\frac{zp'_z(z,\zeta)}{q'_z(\xi,\zeta)}\right|\leq n$, where $z\in U$, $\xi\in\partial U\backslash E(q(z,\zeta))$, $n\geq 2$;

(ii) $q(z,\zeta)+zq'_z(z,\zeta)+z^2q''_{z^2}(z,\zeta)=\frac{\gamma}{z^\gamma}\int_0^z h(t,\zeta)t^{\gamma-1}dt$, $\gamma\in\mathbb{C}$, $Re\gamma>0$, $z\in U$, $\zeta\in\overline{U}$.

If $p(z,\zeta)\in Q_\zeta(a)$ and

$$p(z,\zeta)+zp'_z(z,\zeta)+z^2p''_{z^2}(z,\zeta)+\frac{2zp'_z(z,\zeta)+3z^2p''_{z^2}(z,\zeta)+z^3p'''_{z^3}(z,\zeta)}{\gamma}\in H\left(U\times\overline{U}\right),$$

then

$$p(z,\zeta)+zp'_z(z,\zeta)+z^2p''_{z^2}(z,\zeta)+\frac{2zp'_z(z,\zeta)+3z^2p''_{z^2}(z,\zeta)+z^3p'''_{z^3}(z,\zeta)}{\gamma}\prec\prec$$

$$q(z,\zeta)+zq'_z(z,\zeta)+z^2q''_{z^2}(z,\zeta)+\frac{2zq'_z(z,\zeta)+3z^2q''_{z^2}(z,\zeta)+z^3q'''_{z^3}(z,\zeta)}{\gamma} \quad (2)$$

implies

$$p(z,\zeta)\prec\prec q(z,\zeta),$$

where $z\in U$, $\zeta\in\overline{U}$ and $q(z,\zeta)$ is said to be the best dominant.

Proof. The functions $p(z,\zeta)$, $q(z,\zeta)$ and $h(z,\zeta)$ may be assumed to be satifying the conditions of Lemma 1 and the condition $q'_z(z,\zeta)\neq 0$ for $\xi\in\partial U\backslash E(q(z,\zeta))$. Otherwise, the functions can be replaced by $p_\rho(z,\zeta)=p(\rho z,\zeta)$, $q_\rho(z,\zeta)=q(\rho z,\zeta)$ and $h_\rho(z,\zeta)=h(\rho z,\zeta)$, respectively, with $0<\rho<1$ and those functions have the necessary properties on $U\times\overline{U}$.

Hence, Lemma 1 will be applied for the proof of this result, also considering the definition given for the class of admissible functions.

Define now the function $\psi:\mathbb{C}^4\times U\times\overline{U}\to\mathbb{C}$ as

$$\psi(r,s,t,u,z,\zeta)=r+s+t+\frac{2s+3t+u}{\gamma},\ \ r,s,t,u\in\mathbb{C},\ \mathrm{Re}\gamma>0. \quad (3)$$

Taking $r=p(z,\zeta)$, $s=zp'_z(z,\zeta)$, $t=z^2p''_{z^2}(z,\zeta)$, $u=z^3p'''_{z^3}(z,\zeta)$, the function in (3) becomes:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right) = \tag{4}$$
$$p(z,\zeta) + zp'_z(z,\zeta) + z^2p''_{z^2}(z,\zeta) + \frac{2zp'_z(z,\zeta) + 3z^2p''_{z^2}(z,\zeta) + z^3p'''_{z^3}(z,\zeta)}{\gamma}.$$

Using (4), strong differential subordination (2) becomes:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right) \prec\prec \tag{5}$$
$$q(z,\zeta) + zq'_z(z,\zeta) + z^2q''_{z^2}(z,\zeta) + \frac{2zq'_z(z,\zeta) + 3z^2q''_{z^2}(z,\zeta) + z^3q'''_{z^3}(z,\zeta)}{\gamma},$$

$\mathrm{Re}\gamma > 0$.

Using relation (ii), we can write:

$$z^\gamma\left[q(z,\zeta) + zq'_z(z,\zeta) + z^2q''_{z^2}(z,\zeta)\right] = \gamma\int_0^1 h(t,\zeta)\cdot t^{\gamma-1}dt. \tag{6}$$

By differentiating (6) with respect to z, making simple calculations yield:

$$q(z,\zeta) + zq'_z(z,\zeta) + z^2q''_{z^2}(z,\zeta) + \frac{2zq'_z(z,\zeta) + 3z^2q''_{z^2}(z,\zeta) + z^3q'''_{z^3}(z,\zeta)}{\gamma} = h(z,\zeta). \tag{7}$$

By applying (7), the strong differential subordination (5) can be written as:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right) \prec\prec h(z,\zeta),$$

which can be interpreted in view of Remark 1, part a), as:

$$\left\{\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right)\right\} \subset h\left(U\times\overline{U}\right).$$

Considering $z = z_0 \in U$, we write:

$$\left\{\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right)\right\} \subset h\left(U\times\overline{U}\right).$$

Assume now that $p(z,\zeta) \not\prec\prec q(z,\zeta)$. In this situation, Lemma 1 shows that there exist $z_0 = r_0e^{i\theta_0} \in U$ and $\xi_0 \in \partial U\backslash E(q(z,\zeta))$ such that

$$p(z_0,\zeta) = q(\xi_0,\zeta),\ z_0p'_z(z_0,\zeta) = m\xi_0q'_z(\xi_0,\zeta),$$
$$t = z_0^2p''_{z^2}(z_0,\zeta),\ \ u = z_0^3p'''_{z^3}(z_0,\zeta), \tag{8}$$

satisfy the conditions of Lemma 1.

By replacing $r = q(\xi_0,\zeta)$, $s = m\xi_0q'_z(\xi_0,\zeta)$, t and u in the admissibility condition (1), we obtain:

$$\psi\left(q(\xi_0,\zeta), m\xi_0q'_z(\xi_0,\zeta), t, u\right) \notin h\left(U\times\overline{U}\right).$$

Using the equalities given by (8), we have:

$$\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right) \notin h\left(U\times\overline{U}\right),$$

but this contradicts (2). Hence, we must have that

$$p(z,\zeta) \prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Since $q(z,\zeta) \in H\zeta_U(U)$ and is a solution for the equation (7), it follows that $q(z,\zeta)$ is the best dominant for the strong differential subordination (2). □

Remark 3. *This theorem shows that finding the best dominant for a third-order strong differential subordination requires only the existence of a univalent solution for the differential equation associated with the strong differential subordination.*

The next theorem extends the result proved by G.M. Goluzin in 1935 [21] for second-order differential subordinations to fit the theory of third-order strong differential subordination.

Theorem 2. *Let $h(z,\zeta) \in K\zeta$, with $h(0,\zeta) = a \in \mathbb{C}$ for all $\zeta \in \overline{U}$. Consider the functions $p(z,\zeta) \in H[a,n]$, $n \geq 2$, $p(z,\zeta) \neq a$ and $q(z,\zeta) \in Q_\zeta(a)$, $q(z,\zeta) \in H\zeta_U(U)$ satisfying:*

(i) $Re\frac{\xi q''_{z^2}(\xi,\zeta)}{q'_z(\xi,\zeta)} \geq 0$ and $\left|\frac{zp'_z(z,\zeta)}{q'_z(z,\zeta)}\right| \leq n$, where $z \in U$, $\xi \in \partial U \backslash E(q(z,\zeta))$, $n \geq 2$;

(ii) $zq(z,\zeta) \cdot q'_z(z,\zeta) + z^2 q''_{z^2}(z,\zeta) = \int_0^z \frac{h(t,\zeta)}{t}dt$, $z \in U$, $\zeta \in \overline{U}$.

If

$$zp(z,\zeta) \cdot p'_z(z,\zeta) + (zp'_z(z,\zeta))^2 + z^2 p''_{z^2}(z,\zeta)[p(z,\zeta)+2] + z^3 p'''_{z^3}(z,\zeta) \prec\prec$$
$$zq(z,\zeta) \cdot q'_z(z,\zeta) + (zq'_z(z,\zeta))^2 + z^2 q''_{z^2}(z,\zeta)[q(z,\zeta)+2] + z^3 q'''_{z^3}(z,\zeta),$$

implies

$$p(z,\zeta) \prec\prec q(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

with $q(z,\zeta)$ designated as the best dominant of the third-order strong differential subordination (2).

Proof. As seen in the proof of the first theorem, the functions $p(z,\zeta)$, $q(z,\zeta)$ and $h(z,\zeta)$ may be assumed to be satifying the conditions of Lemma 1 on $U \times \overline{U}$ and the condition $q'_z(z,\zeta) \neq 0$ for $\xi \in \partial U \backslash E(q(z,\zeta))$.

By differentiating (ii) with respect to z, we have

$$q(z,\zeta) \cdot q'_z(z,\zeta) + z^2 (q'_z(z,\zeta))^2 + z^2 q(z,\zeta) q''_{z^2}(z,\zeta) + 2z^2 q''_{z^2}(z,\zeta) +$$
$$z^3 q'''_{z^3}(z,\zeta) = h(z,\zeta). \tag{9}$$

By applying (9), third-order strong differential subordination (2) becomes:

$$zp(z,\zeta) \cdot p'_z(z,\zeta) + [zp'_z(z,\zeta)]^2 + z^2 p''_{z^2}(z,\zeta)[p(z,\zeta)+2] + z^3 p'''_{z^3}(z,\zeta) \prec\prec h(z,\zeta). \tag{10}$$

For finalizing the proof of this theorem, define the function $\psi : \mathbb{C}^4 \times U \times \overline{U} \to \mathbb{C}$ as

$$\psi(r,s,t,u,z,\zeta) = r \cdot s + s^2 + t(r+2) + u, \ \ r,s,t,u \in \mathbb{C}. \tag{11}$$

Taking $r = p(z,\zeta)$, $s = zp'_z(z,\zeta)$, $t = z^2 p''_{z^2}(z,\zeta)$, $u = z^3 p'''_{z^3}(z,\zeta)$, relation (11) becomes:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2 p''_{z^2}(z,\zeta), z^3 p'''_{z^3}(z,\zeta)\right) = \tag{12}$$
$$p(z,\zeta) \cdot zp'_z(z,\zeta) + [zp'_z(z,\zeta)]^2 + z^2 p''_{z^2}(z,\zeta)[p(z,\zeta)+2] + z^3 p'''_{z^3}(z,\zeta).$$

Using (12), the third-order strong differential subordination (10) becomes:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right) \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}. \quad (13)$$

Since $h(z,\zeta)\in K\zeta$ we have that $h(z,\zeta)\in H\zeta_U(U)$ and applying part a of Remark 1 we can write an equivalent form of (13):

$$\left\{\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right)\right\}\subset h\left(U\times\overline{U}\right). \quad (14)$$

Considering $z=z_0\in U$, from (14) we have:

$$\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right)\in h\left(U\times\overline{U}\right). \quad (15)$$

Assume now that $p(z,\zeta)\not\prec\prec q(z,\zeta)$. Then, according to Lemma 1 there exist $z_0\in U$ and $\xi_0\in\partial U\backslash E(q(z,\zeta))$ such that:

$$\begin{gathered} p(z_0,\zeta)=q(\xi_0,\zeta),\ z_0p'_z(z_0,\zeta)=m\xi_0q'_z(\xi_0,\zeta), \\ t=z_0^2p''_{z^2}(z_0,\zeta),\ \ u=z_0^3p'''_{z^3}(z_0,\zeta), \end{gathered} \quad (16)$$

satisfy the conditions of Lemma 1.

By replacing $r=q(\xi_0,\zeta)$, $s=m\xi_0q'_z(\xi_0,\zeta)$, $t=z_0^2p''_{z^2}(z_0,\zeta)$, $u=z_0^3p'''_{z^3}(z_0,\zeta)$ in the admissibility condition from Definition 3, we have:

$$\psi\left(q(\xi_0,\zeta), m\xi_0q'_z(\xi_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right)\notin h\left(U\times\overline{U}\right).$$

Using the equalities seen in (16), relation (2) is written as:

$$\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right)\notin h\left(U\times\overline{U}\right),$$

which contradicts (15). Hence, we must have that

$$p(z,\zeta)\prec\prec q(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}.$$

Since $q(z,\zeta)\in H\zeta_U(U)$ and is a solution for the differential equation (9), it follows that $q(z,\zeta)$ is the best dominant for the third-order strong differential subordination (2). □

Example 1. *Using the outcome of Theorem 1, we can write:*

Let $h(z,\xi)=1+2z\xi$, $z\in U$, $\xi\in\overline{U}$, $h(z,\xi)\in K\xi$, $h(0,\xi)=1\in\mathbb{C}$, $p(z,\xi)=1+z^3\xi$, $q(z,\xi)=1+z\xi$, $z\in U$, $\xi\in\overline{U}$, $\gamma=1$ *satisfying:*

(i) $Re\frac{\xi q''_{z^2}(z,\xi)}{q'_z(z,\xi)}=Re\frac{0}{2\xi}=0\geq 0$ *and* $\left|\frac{z\cdot 3z^2\xi}{\xi}\right|=3\left|z^3\right|\leq 3$, $z\in U$, $\xi\in\partial U\backslash E(q(z,\xi))$;

(ii) $(1+z\xi)+z\xi=\frac{1}{z}\int_0^z(1+2t\xi)\,dt$, $\gamma=1$.

If $\left(1+z^3\xi\right)+z\left(3z^2\xi\right)+z^2\cdot 6z\xi+2z\left(3z^2\xi\right)+3z^2\cdot 6z\xi+z^3\cdot 6\xi=1+z^3\xi+3z^2\xi+6z^3\xi+6z^3\xi+18z^3\xi+6z^3\xi=1+40z^4\xi$, *is analytic in* $U\times\overline{U}$, *then*

$$1+40z^4\xi\prec\prec 1+z\xi+z\xi+2z\xi=1+4z\xi,$$

implies

$$1+z^3\xi\prec\prec 1+z\xi, \quad z\in U,\ \xi\in\overline{U},$$

and $q(z)=1+z\xi$ *is designated as the best dominant.*

3. Conclusion

The new results established in this investigation are contained in Section 2 of the paper, after the necessary notions and previously established results necessary for the investigation are presented. The line of research followed by this study concerns the development of the newly initiated theory of third-order strong differential subordination. Having seen the new recent results obtained by researchers concerning classical third-order differential subordination theory, and considering the nice developments involving the theory of strong differential subordination, this study extends previously known lemmas established in [20, 21], popular in researches in geometric function theory, providing new tools for improving the knowledge related to third-order strong differential subordination theory, recently initiated by the publication [10]. The new results obtained here are given in Theorem 1 and Theorem 2. A numerical example is provided hoping to inspire certain applications for particular functions to be used as best dominants of third-order strong differential subordinations, which could result in obtaining interesting consequences with significant geometrical interpretations. Nevertheless, the main idea of the study doesn't focus on numerical examples but on providing new means of investigation in the field.

Since the initial lemmas that have motivated this study presented as Lemma 2 and 3 concerning second-order differential subordination theory have facilitated major developments of that topic, it is expected that the new results proved during this investigation to have the same effect on motivating future research in third-order strong differential subordination theory.

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