Implementing Real Options Valuation under Macroeconomic Risk and Normally Distributed Cash Flows

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ABSTRACT

The paper highlights the encountered problems in implementing real options under more realistic assumptions such as business cycle risk and normally distributed cash flows. The problems considered include (i) estimating empirical distribution of cash flows from real option investments; (ii) investment decisions across business cycles, and (iii) calculating the probability of investing with the above stated rich features. To this end, we estimate operating cash flows of US corporate firms using a Markov chain model under both geometric and arithmetic Brownian motions assumptions for cash flows and develop a valuation model of real option with normally distributed cash flows. Associated investment valuation models incorporating these estimates reveal that critical cash flow levels significantly differ across models and regimes.

Key words: *Macroeconomic Risk, Regime Switching, Real Options, Arithmetic Brownian Motion, Geometric Brownian Motion*

JEL Codes: D92; E22; E32; G31

1. INTRODUCTION

International financial organizations, such as the IMF, World Bank, and BIS, have flagged a declining investment trend, especially since the global financial crisis of 2008-09, as a challenge for major economies. For instance, Gutierrez and Phillipon (2017) reports that US investment averaged about 20 percent of corporate operating revenues between 1959 and 2001, whereas it averaged only 10 percent from 2002 to 2015. Rising uncertainty has been cited as a key factor behind this trend (Bloom, 2014). The financial option literature also highlights the role of uncertainty in resolving pricing puzzles like volatility smiles and option smirks Liu et al. (2005). Business cycles and financial crises increase investor concern about uncertainty to mitigate potential losses.

Another critical motivation for this study is the need to analyze massive green investments required to achieve net-zero objectives. As nations and corporations intensify their commitments to

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sustainable practices, understanding how investment models can handle such large-scale, long-term commitments under various uncertainties is crucial.

This paper modifies the conventional real options approach to incorporate normally distributed cash flows and macroeconomic risk. Although cash flows are often modeled as log-normally distributed for tractability, evidence supports the use of normal distributions (Burg, 2018). Normal distributions are useful for deriving closed formulas in spread options, mergers, and portfolios (Leland, 2007). They also accommodate negative values, providing a more accurate representation compared to log-normal distributions, which are typically used for their mathematical simplicity but may not fully capture real-world cash flow dynamics.

We extend the works of Xin Guo et al. (2005) and Driffill et al. (2013) to analyze the impact of normally distributed cash flows and macroeconomic risks on investment decisions Arnold, Wagner, and Westermann (2013). While Brownian motion models offer mathematical tractability, they may underestimate risks and fail to capture sudden changes or autocorrelation in cash flows. To address these limitations, future research could explore more sophisticated models, such as jump-diffusion or GARCH models, to better reflect real-world dynamics.

This study examines how firms' investment behavior differs under normally distributed versus lognormally distributed cash flows and how business cycles impact these behaviors. Our approach includes:

- Calculating regime-specific investment-triggering cash flow levels for both normal and lognormal distributions.
- Determining the regime-specific probability of investing under each cash flow distribution.
- Comparing results to highlight differences in investment behavior across distributions and business cycle regimes.

The numerical calculations reveal that investments are more likely under lognormally distributed cash flows compared to normally distributed ones. Specifically:

- In a boom regime:
  - Lognormal: Probability of investing = 0.206
  - Normal: Probability of investing = 0.139
- In a recession regime:
  - Lognormal: Probability of investing = 0.004
  - Normal: Probability of investing = 0.001

These results emphasize the significant impact of cash flow distribution assumptions on investment decisions and the importance of incorporating business cycle considerations. The lognormal distribution's higher investment probabilities in boom periods reflect its right-skewed nature, which suggests a higher potential for returns.

Our paper contributes to the literature by (i) considering normally distributed cash flows, (ii) estimating irreversible investment model parameters using US corporate cash flow data, and (iii) computing critical cash flow levels and investment probabilities. While our focus is on investment decisions across business cycles, exploring various real-world scenarios will enhance the robustness and applicability of our model.

Section 2 reviews related literature, Section 3 outlines the model, Section 4 presents the numerical analysis, and the final section concludes.

2. LITERATURE REVIEW

The real options approach to modeling corporate investment captures the flexibility inherent in many investment decisions, contrasting with traditional discounted cash flow (DCF) methods that treat investments as static and irreversible. Early works by Myers (1977) explored the value of delaying investment under uncertainty, while McDonald and Siegel (1986) laid the groundwork for real options theory by analyzing investment timing in the presence of geometric Brownian motion. Dixit and Pindyck (1994) further developed this framework, highlighting how uncertainty creates an opportunity cost of immediate investment versus waiting for new information.

Extensions of the real options model have incorporated more realistic features of investment decisions. Notably, regime-switching models Bollen (1998, 1999), Guo et al. (1995), Elliott et al. (2009), Chen (2010), Bhamra et al. (2010a, 2010b), Arnold et al. (2013) and Driffill et al. (2013) account for changes in economic conditions over time, allowing for discrete shifts in parameters like growth rates and volatility. While geometric and arithmetic Brownian motions provide a simplified yet robust framework, they may not capture all real-world complexities.

Key contributions to the real options framework include: - McDonald and Siegel (1986), who showed that uncertainty increases the value of waiting, leading firms to delay investments until the project value significantly exceeds the investment cost. - Dixit and Pindyck (1994), who emphasized the interaction of irreversibility, uncertainty, and the ability to delay investment, highlighting that standard NPV calculations often underestimate the true value of investment opportunities. - Trigeorgis (1996), who extended the framework to include options to abandon, expand, or switch, demonstrating the value added by flexibility in investment decisions. - Schwartz and Moon (2000), who applied real options to valuing internet companies, incorporating stochastic processes for revenues, costs, and growth rates to address high uncertainty and growth potential.

Research on macroeconomic impacts includes: - Guo et al. (2005), who developed a model incorporating regime-switching in macroeconomic conditions, showing significant effects on optimal investment thresholds and opportunity values. - Bloom (2014), who highlighted how time-varying uncertainty drives business cycles and affects firm-level investment decisions, showing how uncertainty shocks can delay investment and contribute to economic fluctuations. - Wang, Chen, and Huang (2014), who examined the impact of policy uncertainty on investment behavior, demonstrating that increased policy uncertainty raises the value of waiting, particularly for firms facing higher irreversibility or competitiveness.

Despite these advancements, a gap remains in empirical estimation of cash flow distributions across different business cycle regimes and their impact on investment decisions. Most studies assume a specific distribution (often log-normal) without empirical testing or consideration of how distributions vary across economic conditions.

Our paper addresses this gap by empirically estimating the parameters of normally and log-normally distributed cash flows across business regimes. By calculating regime-specific investment-triggering cash flow levels and investment probabilities, our work bridges theoretical models with empirical realities. This approach enhances understanding of how distributional assumptions and business cycle dynamics influence investment decisions, providing valuable insights for both academics and practitioners.

In summary, our research extends the real options literature by offering a regime-specific analysis of investment under uncertainty, contributing to a more comprehensive understanding of firm investment behavior across the business cycle.

3. THE MODEL

Consider a firm that undertakes irreversible investment decisions under uncertain economic conditions stemming from continuous randomness and macroeconomic factors, such as regime shifts between booms and recessions. These uncertainties are characterized by two types of independent shocks: (i) small but continuous shocks modeled by a Brownian motion process $\mathrm{Z(t)}$, and (ii) large but infrequent shocks represented by a two-state Markov chain process $\{\mathrm{s_t}\}$, where the two regimes are a boom (s = B) and a recession (s = R). The stochastic processes governing the firm's cash flows and the market price of risk are the primary determinants influencing its decision-making process. It is assumed that the cash flow process follows either a normal or a log-normal distribution, while the market price of risk process is chosen in accordance with the distribution of the cash flow process. Investors hedge both risks.

We assume a constant probability setting to govern switching between the two regimes. Hence, the probability of the economy switches from state B to state R within a short period $\Delta\mathrm{t}$ approximately equals $\mathrm{h_B^{\mathbf{P}}}\Delta\mathrm{t}$ while the probability of staying in state B is $1-\mathrm{h_B^{\mathbf{P}}}\Delta\mathrm{t}$. The dynamics of the Markov chain process $\mathrm{s_t}$ is given by

$$ds_t = \mathbf{H}s_t dt + dN_t, \qquad s_0 = i, \tag{3.1}$$

where **H** is the constant intensity matrix $\mathrm{H} = \left(\mathrm{h_{ij}^{\mathbf{Q}}}\right) = \begin{bmatrix} -\mathrm{h_B^{\mathbf{P}}} & \mathrm{h_B^{\mathbf{P}}} \\ \mathrm{h_R^{\mathbf{P}}} & -\mathrm{h_R^{\mathbf{P}}} \end{bmatrix}$ and dN a Poisson process.

We assume an arithmetic Brownian motion (ABM) process to model cash flows following a normal distribution, while a geometric Brownian motion (GBM) process is employed to capture cash flows that exhibit a log-normal distribution.

The regime-switching diffusion processes for normally and log-normally distributed cash flows are given by:

$$ABM: \qquad dX_t = \mu_{s_t}^{\boldsymbol{P}} dt + \sigma_{s_t} dZ_t^{\boldsymbol{P}}, \tag{3.2}$$
$$GBM: \qquad dX_t = \mu_{s_t}^{\boldsymbol{P}} X_t dt + \sigma_{s_t} X_t dZ_t^{\boldsymbol{P}}, \tag{3.3}$$

where $dZ_t^{\boldsymbol{P}}$ is the increment of a standard pyhsical Wiener process, representing the continuous shocks. $\mu_{s_t}^{\boldsymbol{P}}$ denotes the expected regime-specific change in the cash flow, and σ_{s_t} represents the regime-specific volatility of the cash flow. s_t is the regime-switching shock process, with $s_0 = i$ as the initial regime. The (phsical) drift ($\mu_{s_t}^{\boldsymbol{P}}$) and diffusion ($\sigma_{s_t}$) parameters adjust to reflect shifts in macroeconomic conditions. The adjustment is governed by a Markov chain process, which models the regime-switching shocks.

To risk-neutralise the model, we utilize the following stochastic discount factor[1]

$$\frac{dM(s_t)}{M(s_t)} = -r_{s_t}^f dt - \lambda_{s_t} dZ_{m,t}^{\boldsymbol{P}} - \left[exp(\kappa_{s_t}) - 1\right][ds_t - Hs_t dt], \tag{3.4}$$

where λ is the risk price for Brownian shock affecting cash flows and κ_i is the relative jump size of the discount factor when the Markov chain leaves state i implying that $exp(\kappa_{s_i}) - 1$ is the price of switching risk from regime i to j. The model implies that $\kappa_i = 1/\kappa_j$.[2]

The risk neutralisation requires that ${\mu^{\boldsymbol{Q}}}_B = \mu_B^{\boldsymbol{P}} - \lambda_B \rho \sigma_B \sigma_M$ and $\mu_R^{\boldsymbol{Q}} = \mu_R^{\boldsymbol{P}} - \lambda_R \rho \sigma_R \sigma_M$ for the parameters of Brownian motion processes. Here, σ_M is the standard deviation for the market portfolio and ρ is the correlation coefficient between market portfolio and investment project in hand. Given λ_i, ρ and σ_M one can convert μ_i^P under the physical probability measure to its risk-neutral counterpart $\mu_i^{\boldsymbol{Q}}$ in each regime $i = B, R$. We apply a similar conversion to the transition probability terms: $h_i^{\boldsymbol{Q}} = e^{\kappa_i} h_i^{\boldsymbol{P}}$ for $i = B, R$.

We evaluate irreversible investment opportunities faced by investors. The irreversible nature makes the investment timing crucial, as a critical cash flow level exists at which the investor should initiate the investment. First, we explain the valuation of the investment project itself. Then, we consider flexible investment opportunities within this valued project using a regime-switching framework. Investors aim to optimally time the investment initiation, incurring a cost I regardless of the economic regime. The optimal investment timing corresponds to a specific cash flow value, serving as a threshold beyond which investing becomes optimal. However, for models with normally or lognormally distributed cash flows, such critical cash flow levels may not be directly comparable. Therefore, we also compute the probability of investing to facilitate comparisons.

3.1. Valuation of Projects with Regime Shifts

Consider a project generating perpetual cash flows upon its acceptance. Cash flows follow a normal distribution rather than the commonly adopted lognormal one dictated by its mathematical tractability. Cash flows are also subject to macroeconomic risk stemming from shifts in conditions between benign and adverse ones, e.i., business cycles of boom and recession regimes. In other words, there are two types of independent shocks in the economy: (i) small but continuous shocks

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[1] Bhamra, et al. (2010a), Bhamra, et al. (2010b), Chen (2010) and Driffill et al. (2013) showed that there exists a stochastic discount factor that prices both Brownian and regime-switching risks in the economy.
[2] For a careful derivation of this result see Driffill et al. (2013).

generated by a Brownian motion $Z(t)$ and (ii) large but infrequent shocks by a marked point process, more precisely two-state Markov chain process $\{s_t\}$. That is, the two regimes are a boom and recession: $s \in \{B, R\}$. Investors hedge both risks using traded assets thanks to their correlation with the project's cash flows X_t.

Endowed with the cash flow, Markov Chain and pricing processes, we can now derive expressions to value claims on the cash flow. We start with the no-arbitrage valuation of the project's cash flows (assets-in-place) without the option to invest. The regime-switching version of the fundamental pricing equation (see for example) is given by:

$$d\big(M(s_t)V(s_t)\big) + M(s_t)x(t)dt = 0. \tag{3.5}$$

The following proposition provides the solution of the equations for the value of the project's cash flows (see Appendix A for proof).

Proposition 1. (Project Valuation under the Regime-Switching ABM Process) The value of project's cash flows (assets-in-place) under the regime-switching ABM process is

$$V^i = \bar{A}^i + \bar{B}^i x_0,$$

with

$$\begin{bmatrix} V^B \\ V^R \end{bmatrix} = \underbrace{\begin{bmatrix} r_B^f + h_B^{\boldsymbol{Q}} & -h_B^{\boldsymbol{Q}} \\ -h_R^{\boldsymbol{Q}} & r_R^f + h_R^{\boldsymbol{Q}} \end{bmatrix}^{-1} \begin{bmatrix} \mu_B^{\boldsymbol{Q}} \bar{B}_B \\ \mu_R^{\boldsymbol{Q}} \bar{B}_R \end{bmatrix}}_{\bar{A}} + \begin{bmatrix} \bar{B}_B \\ \bar{B}_R \end{bmatrix} x_0, \tag{3.6}$$

$$\bar{B}_i = \frac{1}{r_i^p}, \qquad r_i^p = r_i^f + \frac{r_j^f - r_i^f}{\tilde{p} + r_j^f}\tilde{p}\tilde{f}_j, \qquad i = B, R$$

where r_i^p is the perpetual risk-free rate and $\tilde{p} = h_B^{\boldsymbol{Q}} + h_R^{\boldsymbol{Q}}$ and x_0 is the initial cash flow. Here, $\mu_{s_t}^{\boldsymbol{Q}} = \mu_{s_t}^{\boldsymbol{P}} - \rho_{s_t}\lambda_{s_t}\sigma_{s_t}$ is the risk-neutral drift term and $h^{\boldsymbol{Q}}(s_t)$ is the risk-neutral transition probability term. The term $\rho_{s_t} = 1/dt\, E(dZdZ_m)$ captures the regime-specific correlation between the cash flow of the project and the pricing process (market portfolio). λ_{s_t} is the regime-specific risk price for systematic Brownian shocks from Z^m. To calculate the project value we simply subtract the cost of investment I: $V^i - I$ for $i = B, R$.

The project value under lognormally distributed cash flows takes the following form:

$$V^i = \bar{B}^i x_0$$

where $\bar{B}^i = r_i^f - \mu_i^{\boldsymbol{Q}} + \frac{(r_j^f - \mu_j^{\boldsymbol{Q}}) - (r_i^f - \mu_i^{\boldsymbol{Q}})}{r_j^f - \mu_j^{\boldsymbol{Q}} + \tilde{p}}\tilde{p}\tilde{f}_j, \qquad i = B, R$

3.2. Valuation of the Option to Invest with Regime Shifts

We now consider investment opportunities in the project we have just valued using the regime-switching framework. There also exits flexibility in investment opportunities. Investors try to resolve uncertainty to optimal time the initiation of their investment which costs I irrespective of the economic regime. The optimal time effectively corresponds to a cash flow value. This critical value is a threshold at which it is optimal to invest. The presence of regime-switching macroeconomic conditions generates two of such threshold levels. We first value the option to invest under our Markov chain model and then determine the critical cash flow values. To this end, we solve the following second-order differential equation (reproduced from *Equation A.9* in Appendix A):

$$\begin{aligned} -r_B^f V^B + V_x^B \mu_B^{\mathbf{Q}} + V_{xx}^B \sigma_B^2 + h_B^{\mathbf{Q}}(V^R - V^B) + x(t) &= 0, \\ -r_R^f V^R + V_x^R \mu_R^{\mathbf{Q}} + V_{xx}^R \sigma_R^2 + h_R^{\mathbf{Q}}(V^B - V^R) + x(t) &= 0. \end{aligned}$$

With optionality features at present, the solution of this equation requires a homogeneous solution besides the particular solution obtained in the previous subsection. The value of the option to invest across the boom and recession regimes is given by:

$$\begin{aligned} F^B(x) &= A_1^B e^{\gamma_3 x} + A_2^B e^{\gamma_4 x}, \\ F^R(x) &= A_1^R e^{\gamma_3 x} + A_2^R e^{\gamma_4 x}, \end{aligned}$$

where A_1^B, A_2^B, A_1^R, and A_2^R are the constants determined from the boundary conditions and γ_3 and γ_4 will be solved from the Cramer-Lunderberg equation (quadric polynomial) (see, for example, Guo (2001) for a proof). Once investors exercised the option to invest, the value of $F^i(x),\ i = B, R$ takes their project values as in Proposition 1:

$$\begin{aligned} F^B(x) &= \bar{A}^B + \bar{B}^B x_0, \\ F^R(x) &= \bar{A}^R + \bar{B}^R x_0. \end{aligned}$$

Solving the model for the constants A and γ_3 requires an involved process. It stems from the fact that there is an additional region besides the ones described above. This third one is called the transient region associated with shifts from a recession to a boom regime. There are two (regime-specific) threshold values of cash flows at which investors decide on exercising the option. We label them x^B and x^R and assume that $x^B < x^R$. Therefore, the regions are (i) the continuation region $x < x^B < x^R$, (i) the transient region $x^B \leq x < x^R$ and (i) the investment region $x^R \leq x$. Proposition 2 states the value of the option over these three regions.

Proposition 2. (Option to Invest Valuation) At given exercise boundaries $[x^B, x^R]$, the value of the option to invest in regime i is written as where γ_3, γ_4 are the positive roots of the Cramer-Lundberg equation,

$$\begin{aligned} F^B(\gamma)F^R(\gamma) &= h_B^{\boldsymbol{Q}} h_R^{\boldsymbol{Q}} \\ where \quad F^B(\gamma) &= \left(r_B^f + h_B^{\boldsymbol{Q}}\right) - \mu_B^{\boldsymbol{Q}}\gamma - 1/2\, \sigma_B^2 \gamma^2 \\ F^R(\lambda) &= \left(r_R^f + h_R^{\boldsymbol{Q}}\right) - \mu_R^{\boldsymbol{Q}}\gamma - 1/2\, \sigma_R^2 \gamma^2, \end{aligned}$$

$$C_3 = \frac{h_R^{\boldsymbol{Q}} \bar{B}_B}{r_R^f + h_R^{\boldsymbol{Q}}} \quad and \quad C_4 = \frac{\mu_R^{\boldsymbol{Q}} C_3 + h_R^{\boldsymbol{Q}}(\bar{A}_B - I)}{r_R^f + h_R^{\boldsymbol{Q}} - \mu_R^{\boldsymbol{Q}}},$$

$$A_j^R = l_j A_j^B,$$

with Appendix B solves the system of equations[3]

Under the assumption of cash flows following a regime-switching geometric Brownian motion, the option to invest problem and its solution are modified as follows. The second-order stochastic differential equations take the form of

$$\begin{aligned} -r_B^f V^B + V_x^B \mu_B^{\boldsymbol{Q}} x(t) + V_{xx}^B \sigma_B^2 x^2(t) + h_B^{\boldsymbol{Q}}(V^R - V^B) + x(t) &= 0, \\ -r_R^f V^R + V_x^R \mu_R^{\boldsymbol{Q}} x(t) + V_{xx}^R \sigma_R^2 x^2(t) + h_R^{\boldsymbol{Q}}(V^B - V^R) + x(t) &= 0. \end{aligned}$$

The regime specific homogeneous solutions:

$$\begin{aligned} F^B(x) &= A_1^B x^{\gamma_3} + A_2^B x^{\gamma_4}, \\ F^R(x) &= A_1^R x^{\gamma_3} + A_2^R x^{\gamma_4}. \end{aligned}$$

At given exercise boundaries $[x^B, x^R]$, the value of the option to invest in regime i is written as

$$F^i(x) = \begin{cases} A_1^i x^{\gamma_3} + A_2^i x^{\gamma_4}, & x < x^B, & i = B, R, \\ C_1 x^{\xi_1} + C_2 x^{\xi_2} + C_3 x + C_4, & x^B \leq x < x^R, & i = R \\ \bar{B}^i x - I, & x^R \leq x, & i = B, R. \end{cases}$$

The Cramer-Lundberg equation is given by

$$\begin{aligned} F^B(\lambda)F^R(\lambda) &= h_B^{\boldsymbol{Q}} h_R^{\boldsymbol{Q}} \\ where \quad F^B(\lambda) &= \mu_B^{\boldsymbol{Q}}\lambda + 1/2\, \sigma_B^2 \lambda(\lambda - 1) - \left(r_B^f + h_B^{\boldsymbol{Q}}\right) \\ F^R(\lambda) &= \mu_R^{\boldsymbol{Q}}\lambda + 1/2\, \sigma_R^2 \lambda(\lambda - 1) - \left(r_R^f + h_R^{\boldsymbol{Q}}\right), \end{aligned}$$

With

$$C_3 = h_R^{\boldsymbol{Q}} \frac{\bar{B}_B}{r_R^f - \mu_R^{\boldsymbol{Q}} + h_R^{\boldsymbol{Q}}} \quad and \quad C_4 = -h_R^{\boldsymbol{Q}} \frac{I}{r_R^f + h_R^{\boldsymbol{Q}}},$$

[3] Bensoussan et al. (2012) provides the existence and uniqueness of the solution. to obtain $[A_1^B, A_2^B, A_1^R, A_2^R, x^B, x^R]$.

$$\xi_{1,2} = \frac{1}{2} - \frac{\mu_R^{\boldsymbol{Q}}}{\sigma_R^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\mu_R^{\boldsymbol{Q}}}{\sigma_R^2}\right)^2 + \frac{2\left(r_R^f + h_R^{\boldsymbol{Q}}\right)}{\sigma_R^2}},$$

$$A_j^R = l_j A_j^B,$$

$$l_j = \frac{A_j^R}{A_j^B} = \frac{1}{h_B^{\boldsymbol{Q}}}\left(h_B^{\boldsymbol{Q}} + r_B^f - \mu_B^{\boldsymbol{Q}}\gamma_j - 1/2\,\sigma_B^2\gamma_j\left(\gamma_j - 1\right)\right), \qquad j = 3,4.$$

3.3. Probability of Investment

This section explains how to compute the probability of investing. Unfortunately, models with regime shifts do not have closed-form solutions. We, therefore, briefly explain the derivation for single-regime cases. It resembles the case of a perpetual American call option holder who wants to calculate the probability of exercising the option within the next T years. We derive the probability expression by summing two probabilities: (i) the probability of the path that ends above the critical investment level $x(T) = x^*$ at time T, and (ii) the probability of the path that ends below the level $x(T) = x^*$ at time T but has crossed the level $x(t) = x^*$ at some time before T.[4] Following the derivation in Harrison (2013), we derive the corresponding probability to invest over the period [0,T] for the ABM process as follows:[5]

$$\boldsymbol{P}(T_{x^*}, 0 < T) = \Phi\left(\frac{(x_0 - x^*) + \mu^{\boldsymbol{P}}T}{\sigma\sqrt{T}}\right) + exp\left(\frac{2\mu^{\boldsymbol{P}}(x^* - x_0)}{\sigma^2}\right)\Phi\left(\frac{(x_0 - x^*) - \mu^{\boldsymbol{P}}T}{\sigma\sqrt{T}}\right). \qquad (3.7)$$

where x^* is the investment trigger point and $\Phi(\cdot)$ denotes the standard normal cumulative density function. Similarly, its GBM counterpart is:

$$\boldsymbol{P}(T_{x^*}, 0 < T) = \Phi\left(\frac{ln\left(\frac{x_0}{x^*}\right) + (\mu^{\boldsymbol{P}} - 1/2\,\sigma^2)T}{\sigma\sqrt{T}}\right) + \left(\frac{x^*}{x_0}\right)^{\frac{2\mu^{\boldsymbol{P}}}{\sigma^2}-1}\Phi\left(\frac{ln\left(\frac{x_0}{x^*}\right) - (\mu^{\boldsymbol{P}} - 1/2\,\sigma^2)T}{\sigma\sqrt{T}}\right). \qquad (3.8)$$

While the reflected Brownian motion method offers a closed-form solution for the probability of a Brownian particle reaching a specific level, the inverse Laplace transform of the first passage time distribution provides an alternative approach that can potentially be more efficient. The Laplace transform of the first-passage time is given by

[4] Harrison (2013, equation 1.51, page 15) obtains the probability associated with this case based on reflected Brownian motion, which is the second term on the right-hand side of Eq. (1.47) in his book. For a rigorous derivation, see especially his Chapter 6. Sarkar (2000) also uses the expression for the Geometric Brownian Motion (GBM) case.

[5] This is exactly the equation 12 of Hieber (2014).

$$\Psi = exp\left(\frac{\mu^{\boldsymbol{P}}(x^* - x_0)}{\sigma^2}\left(1 - \sqrt{1 + \frac{2\sigma^2 r}{\mu^{\boldsymbol{P}}}}\right)\right).$$

By ınverting this Laplace transform equation using the Laplace inversion tables one can obtain the first-passage time probabilities of Brownian motion, i.e., *Equation 3.7* and *8*.

Let T_{BR} denote the first time the cash flow process x_t hits either of the two barriers x^B or x^R, with T_{BR}^+ representing the case when $x_{T_{BR}} > x^B$, and T_{BR}^- when $x_{T_{BR}} \leq x^B$. The Laplace transform of the first-passage time distribution is defined as $\Psi^{\pm}(r) = \boldsymbol{E}\left[exp\left(-rT_{BR}^{\pm}\right)\right]$. Several numerical methods are available to invert these Laplace transform expressions, including:

- The Wiener-Hopf factorization technique Hieber (2014)
- The Brownian Bridge method (Hieber 2014)
- Numerical integration approaches (Hieber 2014)
- Lattice-based methods Hieber (2014)
- Monte Carlo simulations Hieber (2014)

Each method has its own advantages and limitations, and the choice depends on factors such as the required accuracy, computational efficiency, and the specific problem characteristics. For instance, Monte Carlo simulations can be computationally intensive but offer flexibility in handling complex processes, while lattice-based methods may be more efficient for certain problems but require discretization of the underlying process. In this paper, we employ a Monte Carlo simulation method involving the simulation of continuous-time Brownian-Markov chain processes to obtain the first-passage time distributions. This approach allows us to handle the specific characteristics of our cash flow process while maintaining a desired level of accuracy.

3.4. The One Regime Model

This subsection presents the single-regime model as a special case of models with regime shifts. Table 1 provides detailed expressions for both models, considering cash flows following either an arithmetic Brownian motion (ABM) or a geometric Brownian motion (GBM) specification.

The project value under one regime case reduces to $\frac{\mu}{(r^f)^2} + \frac{0}{r^f} - I$.

In a single regime model, the real option valuation with ABM and GBM processes simplifies to:

	ABM Model	GBM Model
Process	$dx(t) = \mu^{\boldsymbol{Q}}dt + \sigma dZ^{\boldsymbol{Q}}$	$dx(t) = \mu^{\boldsymbol{Q}}x(t)dt + \sigma x(t)dZ^{\boldsymbol{Q}}$
Project value	$\frac{\mu^{\boldsymbol{Q}}}{(r^f)^2} + \frac{x_0}{r^f} - I$	$\frac{x_0}{r^f - \mu^{\boldsymbol{Q}}} - I$
Option value	$F(x) = Ae^{\lambda x_0}$	$F(x) = Ax_0^{\lambda}$
	$A = e^{-\lambda x^*}\left[\frac{\mu^{\boldsymbol{Q}}}{(r^f)^2} + \frac{x^*}{r^f} - I\right]$	$A = \frac{(x^*)^{1-\lambda}}{r^f - \mu^{\boldsymbol{Q}}} - (x^*)^{-\lambda}I$

	$\lambda = -\frac{\mu^{\boldsymbol{Q}}}{\sigma^2} + \sqrt{\left(\frac{\mu^{\boldsymbol{Q}}}{\sigma^2}\right)^2 + \frac{2r^f}{\sigma^2}}$	$\lambda = \frac{1}{2} - \frac{\mu^{\boldsymbol{Q}}}{\sigma^2} + \sqrt{\left(\frac{\mu^{\boldsymbol{Q}}}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r^f}{\sigma^2}}$
Critical value	$x^* = \frac{1}{\lambda} + r^f I - \frac{\mu^{\boldsymbol{Q}}}{r^f}$	$x^* = \frac{\lambda}{\lambda - 1}[r^f - \mu^{\boldsymbol{Q}}]I$

Table 1. Expressions of One Regime Models

4. ESTIMATION AND NUMERICAL ANALYSIS

Quarterly data on operating income comes from Datastream for the period 1989:2 - 2021:1, covering four US economic recessions. Data is available for 314 US corporate firms. Furthermore, we applied a data clustering algorithm, Density-based spatial clustering of applications with noise (DBSCAN), to remove outliers. It reduced the total firm number to 278. To convert nominal values into real ones, we use the US consumer price index (CPI).

We utilised the R package MSwM to estimate the parameters of regime-switching geometric and arithmetic operating cash flow processes. Table 2 reports the estimated values for $\mu_B^{\boldsymbol{Q}}, \mu_R^{\boldsymbol{Q}}, \sigma_B, \sigma_R, h_B^{\boldsymbol{Q}}$ and $h_R^{\boldsymbol{Q}}$[6]. To ensure the robustness of our results, we have also estimated our models using two additional statistical software packages, Stata and Python's StatModels. Table A1 and A2 in Appendix C present a comparative summary of the findings across these different implementations.

Plugging these estimates into the respective real options models yields the following investment triggering cash flows across regimes:

Another way of comparing the alternative ABM and GBM models is to calculate the probability of investing within a particular period. We take five years as the length of the period and simulate operating cash flows over this span using both ABM and GBM stochastic processes. We count instances whenever the simulated value passes the regime-specific value within each regime. These counted instances enable us to calculate the probability of investing by dividing them by the total number of simulations. Table 4 reports these cumulative probabilities. The table also shows the average timing of investment over the five years. It is needless to say that the results in Table 4 depending on the relative distance of initial operating cash flows from its critical values given in Table 3. As stated in Table 3 in both model and regimes, we take the initial value of x as 50.

Again, the cumulative probability of investing and average expected investment times in Table 4 reveals that firms with the GBM operating cash flow assumption are likely to invest much earlier than those holding the ABM assumption.

5. CONCLUSION

Our analytical models and numerical results underscore the critical importance of accurately determining the probability distribution of cash flows from investments and incorporating

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[6] Similar estimates for the single regime case are $\mu^{GBM} = 0.165,\ \mu^{ABM} = 0.85,\ \sigma^{GBM} = 0.89,\ \sigma^{ABM} = 75.2$. as well as plausible values for the interest rate r_B^f and r_R^f .[^4]

macroeconomic risks. The numerical calculations reveal a significant finding: investments are more likely under lognormally distributed cash flows compared to normally distributed cash flows. This insight highlights the substantial impact that distributional assumptions can have on investment decisions.

Several promising avenues for future research emerge from this study:

- Incorporating uncertainty aversion could provide valuable insights into how risk perception influences investment decisions, potentially explaining observed patterns of investment hesitancy.
- Considering firm-specific leverage conditions could yield more nuanced, individualized results, enhancing the model's applicability to diverse corporate scenarios.
- Integrating idiosyncratic aspects related to firm characteristics or industry-specific factors could improve the model's granularity and predictive power.
- Applying our proposed valuation model to various real-world investment scenarios could test its robustness and practical utility.
- Refining the estimation methodology could significantly enhance the model's accuracy. Two potential approaches stand out:
    a) Employing filtering and parameter estimation techniques from Hidden Markov Models, as demonstrated by Elliott et al. (1995).
    b) Utilizing Gibbs sampling for Monte Carlo Markov Chain models, offering an alternative perspective on parameter estimation and uncertainty quantification.

Furthermore, our regime-switching framework has broader applications beyond investment decisions. It can be leveraged to analyze other critical financial decisions across business cycles, including capital structure optimization, dividend policy formulation, and scope decisions.
In conclusion, this study not only provides valuable insights into the relationship between cash flow distributions, business cycles, and investment behavior but also lays the groundwork for a more comprehensive understanding of corporate financial decision-making under varying economic conditions. As we continue to refine and expand these models, we move closer to a more robust and nuanced framework for financial analysis and decision-making in a complex, cyclical economic environment.

6. APPENDIX

6.1. Proof of Proposition 1

Proof. Applying Ito's formula to the pricing equation in Equation 3.5 in the main text

$$d\big(M(s_t)V(s_t)\big) + M(s_t)x(t)dt = 0$$

Yields

$$V(s_t)dM(s_t) + M(s_t)dV(s_t) + dV(s_t)dM(s_t) + M(s_t)x(t)dt = 0.$$

After substituting Equation 3.4 and noting that

$$dV(s_t) = V_x(s_t)dx(t) + V_{xx}(s_t)\big(dx(t)\big)^2 + ds_t \Delta V(s_t),$$

we write the expected value of the above equation over two regimes as:

$$\begin{aligned} -r_B^f V^B + V_x^B \mu_B^{\mathbf{Q}} + V_{xx}^B \sigma_B^2 + h_B^{\mathbf{Q}}(V^R - V^B) + x(t) &= 0, \\ -r_R^f V^R + V_x^R \mu_R^{\mathbf{Q}} + V_{xx}^R \sigma_R^2 + h_R^{\mathbf{Q}}(V^B - V^R) + x(t) &= 0. \end{aligned} \quad (A.9)$$

By postulating $V^i = \bar{A}_i + \bar{B}_i x$ for $i = B, R$ we obtain the value of assets-in-place under the regime switching ABM process:

$$\begin{bmatrix} V^B \\ V^R \end{bmatrix} = \underbrace{\begin{bmatrix} r_B^f + h_B^{\mathbf{Q}} & -h_B^{\mathbf{Q}} \\ -h_R^{\mathbf{Q}} & r_R^f + h_R^{\mathbf{Q}} \end{bmatrix}^{-1} \begin{bmatrix} \mu_B^{\mathbf{Q}} \bar{B}_B \\ \mu_R^{\mathbf{Q}} \bar{B}_R \end{bmatrix}}_{\bar{A}} + \begin{bmatrix} \bar{B}_B \\ \bar{B}_R \end{bmatrix} x_0, \quad (A.10)$$

With

$$\bar{B}_i = \frac{1}{r_i^p}, \qquad r_i^p = r_i^f + \frac{r_j^f - r_i^f}{\tilde{p} + r_j^f} \tilde{p} \tilde{f}_j, \qquad i = B, R$$

where r_i^p is the perpetual risk-free rate, $\tilde{p} = h_B^{\mathbf{Q}} + h_R^{\mathbf{Q}}$, $\tilde{f}_i = \big(h_R^{\mathbf{Q}}/\tilde{p}, h_B^{\mathbf{Q}}/\tilde{p}\big)$ and x_0 is the initial cash flow. [QED].

6.2. Proof of Proposition 2

In the continuation region $x < x^B < x^R$, the option values satisfy the following system in compact matrix form:

$$0.5\mathbf{\Sigma}\mathbf{F_{xx}} + \mathbf{M}\mathbf{F_x} + (\mathbf{H} - \mathbf{R})\mathbf{F} = 0, \quad (A.11)$$

where $\mathbf{R} = diag(r_B, r_R)$, $\mathbf{M} = diag\big(\mu_B^{\mathbf{Q}}, \mu_R^{\mathbf{Q}}\big)$ and $\mathbf{\Sigma} = diag(\sigma_B^2, \sigma_R^2)$. The value functions are the vector $\mathbf{F}$, and the first and second derivatives of the value functions $\mathbf{F_x}$ and $\mathbf{F_{xx}}$, respectively.
Jobert and Rogers (2006) formulates the homogeneous part of Equation A.11 as a quadratic eigenvalue problem with solution:

$$\begin{aligned} F^B &= \mathbf{\Sigma}_{j=1}^4 A_j^B b_j^B e^{\gamma_j^B} \\ F^R &= \mathbf{\Sigma}_{j=1}^4 A_j^R b_j^R e^{\gamma_j^R x}, \end{aligned}$$

where b_j^i and γ_j^i are solutions to the following standard eigenvalue problem:

$$\begin{bmatrix} \mathbf{0} & \mathbb{I} \\ -2\mathbf{\Sigma}^{-1}(\mathbf{H} - \mathbf{R}) & -2\mathbf{\Sigma}^{-1}\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{d} \end{bmatrix} = \boldsymbol{\gamma} \begin{bmatrix} \mathbf{b} \\ \mathbf{d} \end{bmatrix},$$

where $\mathbb{I}$ is an 2×2 identity matrix. We follow Jobert and Rogers (2006) to obtain:

$$0 = 0.5\Sigma\gamma^2\mathrm{b} + \mathrm{M}\gamma\mathrm{b} + (\mathrm{H} - \mathrm{R})\mathrm{b},$$

which yields the Cramer-Lundberg equation

$$\begin{aligned} F^B(\gamma)F^R(\gamma) &= h_B^{\mathbf{Q}} h_R^{\mathbf{Q}} \\ \text{where} \qquad F^B(\gamma) &= \left(r_B^f + h_B^{\mathbf{Q}}\right) - \mu_B^{\mathbf{Q}}\gamma - 1/2\,\sigma_B^2\gamma^2 \\ F^R(\gamma) &= \left(r_R^f + h_R^{\mathbf{Q}}\right) - \mu_R^{\mathbf{Q}}\gamma - 1/2\,\sigma_R^2\gamma^2, \end{aligned}$$

giving rise to a solution of the fourth order equation with $-\infty < \gamma_3 < \gamma_4 < 0 < \lambda_3 < \lambda_4 < +\infty$. Since we require $\lim_{x \searrow x^i} F^i(x) = 0, \;\; i = B, R$ we eliminate the terms with negative roots $A_3^i = A_4^i = 0$ for $i = B, R$. As for the solution of $\mathbf{b}$ we have the freedom to set the first term to unity $b_1^B = b_2^B = 1$, so that we can relate A_j^R's to A_j^B's as follows:

$$l_j = \frac{A_j^R}{A_j^B} = \frac{1}{h_B^{\mathbf{Q}}}\left(h_B^{\mathbf{Q}} + r_B^f - \mu_B^{\mathbf{Q}}\gamma_j - 1/2\,\sigma_B^2\gamma_j^2\right), \qquad j = 3,4. \qquad (A.12)$$

Hence, $A_j^R = l_j A_j^B$.

In the region where $x^B \leq x < x^R$, the claim is alive only in the R-regime, and in this case it satisfies the ODE:

$$\frac{\sigma_R^2}{2}F_{xx}^R + \mu_R^{\mathbf{Q}}F_x^R + h_R^{\mathbf{Q}}(V^B - I - F^R) - r_R^f F^R = 0 \qquad (A.13)$$

The homogeneous solution to Equation A.13 is $C_1e^{\xi_1 x} + C_2e^{\xi_2 x}$, where $\xi_1 > 0$ and $\xi_2 < 0$ are the two roots of the quadratic equation $\mu_h^{\mathbf{Q}}\xi + \frac{\sigma_R^2}{2}\xi^2 = \left(r_R^f + h_R^{\mathbf{Q}}\right)$ with

$$\xi_{1,2} = -\frac{\mu_R^{\mathbf{Q}}}{\sigma_R^2} \pm \sqrt{\left(\frac{\mu_R^{\mathbf{Q}}}{\sigma_R^2}\right)^2 + \frac{2\left(r_R^f + h_R^{\mathbf{Q}}\right)}{\sigma_R^2}}. \qquad (A.14)$$

The particular solution to Equation A.13 takes the form of $C_3x + C_4$ with the solutions

$$C_3 = \frac{h_R^{\mathbf{Q}}\bar{B}_B}{r_R^f + h_R^{\mathbf{Q}}}, \quad \text{and} \quad C_4 = \frac{\mu_R^{\mathbf{Q}}C_3 + h_R^{\mathbf{Q}}(\bar{A}_B - I)}{r_R^f + h_R^{\mathbf{Q}} - \mu_R^{\mathbf{Q}}}.$$

In the investment region $x^B < x^R < x$ the value of the option across regimes:

$$F^i = V^i - I, \qquad i = B, R.$$

Finally, we summarise the solutions as follows:

$$
\begin{aligned}
F^B &= A_1^B e^{\gamma_3 x} + A_2^B e^{\gamma_4 x} && x \leq x^B \\
F^R &= A_1^R e^{\gamma_3 x} + A_2^R e^{\gamma_4 x} && x \leq x^B \\
F^B &= \bar{A}^B + \bar{B}^B x - I && x^B < x \leq x^R \\
F^R &= C_1 x^{\xi_1} + C_2 e^{\xi_2 x} + C_3 x + C_4 && x^B < x \leq x^R \\
F^B &\leq \bar{A}^B + \bar{B}^B x - I && x^R < x \\
F^R &\leq \bar{A}_R + \bar{B}^R x - I && x^R < x
\end{aligned}
$$

To determine the constants A_i, B_i, C_i (for i=1,2) we use the value-matching and smooth-pasting conditions at the threshold boundaries x^B and x^R:

$$
\begin{aligned}
\lim_{x \nearrow x^B} F^B(x) &= \bar{A}^B + \bar{B}^B x^B - I \\
\lim_{x \nearrow x^B} \partial_x F^B(x) &= \bar{B}^B \\
\lim_{x \nearrow x^R} F^R(x) &= \bar{A}^R + \bar{B}^R x^R - I \\
\lim_{x \nearrow x^R} \partial_x F^R(x) &= \bar{B}^R
\end{aligned}
$$

The solution also requires the transient boundary conditions associated with shifts from a recession to a boom regime:

$$
\begin{aligned}
\lim_{x \searrow x^B} F^R(x) &= \lim_{x \nearrow x^B} F^R(x) \\
\lim_{x \searrow x^B} \partial_x F^R(x) &= \lim_{x \nearrow x^B} \partial_x F^R(x)
\end{aligned}
$$

Recall that $F(x) = C_1 e^{\xi_1 x} + C_2 e^{\xi_2 x} + C_3 x + C_4$ gives the option value in the region $x^B \leq x < x^R$. Hence, we write the above six conditions as follows:

$$
\begin{aligned}
A_1^B e^{\gamma_3 x^B} + A_2^B e^{\gamma_4 x^B} &= \bar{A}^B + \bar{B}^B x^B - I. \\
\gamma_3 A_1^B e^{\gamma_3 x^B} + \gamma_4 A_2^B e^{\gamma_4 x^B} &= \bar{B}^B \\
C_1 e^{\xi_1 x^R} + C_2 e^{\xi_2 x^R} + C_3 x^R + C_4 &= \bar{A}^R + \bar{B}^R x^R - I \\
\xi_1 C_1 e^{\xi_1 x^R} + \xi_2 C_2 e^{\xi_2 x^R} + C_3 &= \bar{B}^R. \\
C_1 e^{\xi_1 x^B} + C_2 e^{\xi_2 x^B} + C_3 x^B + C_4 &= l_1 A_1^B e^{\gamma_3 x^B} + l_2 A_2^B e^{\gamma_4 x^B} \\
\xi_1 C_1 e^{\xi_1 x^B} + \xi_2 C_2 e^{\xi_2 x^B} + C_3 &= \gamma_3 l_1 A_1^B e^{\gamma_3 x^B} + \gamma_4 l_2 A_2^B e^{\gamma_4 x^B}
\end{aligned}
$$

A more stable way of solving this system of equations (Guo and Zhang (2004) and Bensoussan et al. (2012)) is to pursue a 2-by-2 procedure. We obtain A_1 and A_2 from the first pair of equations; C_1 and C_2 from the second pair and finally x^B and x^R from the last one (see Arnold et al. (2013) for an alternative approach.) QED.

6.3. Robustness Checks: Estimation Results from Alternative Statistical Packages

R Code	AIC	BIC	logLik
GBM	144.61	159.76	-70.30
ABM	1133.16	1148.31	-554.58

(a) R Code

Stata	AIC	BIC	logLik
GBM	1.43	1.49	-80.04
ABM	9.51	9.65	-564.58

(b) Stata

Statmodels	AIC	BIC	logLik
GBM	152.84	169.57	-70.42
ABM	1140.41	1157.13	-564.20

(c) Statmodels

Table 2. Predictive Power of the GBM and ABM Regime-Switching Real Options Models

	$\mu_1^{\mathbf{Q}}$	$\mu_2^{\mathbf{Q}}$	σ_1	σ_2	p_{11}	p_{22}
GBM	0.350	0.203	0.254	0.824	0.937	0.850
ABM	4.950	-0.686	20.230	97.162	0.987	0.839

(a) R Code

	$\mu_1^{\mathbf{Q}}$	$\mu_2^{\mathbf{Q}}$	σ_1	σ_2	p_{11}	p_{22}
GBM	0.350	0.207	0.063	0.663	0.925	0.863
ABM	5.026	-0.982	404.154	9110.400	0.978	0.889

(b) Statmodels

	$\mu_1^{\mathbf{Q}}$	$\mu_2^{\mathbf{Q}}$	σ_1	σ_2	p_{11}	p_{22}
GBM	0.423	-0.569	0.400	0.469	0.956	0.673
ABM	7.375	-9.012	17.779	80.083	0.956	0.837

(c) Stata

Table 3. Parameter Estimates of the GBM and ABM Regime-Switching Real Options Models

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