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SECOND MODULES RELATIVE TO SUBCLASSES OF PRERADICALS OF R-MOD

Luis Fernando García Mora and Hugo Alberto Rincón-Mejía

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Abstract. We study the concept of second module and extend it to more general environments. We also provide descriptions of simple left semiartinian, left local rings, semisimple and simple rings in terms of their $\mathscr{A}$-second modules with respect to a preradical class.

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1. Introduction

In recent times, some concepts associated with primality for R-modules have been introduced and studied, providing valuable insights into various mathematical elements. Of particular interest is the dual notion of prime submodule, introduced by S. Yassemi in [10] and further extended to lattice theory by J. Abuhlail and H. Hroub in [1]. A nonzero submodule N of ${}_RM$ is second if $IN = 0$ or $IN = N$ for each I ideal of R. This is equivalent to $Ann_R(N) = Ann_R(N/K)$ for every proper submodule K of N, where $Ann_R(M)$ is the left annihilator of M in R, see Remark 2.4. On the other hand, the lattice of preradicals in R-*Mod* has proven been highly useful in characterizing and describing different types of rings and modules, for examples see Theorems 9, 11 and 13 of [6].

In this paper, we define and study two extensions of the concept of second module. One involves actions of a subclass $\mathscr{A}$ of R-**pr** on a lattice of submodules of a module, which allows us to introduce $\mathscr{A}$-second modules. The other method involves the R-**pr**-annihilator of a module, which allows us to introduce strongly second modules.

In Sections 2 and 3, we introduce the fundamental concepts required to define R-**pr**-second modules, $\mathscr{A}$-second modules and strongly second modules. In Section 4, we define R-**pr**-second modules and give examples of R-**pr**-seconds modules. In Section 5, we characterize the R-**pr**-second modules over a principal ideal domain, perfect rings and left semiartinian rings. In Section 6, we prove that each R-**pr**-second module is a second module, and we give a counterexample for the converse in Example 6.3. In Section 7, we characterize the R-*id*-second modules and R-*rad*-second modules, where R-*id* is the class of idempotent preradicals in R-*Mod* and R-*rad* is the class of radicals in R-*Mod*. In Section 8, we prove that each strongly second module is a second module, we give a counterexample for the converse in Example 8.11 and characterize rings in which each module is strongly second. In Section 9, we characterize the rings for which each module is R-*id*-second, the rings for which each module is R-*rad*-second and the rings for which each module is R-**pr**-second.

2. Preliminaries

Throughout this paper, all rings will be assumed to possess an identity element. Every simple R-module can be represented by a module of the form R/I, where I is a maximal left ideal of R. We can pick a collection of representatives of the isomorphic classes of simple modules, which we will refer to as R-*simp*. Recall that a left R-module M is semisimple if each of its left submodules is a direct summand. This happens precisely when M coincides with the sum of its simple left submodules, which is called the socle of M and is denoted $soc(M)$. Moreover, a left semisimple R-module M is homogeneous if any two simple submodules S and S' of M are isomorphic.

Recall that a ring R is a left V-ring if every simple left R-module is injective. Moreover, R is a left semiartinian ring if every nonzero R-module has a nonzero socle. When every two simple left R-modules are isomorphic, R is a left local ring. A ring R is a semisimple ring when it is semisimple when viewed as a module over itself. A ring R is a left perfect ring if every left R-module has a projective cover.

The lattice of submodules of a module. The category of left R-modules is denoted by R-*Mod*. A preradical σ on R-Mod is a functor σ : R-*Mod* $\to$ R-*Mod* such that:

- $\sigma(M) \leq M$ for each $M \in R\text{-}Mod$.
- For each R-morphism $f\colon M \to N$, the following diagram is commutative:

$$\begin{tikzcd} \sigma(M) \arrow[r, "f\restriction"] \arrow[d, hook] & \sigma(N) \arrow[d, hook] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

R-**pr** denotes the collection of all preradicals in R-*Mod*. Recall that for each $\beta \in R$-**pr** and each family of R-modules $\{M_i\}_{i\in I}$, we have $\beta(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} \beta(M_i)$, see Proposition I.1.2 of [3]. Recall that $\sigma \in R$-**pr** is idempotent if $\sigma \circ \sigma = \sigma$. σ is radical if $\sigma(M/\sigma(M)) = 0$, for each $M \in R$-Mod. σ is a left exact preradical if it is a left exact functor. σ is t-radical if $\sigma(M) = \sigma(R)M$. Note that σ is a t-radical if and only if σ preserves epimorphisms; and σ is a left exact preradical if and only if for each submodule N of a module M we have $\sigma(N) = \sigma(M) \cap N$. We will denote R-*id*, R-*rad*, R-*lep*, and R-*ler*, the collections of idempotent preradicals, radicals, left exact preradicals, and left exact radicals, respectively.

If ${}_RM$ is a left R-module, it is well known that the set of submodules of M, which we shall denote $L({}_RM)$, is a complete lattice, where the supremum and infimum of a family $\{N_i\}_{i\in I}$ of submodules of ${}_RM$ are

$$\sum_{i\in I} N_i = \left\{ \sum_{i=1}^{n} x_i \mid n \in \mathbb{N},\ x_j \in N_j \ \textit{and}\ j \in I \right\} \text{ and } \bigcap_{i\in I} N_i,$$

respectively. See Proposition 2.5 of [2].

Preradicals. We introduce the basic definitions and results of preradicals in R-Mod. For more information on preradicals, see [3], [6], [7], [8], and [9]. By Theorem 7 of [6], R-**pr** is a big lattice where

(1) the order in R-**pr** is given by $\alpha \preceq \beta$ if $\alpha(M) \leq \beta(M)$ for every $M \in R$-*Mod*;
(2) for any family of preradicals $\{\sigma_i\}_{i\in I}$ in R-*Mod* the supremum and infimum for the family are given respectively by
   - $(\bigvee_{i\in I} \sigma_i)(M) = \sum_{i\in I} \sigma_i(M)$, and
   - $(\bigwedge_{i\in I} \sigma_i)(M) = \bigcap_{i\in I} \sigma_i(M)$.

We say that a submodule ${}_RN$ of ${}_RM$ is fully invariant in M if $f(N) \leq N$ for all $f \in End({}_RM)$. It is easy to see that the set of fully invariant submodules of ${}_RM$ is a complete sublattice of $Lat(M)$. We will denote $Lat_{f.i.}(M)$, the lattice of fully invariant submodules of M, and $N_{f.i.} \leq_R M$ will mean that N is a fully invariant submodule of M.

Recall for a fully invariant submodule N_R of ${}_RM$ the preradicals α_N^M and ω_N^M were defined in Definition 4 of [6].

Definition 2.1. Let N_R be a fully invariant submodule of M_R and $U \in R$-*Mod*.

- $\alpha_N^M(U) = \sum\{f(N) | f \in Hom_R(M, U)\}$ and
- $\omega_N^M(U) = \bigcap\{f^{-1}(N) | f \in Hom_R(U, M)\}$.

Remark 2.2. Let M be an R-module and let N be a fully invariant submodule of M. By Proposition 5 of [6], $\alpha_N^M : R\text{-}Mod \to R\text{-}Mod$ is the least preradical ρ such that $\rho(M) = N$, and $\omega_N^M : R\text{-}Mod \to R\text{-}Mod$ is the largest preradical ρ such that $\rho(M) = N$. It is easy to see that $\{\rho \in R\text{-}\mathbf{pr} \mid \rho(M) = N\} = [\alpha_N^M, \omega_N^M]$, an interval in R-pr.

We can see that $\alpha_M^M(L)$ is the trace of M in L, that is, $\alpha_M^M = tr_M$. Furthermore, $\omega_0^M(L)$ is the reject of M in L, and is the smallest submodule of L such that $L/\omega_0^M(L)$ embeds in a product of copies of M.

It should be noted that $soc : R\text{-}Mod \to R\text{-}Mod$ is a preradical which is idempotent, and $soc(M)$ is the biggest semisimple submodule of M. Additionally, $soc = \vee\{\alpha_S^S \mid S \in R\text{-}simp\}$.

Second modules. We will begin by recalling the concepts defined in module theory as an introduction to the concepts that will be presented in more general terms.

In [1], the authors define a second module as follows.

Definition 2.3. An R-module M is second if, for every I ideal of R, $IM = 0$ or $IM = M$.

Remark 2.4. The following statements are equivalent for a left R-module ${}_RM$:

(1) For each ideal I of R, $IM = 0$ or $IM = M$.
(2) $Ann_R(M) = Ann_R(M/N)$ for all $N \lneq M$.

Proof. If $IM \lneq M$ for some ideal I, then $I \leq Ann_R(M/IM) = Ann_R(M)$. Therefore, $IM = 0$.

On the other hand, assume that $Ann_R(M) = Ann_R(M/N)$ for all $N \lneqq M$. If $IM \lneqq M$, then as $I \leq Ann_R(M/IM) = Ann_R(M)$, then $IM = 0$. □

In [4], the authors give the following definition, which is equivalent to Definition 2.3, by Remark 2.4.

Definition 2.5. An R-module M is second if $Ann_R(M) = Ann_R(M/N)$ for all $N \lneq M$.

3. $\mathscr{P}$-second elements of a lattice

We use the concepts of actions of partial orders in lattices, and the notion of second elements of a lattice, which were introduced in [1]. We will study some instances of these actions. As the lattices are particular kinds of posets (short for partially ordered sets), we can explore the actions of lattices in other lattices as particular cases of the above definition. We examine in particular $Lat(M)$, the lattice of left R-submodules of ${}_RM$, R-**pr**, the lattice of preradicals in R-mod, and

$Lat_{fi}(M)$, the lattice of fully invariant submodules of M. All of these are examples of lattices that can either act on or be acted upon by other posets. We distinguish $Lat({}_\bullet R)$, the lattice of left ideals of R, from $Lat(R)$, which will denote the lattice of two-sided ideals of R-mod.

Definition 3.1. A lattice $\mathscr{L}$ is bounded if it contains elements 0 and 1 such that all elements x in $\mathscr{L}$ are between 0 and 1, with 0 being the smallest element and 1 the greatest element.

Definition 3.2. Let $\mathscr{L} = (L, \leq, \vee, \wedge)$ be a lattice, and let $\mathscr{P} = (P, \leq')$ be a poset. A $\mathscr{P}$-action on $\mathscr{L}$ is a function $\rightharpoonup : \mathscr{P} \times \mathscr{L} \to \mathscr{L}$ satisfying the following conditions for all $s, t \in P$ and $x, y \in L$:

(1) $s \leq' t \Rightarrow s \rightharpoonup x \leq t \rightharpoonup x$.
(2) $x \leq y \Rightarrow s \rightharpoonup x \leq s \rightharpoonup y$.
(3) $s \rightharpoonup x \leq x$.

Example 3.3. The lattice $Lat(R)$ acts on $Lat(M)$.

$$Lat(R) \times Lat(M) \xrightarrow{\rightharpoonup} Lat(M)$$

$$(I, N) \longmapsto IN,$$

Example 3.4. The lattice R-**pr** acts on $Lat(M)$.

$$R\text{-}\mathbf{pr} \times Lat(M) \xrightarrow{\rightharpoonup} Lat(M)$$

$$(r, N) \longmapsto r(N),$$

Example 3.5. The lattice R-pr acts on R-pr by composition.

$$R\text{-}\mathbf{pr} \times R\text{-}\mathbf{pr} \xrightarrow{\circ} Lat(M)$$

$$(\rho, \sigma) \longmapsto \rho \circ \sigma.$$

Remark 3.6. Let $\mathscr{P} = (P, \leq')$ be a poset, $\mathscr{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathscr{P}$-action $\rightharpoonup : \mathscr{P} \times \mathscr{L} \to \mathscr{L}$ and $\mathscr{Q}$ be a suborder of $\mathscr{P}$. Then, the restriction of $\rightharpoonup$ to $\mathscr{Q} \times \mathscr{L}$ is a $\mathscr{Q}$-action.

Definition 3.7. Let $\mathscr{P} = (P, \leq')$ be a poset, $\mathscr{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with an $\mathscr{P}$-action $\rightharpoonup : \mathscr{P} \times \mathscr{L} \to \mathscr{L}$ and $x \in L \setminus \{0\}$. We say that x is $\mathscr{P}$-second if for every $s \in P$:

$$s \rightharpoonup x = 0 \text{ or } s \rightharpoonup x = x.$$

When this definition is applied to the action described in Example 3.3, the usual definition of second modules is obtained, see Definition 2.3.

We omit the easy proof of the following proposition.

Proposition 3.8. *Each order-preserving function $f\colon \mathscr{P} \to \mathscr{Q}$ induces a correspondence between $\mathscr{Q}$-actions on $\mathscr{L}$ and $\mathscr{P}$-actions on $\mathscr{L}$. Explicitly, $\mathscr{Q} \times \mathscr{L} \xrightarrow{\rightharpoonup} \mathscr{L} \mapsto \mathscr{P} \times \mathscr{L} \xrightarrow{\rightharpoonup_f} \mathscr{L}$, where $a \rightharpoonup_f x := f(a) \rightharpoonup x$.*

Lemma 3.9. *Let $\mathscr{P} = (P, \leq')$ be a poset, $\mathscr{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathscr{P}$-action $\rightharpoonup\colon \mathscr{P} \times \mathscr{L} \to \mathscr{L}$ and let $f : \mathscr{Q} \to \mathscr{P}$ be an order preserving function. Then any $\mathscr{P}$-second respect $\rightharpoonup$ element of $\mathscr{L}$ is $\mathscr{Q}$-second respect $\rightharpoonup_f$.*

In particular, note that we can restrict a $\mathscr{P}$-action in $\mathscr{L}$ to a subset $\mathscr{Q}$ of $\mathscr{P}$. It is easy to prove the following corollary.

Corollary 3.10. *Let $\mathscr{P} = (P, \leq')$ be a poset, $\mathscr{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathscr{P}$-action $\rightharpoonup\colon \mathscr{P} \times \mathscr{L} \to \mathscr{L}$ and let $\mathscr{Q}$ be a subset of $\mathscr{P}$. Then any $\mathscr{P}$-second element of $\mathscr{L}$ is $\mathscr{Q}$-second.*

Definition 3.11. Let $\mathscr{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and $y, x \in L$ such that $y \leq x$. We denote $[y, x]$ the bounded lattice of all $z \in L$ such that $y \leq z \leq x$.

Lemma 3.12. *Let $\mathscr{P} = (P, \leq')$ be a poset, $\mathscr{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathscr{P}$-action $\rightharpoonup\colon \mathscr{P} \times \mathscr{L} \to \mathscr{L}$. All atoms of $\mathscr{L}$ are $\mathscr{P}$-second.*

Proof. Let a be an atom of $\mathscr{L}$. Let $s \in \mathscr{P}$ and $z \leq x$. As $s \rightharpoonup a \leq a$ and a is an atom, $s \rightharpoonup a = 0$ or $s \rightharpoonup a = a$. Therefore, a is $\mathscr{P}$-second. $\square$

4. R-pr-second modules

We now apply Definition 3.7 to the action described in Example 3.4 to define second modules with respect to preradicals.

Definition 4.1. Let M be an R-module and $N \in \mathscr{L}(M) \setminus \{0\}$. We say that N is an R-**pr**-second submodule of M if for every $\alpha \in R$-**pr**,

$$\alpha(N) = 0 \text{ or } \alpha(N) = N.$$

Example 4.2. Let $p, n \in \mathbb{N}$ such that p is a prime and $p|n$. $(n/p)\mathbb{Z}_n$ is a $\mathbb{Z}$-**pr**-second submodule of $\mathbb{Z}_n$, due to the Lemma 3.12.

Lemma 4.3. *Let M be an R-module. Every simple submodule of M is an R-**pr**-second submodule of M.*

Proof. It follows from Lemma 3.12. $\square$

Notice that for any $S, M \in R$-Mod with S simple, $\alpha_S^S(M) = \sum\{f(S) \mid f\colon S \to M\}$ is a semisimple homogeneous R-module.

Lemma 4.4. *Let M be an R-module and N be a submodule of M with nonzero socle. Then N is an R-**pr**-second submodule of M iff $N \cong S^{(I)}$ for some set I and for some simple R-module S.*

Proof. Assume first that $N \cong S^{(I)}$, with S a simple left R-module, and let σ be a preradical. As a preradical commutes with coproducts, $\sigma(S^{(I)}) = ((\sigma(S))^{(I)}$, but this is $_R0$ or $S^{(I)}$, depending on whether $\sigma(S)$ is $_R0$ or S.

Now, assume that N has a nonzero socle, then there exists $S \in R$-*simp* which embeds in N. As there is a nonzero morphism $f\colon S \to N$, then $\alpha_S^S(N) \neq 0$, so $\alpha_S^S(N) = N$, which implies that $N \cong S^{(I)}$ for some set I. □

Remark 4.5. We have that if N is a submodule of M, then N is R-**pr**-second in $\mathscr{L}(M)$ if and only if N is R-**pr**-second in $\mathscr{L}(N)$. In this sense, we will say that an R-module M is R-**pr**-second if it is R-**pr**-second as a submodule of itself.

Proposition 4.6. *The following conditions are equivalent for a nonzero module $_RM$.*

(1) *M is R-**pr**-second.*
(2) *The only fully invariant submodules of $_RM$ are 0 and M.*

Proof. (1) $\Rightarrow$ (2) Let N be a nonzero fully invariant submodule of $_RM$. Then $\omega_N^M \in R$-**pr** and $\omega_N^M(M) = N \neq 0$, so $N = \omega_N^M(M) = M$, as M is R-**pr**-second. Therefore, the only fully invariant submodules of $_RM$ are 0 and M.

(2) $\Rightarrow$ (1) Let $\sigma \in R$-**pr**. Then $\sigma(M)$ is a fully invariant submodule of M, so $\sigma(M) = M$ or $\sigma(M) = 0$. Therefore, M is R-**pr** second. □

Proposition 4.7. *Let R be a ring and I be a two-sided ideal of R. Then for each $\sigma \in R$-**pr**, $\sigma(R/I)$ is a two-sided ideal of R/I.*

Proof. The left R-submodule $\sigma(R/I)$ of R/I is annihilated by I, thus making it a left ideal of R/I. Now, for each $a \in R$, the function $\_ \cdot (a+I)\colon R/I \to R/I$ given by $\_ \cdot (a+I)[b+I] = (b+I)(a+I)$ is an R-morphism, so the following diagram commutes:

$$\begin{tikzcd} \sigma(R/I) \arrow[r, "\_\cdot(a+I)\restriction"] \arrow[d, hook] & \sigma(R/I) \arrow[d, hook] \\ R/I \arrow[r, "\_\cdot(a+I)"] & R/I. \end{tikzcd}$$

Then $\_ \cdot (a+I)(\sigma(R/I)) \subseteq \sigma(R/I)$, so $\sigma(R/I)$ is a right ideal of R/I. Therefore $\sigma(R/I)$ is a two-sided ideal of R/I. □

A consequence of the Correspondence Theorem for rings, of Proposition 4.6, and Proposition 4.7 is the following corollary.

Corollary 4.8. *Let R be a ring and I be a maximal two-sided ideal of R. Then R/I is an R-**pr**-second R-module.*

Corollary 4.9. *Let R be a ring. Then R is an R-**pr**-second left R-module if and only if R is a simple ring.*

Example 4.10. ${}_{\mathbb{Z}}\mathbb{Q}$ is $\mathbb{Z}$-**pr**-second. Let $\sigma \in \mathbb{Z}$-**pr** such that $\sigma(\mathbb{Q}) \neq 0$. Then there exists $0 \neq x \in \sigma(\mathbb{Q})$. Let us take the $\mathbb{Z}$-monomorphism $f\colon \mathbb{Z} \to \mathbb{Q}$ such that $f(1) = x$.
If $\sigma(\mathbb{Q}) \neq \mathbb{Q}$, there exists $0 \neq y \in \mathbb{Q} \setminus \sigma(\mathbb{Q})$. Then take the $\mathbb{Z}$-morphism $g\colon \mathbb{Z} \to \mathbb{Q}$ such that $g(1) = y$. As $\mathbb{Q}$ is a $\mathbb{Z}$-injective module, there exists a $\mathbb{Z}$-morphism $\overline{f}\colon \mathbb{Q} \to \mathbb{Q}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f"] \arrow[d, "g"] & \mathbb{Q} \arrow[dl, "\overline{f}"] \\ \mathbb{Q}. & \end{tikzcd}$$

As σ is a preradical, $\overline{f}(\sigma(\mathbb{Q})) \subseteq \sigma(\mathbb{Q})$. But we have that $x \in \sigma(\mathbb{Q})$ and $\overline{f}(x) = \overline{f}(f(1)) = g(1) = y \notin \sigma(\mathbb{Q})$, a contradiction. Then $\sigma(\mathbb{Q}) = \mathbb{Q}$. Therefore, ${}_{\mathbb{Z}}\mathbb{Q}$ is $\mathbb{Z}$-**pr**-second.

Proposition 4.11. *If ${}_RM$ is an R-**pr**-second module, then ${}_RM$ is generated by each of its nonzero submodules and cogenerated by its nonzero quotients.*

Proof. If $0 \neq N \leq M$, then $tr_N(M) \neq 0$, hence $tr_N(M) = M$, thus M is generated by N.

Now, if $0 \neq M/N$, then $\omega_0^{M/N}(M) \leq N \neq M$, thus $\omega_0^{M/N}(M) = 0$. This is equivalent to saying that M is cogenerated by M/N. □

5. R-pr-second modules over special rings

Recall that for a principal ideal domain R, an R-module M is injective if and only if M is a divisible R-module. Recall also that each nonzero cyclic torsion module has a simple submodule because each torsion element has a nonzero multiple annihilated by a prime element. The following proposition characterizes the R-**pr**-second modules when R is a principal ideal domain.

Proposition 5.1. *Let R be a principal ideal domain and M be a nonzero R-module. The following statements are equivalent:*

(1) *M is R-**pr**-second.*

(2) *M is a semisimple homogeneous module or M is a divisible torsion-free module.*

Proof. (1) $\Rightarrow$ (2) Let M be an R-**pr**-second module. Then $soc(M) = M$ or $soc(M) = 0$. If $soc(M) = M$, then M is a semisimple module, so M is a semisimple homogeneous module according to Lemma 4.4. On the other hand, if $soc(M) = 0$, then M is a torsion-free module, so $IM \neq 0$ for each $0 \neq I \leq R$. Moreover, as M is R-**pr**-second, then $IM = M$ for each $0 \neq I \leq R$, so M is a divisible torsion-free module.

(2) $\Rightarrow$ (1) It is clear that a homogeneous semisimple module is R-**pr**-second, because a preradical commutes with coproducts, and a simple module is R-**pr**-second. Assume that M is a nonzero divisible torsion-free module and take a preradical σ. If $\sigma(M) \neq 0$, let us take $0 \neq x \in \sigma(M)$, and the R-monomorphism $f\colon R \to M$ such that $f(1) = x$.

If $\sigma(M) \neq M$, let us take $0 \neq y \in M \setminus \sigma(M)$, and the R-monomorphism $g\colon R \to M$ such that $g(1) = y$. Now, as M is divisible it is an injective R-module, so there exists an R-morphism $\overline{f}\colon M \to M$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, "\overline{f}"] \\ M. & \end{tikzcd}$$

Then $\overline{f}(x) = \overline{f}(f(1)) = g(1) = y \notin \sigma(M)$ but $x \in \sigma(M)$, thus $\overline{f}(\sigma(M)) \nsubseteq \sigma(M)$, a contradiction. Then $\sigma(M) = M$. Therefore M is R-**pr**-second. □

Proposition 5.2. *Let R be a left perfect ring and M be a nonzero R-module. The following statements are equivalent:*

(1) *M is R-**pr**-second.*

(2) *M is a semisimple homogeneous R-module.*

Proof. (1) $\Rightarrow$ (2) Let M be a nonzero R-**pr**-second module. As R is a left perfect ring, we have that $rad(R)M = rad(M) \neq M$ by 9.7.3(a) [2], so $rad(R)M = 0$ which implies that M is an $R/rad(R)$-module. Moreover, $R/rad(R)$ is semisimple, so M is a semisimple $R/rad(R)$-module which implies that M is semisimple. Therefore M is a semisimple homogeneous R-module by Lemma 4.4.

(2) $\Rightarrow$ (1) It follows from Lemma 4.4. □

Proposition 5.3. *Let R be a left semiartinian ring and M be a nonzero R-module. The following statements are equivalent:*

(1) *M is R-**pr**-second.*
(2) *M is a semisimple homogeneous R-module.*

Proof. (1) $\Rightarrow$ (2) Let M be an R-**pr**-second module. As R is a left semiartinian ring, we have that $soc(M) \neq 0$, then $soc(M) = M$, so M is a semisimple R-**pr**-second R-module. Therefore M is a semisimple homogeneous R-module by Lemma 4.4.

(2) $\Rightarrow$ (1) It follows from Lemma 4.4. $\square$

6. R-pr-second modules vs. second modules

We can contrast the concepts of second module and R-**pr**-second module for a clearer understanding. Recall that each t-radical σ can be described as $\sigma(R) \bullet (-)$, and recall that for each preradical σ, $\sigma(R)$ is a two-sided ideal of R. Thus, there is a lattice isomorphism between $t - Rad$ and $Lat(R)$, sending σ to $\sigma(R)$, see Corollary I.2.11 [3]. So $Lat(R)$ embeds in R-**pr** as a partially ordered set and induces the action described in Example 3.3. Consequently, by Lemma 3.9, the following proposition holds.

Proposition 6.1. *Let M be a left R-module. If M is a R-**pr**-second module, then M is a second module.*

Theorem 6.2. *For a ring R, the following statements are equivalent:*

(1) *R is a simple ring.*
(2) *Any R-module is second.*

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) Let I be an ideal of R. As R is second, $IR = 0$ or $IR = R$, then $I = 0$ or $I = R$. Therefore R is a simple ring. $\square$

Example 6.3. Let F be a field and let R be the ring of linear endomorphisms of the F vector space $F^{(\mathbb{N})}$. Consider $I = \{f \in R \mid rank(f) \in \mathbb{N}\}$. If $f, g \in I$, $h \in R$ and $0 \in R$, then $rank(0) = 0$, $rank(f+g) \leq rank(f) + rank(g)$, $rank(hf) \leq rank(f)$, and $rank(fh) \leq rank(f)$ so $rank(f+g), rank(hf), rank(fh) \in \mathbb{N}$, from which it follows that I is an ideal of R. If $f \in R \setminus I$, then $f(F^{(\mathbb{N})})$ has infinite dimension, thus there exists $\varphi\colon F^{(\mathbb{N})} \to f(F^{(\mathbb{N})})$ an F-isomorphism. Moreover, as every short exact sequence of vector spaces splits, in the exact sequence

$$0 \longrightarrow Ker(f) \hookrightarrow F^{(\mathbb{N})} \xrightarrow{\overline{f}=f|^{f(F^{(\mathbb{N})})}} f(F^{(\mathbb{N})}) \longrightarrow 0,$$

there exists $g\colon f(F^{(\mathbb{N})}) \to F^{(\mathbb{N})}$, an F-morphism such that $\overline{f}g = Id_{f(F^{(\mathbb{N})})}$ and thus there exists an F-subspace W of $F^{(\mathbb{N})}$ such that $F^{(\mathbb{N})} = W \bigoplus f(F^{(\mathbb{N})})$. Then, the following diagram commutes:

$$\begin{tikzcd}
f(F^{(\mathbb{N})}) \arrow[r, "g"] & F^{(\mathbb{N})} \arrow[r, "\overline{f}"] & f(F^{(\mathbb{N})}) \arrow[r, hook, "\iota"] \arrow[d, "\varphi^{-1}"] & F^{(\mathbb{N})} = W \bigoplus f(F^{(\mathbb{N})}) \arrow[dl, "0\oplus\varphi^{-1}"] \\
F^{(\mathbb{N})} \arrow[u, "\varphi"] \arrow[rr, "Id_{F^{(\mathbb{N})}}"] & & F^{(\mathbb{N})}, &
\end{tikzcd}$$

where $\iota\colon f(F^{(\mathbb{N})}) \to F^{(\mathbb{N})}$ is the canonical inclusion. Moreover, $0 \oplus \varphi^{-1}, g\varphi$ are linear endomorphism of $F^{(\mathbb{N})}$, then

$$Id_{F^{(\mathbb{N})}} = \varphi^{-1}\overline{f}g\varphi = ((0 \oplus \varphi^{-1})\iota)\overline{f}(g\varphi) = (0 \oplus \varphi^{-1})(\iota\overline{f})(g\varphi) = (0 \oplus \varphi^{-1})f(g\varphi) \in J,$$

where J is the ideal generated by f. Then $I + J = R$, therefore, I is a maximal ideal of R, so $S = R/I$ is a simple ring.

Now, we consider $S = R/I$. Let β be a base of $F^{(\mathbb{N})}$. Then there exists $\{\beta_i\}_{i\in\mathbb{N}} \subseteq \wp(\beta)$ such that $\beta = \overset{\circ}{\bigcup\limits_{i\in\mathbb{N}}} \beta_i$ and $|\beta_i| = |\beta|$, then ${}_F F^{(\mathbb{N})} = \bigoplus\limits_{i\in\mathbb{N}} W_i$, where $W_i ={}_F \langle\beta_i\rangle$.

Let $\pi_j : F^{(\mathbb{N})} \to \bigoplus\limits_{i=1}^{j} W_i$ denote the canonical projection, and $\iota_j : \bigoplus\limits_{i=1}^{j} W_i \to F^{(\mathbb{N})}$, the canonical inclusion. Let us define $f_j = \iota_j\pi_j$ and $I_j = Ann_R(f_j)$ for any $j \in \mathbb{N}$. For all $j, l \in \mathbb{N}$, if $j < l$, then $\bigoplus\limits_{i=1}^{j} W_i \lneq \bigoplus\limits_{i=1}^{l} W_i$, so that $I_l \leq I_j$, from which it follows that $(I_l + I)/I \leq (I_j + I)/I$.

Now, let $M_j = \bigoplus\{W_i | j < i\}$, $\overline{\pi}_j : F^{(\mathbb{N})} \to M_j$ denote the canonical projection, $\overline{\iota}_j : M_j \to F^{(\mathbb{N})}$, the canonical inclusion, and $g_j = \overline{\iota}_j\overline{\pi}_j$. We have that for any $i, j \in \mathbb{N}$, if $j < i$, $g_j + I \in (I_j + I)/I$ but $g_j + I \notin (I_i + I)/I$, so that $\{(I_i + I)/I\}_{i\in\mathbb{N}}$ is an infinite properly descending chain of submodules of S.

Thus S is not a left artinian ring. So, S is not a left semisimple ring, then $soc(R) \neq R$.

Let ${}_S M = S \oplus K$ with $K \in S - simp$. We have that $soc(M) \neq 0$ and $soc(M) \neq M$. Since $soc(K) = K \neq 0$ and $soc(S) \neq S$, ${}_S M$ is not an S-**pr**-second module. On the other hand, as S is a simple ring, any R-module is second by Theorem 6.2, in particular ${}_S M$ is a second module but it is not an S-**pr**-second module.

Hence, the fact that module ${}_R M$ is second does not necessarily mean that it is R-**pr**-second.

Remark 6.4. By Example 6.3, the converse of Proposition 6.1 is not generally true.

Note that any simple but not semisimple ring R would also work in the above example.

7. Second modules relative to subclasses of R-pr

Let $\mathscr{A} \subseteq R$-**pr**. We can consider the action of $\mathscr{A}$ on the lattice of submodules of a module induced by the inclusion of $\mathscr{A}$ in R-**pr** and the R-**pr** action described in Example 3.4, as in Lemma 3.8, for the following definition:

Definition 7.1. Let M be a nonzero R-module and $\mathscr{A} \subseteq R$-**pr**. We say that M is $\mathscr{A}$-second if for every $\alpha \in \mathscr{A}$:

$$\alpha(M) = 0 \text{ or } \alpha(M) = M.$$

Recall that given $\sigma \in R$-**pr**, $\mathbb{T}_\sigma$ is the class of σ-pretorsion modules and $\mathbb{F}_\sigma$ is the class of σ-pretorsion-free modules. Then, for $\mathscr{A} \subseteq R$-**pr**, we define $\mathbb{T}_{\mathscr{A}} := \bigcap_{r \in \mathscr{A}} \mathbb{T}_r$ and $\mathbb{F}_{\mathscr{A}} := \bigcap_{r \in \mathscr{A}} \mathbb{F}_r$. On the other hand, for $\mathscr{A} \subseteq R$-**pr**, we will denote as $\mathbb{S}_{\mathscr{A}}$ the class of all $\mathscr{A}$-second modules and the zero module. In particular, for $\sigma \in R$-**pr**, we will use $\mathbb{S}_\sigma$ instead of $\mathbb{S}_{\{\sigma\}}$.

Remark 7.2. For any $\sigma \in R$-**pr** and $\mathscr{A} \subseteq R$-**pr**, we have that:

(1) $\mathbb{S}_{\mathscr{A}} = \bigcap_{r \in \mathscr{A}} \mathbb{S}_r$.
(2) $\mathbb{S}_\sigma = \mathbb{T}_\sigma \cup \mathbb{F}_\sigma$.
(3) $\mathbb{T}_{\mathscr{A}} \subseteq \mathbb{S}_{\mathscr{A}}$.
(4) $\mathbb{F}_{\mathscr{A}} \subseteq \mathbb{S}_{\mathscr{A}}$.

Remark 7.3. The R-*ler*-second modules are the decisive modules defined by Golan in [5], Chapter 31.

Proof. By I.5.E2 of [3], for each hereditary torsion theory $(\mathbb{T}, \mathbb{F})$, there exists a left exact radical $\sigma \in R$-**pr** such that $(\mathbb{T}, \mathbb{F}) = (\mathbb{T}_\sigma, \mathbb{F}_\sigma)$. Now, ${}_RM$ is decisive if and only if $M \in \mathbb{T}$ or $M \in \mathbb{F}$ for each hereditary torsion theory $(\mathbb{T}, \mathbb{F})$, which, as mentioned earlier, is equivalent to $\sigma(M) = M$ or $\sigma(M) = 0$ for each $\sigma \in R$-*ler*. $\square$

Example 7.4. Let $R = \mathbb{Z}$, $p \in \mathbb{N}$ be a prime, $\sigma = \alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}$ and $N \in R$-*Mod*. Then we have that $N \in \mathbb{T}_\sigma$ if and only if N is a semisimple homogeneous group whose elements have order p, since $\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}(N)$ is the largest homogeneous semisimple subgroup of M whose elements have order p.

Now, if $N \in R$-Mod has an element a of order p, then $K = \langle a \rangle \cong \mathbb{Z}_p$, so $\sigma(N) \neq 0$. On the other hand, if $\sigma(N) \neq 0$, then there exists $f \in Hom_R(\mathbb{Z}_p, N) \setminus \{0\}$, so $f(1 + p\mathbb{Z})$ is an element of order p of N. Now, we have that $N \in \mathbb{F}_\sigma$ if and only if N has no elements of order p.

Thus $N \in \mathbb{S}_\sigma$ if and only if N has no elements of order p or N is a semisimple homogeneous group whose elements have order p.

Example 7.5. Let $R=\mathbb{Z}$, $\mathscr{A}=\{\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p} | p \text{ is prime}\}$ and $N \in R$-Mod. Then we have $N \in \mathbb{T}_{\mathscr{A}}$ if and only if $N=0$, since from Remark 7.2(1) and the previous example we have N has to be a p-homogeneous semisimple group for all p prime, so N has to be 0.

From Remark 7.2, and the above example, we have that $N \in \mathbb{F}_{\mathscr{A}}$ if and only if N has no elements of order p for all prime p, whence it follows that $N \in \mathbb{F}_{\mathscr{A}}$ if and only if N has no nontrivial elements of finite order, that is N is a torsion-free abelian group.

If $M \in \mathbb{S}_{\mathscr{A}}$ and $M \notin \mathbb{F}_{\mathscr{A}}$, then there exists $p \in \mathbb{Z}$ prime such that $M \in \mathbb{T}_\sigma$ where $\sigma=\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}$, so $M=\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}(M)$, therefore M is a semisimple homogeneous $\mathbb{Z}$-module. Furthermore, if $M \in \mathbb{F}_{\mathscr{A}}$, then $M \in \mathbb{S}_{\mathscr{A}}$ by Remark 7.2(4); and if M is a nonzero semisimple homogeneous $\mathbb{Z}$-module, then M is an $\mathscr{A}$-second module by Lemma 4.4 and Lemma 3.9. Therefore, $\mathbb{S}_{\mathscr{A}}$ is the class of all semisimple homogeneous $\mathbb{Z}$-modules and all torsion-free abelian groups.

Thus, for a nonzero $\mathbb{Z}$-module M, we have that M is **$\mathscr{A}$-second** if and only if M is a semisimple homogeneous group or M is a torsion-free abelian group.

Proposition 7.6. *Let R be a ring and $S \in R$-simp. The following conditions are equivalent:*

(1) $\mathbb{S}_{R\text{-}pr}=\mathbb{T}_{tr_S}$.
(2) $\mathbb{S}_{R\text{-}pr}$ *is a class closed under taking direct sums.*

Proposition 7.7. *Let ${}_RM$. The following conditions are equivalent:*

(1) *M is R-id-second.*
(2) *M is generated by any of its nonzero submodules.*

Proof. (1) $\Rightarrow$ (2) Let $0 \neq N \leq M$. As M is R-*id*-second, $tr_N \in R$-*id* and $tr_N(M) \neq 0$, then $tr_N(M)=M$. Therefore, M is generated by N.

(2) $\Rightarrow$ (1) Let $\sigma \in R$-*id* such that $\sigma(M) \neq 0$. Then there exists a set X and an epimorphism $g\colon \sigma(M)^{(X)} \to M$. As σ is an idempotent preradical,

$$\begin{tikzcd} \sigma(M)^{(X)} \arrow[r, "g"] & M \\ \sigma(\sigma(M)^{(X)}) \arrow[u, hook] \arrow[r, "g|"] & \sigma(M) \arrow[u, hook] \end{tikzcd}$$ is commutative and $\sigma(\sigma(M))=\sigma(M)$.

So $\sigma(\sigma(M)^{(X)})=\sigma(M)^{(X)}$, then $\sigma(M)=M$. Therefore M is R-*id*-second. $\square$

Proposition 7.8. *Let ${}_RM$. The following conditions are equivalent:*

(1) *M is R-rad-second.*

(2) *M is cogenerated by any of its nonzero quotients.*

Proof. (1) $\Rightarrow$ (2) Let $N \lneq M$. As M is R-*rad*-second, $\omega_0^{M/N} \in R\text{-}rad$ and $\omega_0^{M/N}(M) \neq M$, then $\omega_0^{M/N}(M) = 0$. Therefore M is cogenerated by M/N.

(2) $\Rightarrow$ (1) Let $\sigma \in R\text{-}rad$ such that $\sigma(M) \neq M$. Then there exists a set X and a monomorphism $g\colon M \to (M/\sigma(M))^X$. As σ is a radical,

$$\begin{tikzcd} M \arrow[r, "g"] & (M/\sigma(M))^X \\ \sigma(M) \arrow[u, hook] \arrow[r, "g|"] & \sigma((M/\sigma(M))^X) \arrow[u, hook] \end{tikzcd}$$

is commutative and $\sigma(M/\sigma(M)) = 0$. So $\sigma((M/\sigma(M))^X) = 0$, then $\sigma(M) = 0$. Therefore M is R-*rad*-second. $\square$

8. Strongly second modules

As R-**pr** acts in the lattice of submodules of a module M by evaluation, we can define the R-**pr**-annihilator of M, and extend Definition 2.5 to this context. We introduce strongly second modules, and contrast this concept with the concept of second modules. We prove that a strongly second module is a second module in Proposition 8.12, and we show that the converse is not generally true in Remark 8.13. Likewise, we contrast the concepts of strongly second module and of R-*rad*-second module, proving that each strongly second module is R-*rad*-second in Proposition 8.10, and that the converse is not generally true in Remark 8.11. We characterize the strongly second modules in Proposition 8.7 and the strongly second rings in Theorem 8.8. We show that the concepts of strongly second module and of R-**pr**-second module are, in general, independent.

Definition 8.1. Given $M \in R\text{-}Mod$. The R-**pr**-annihilator of M is defined as:

$$Ann_{R\text{-}\mathbf{pr}}(M) = \{r \in R\text{-}\mathbf{pr} \mid r(M) = 0\}.$$

Example 8.2. Let $\mathbb{P}$ be the set of all prime integers and $M = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. We have that $\alpha \in Ann_{R\text{-}\mathbf{pr}}(M)$ if and only if $\alpha(M) = 0$. So that $\underline{0} \preceq \alpha \preceq \omega_0^M$, but $\omega_0^M = rad$. Therefore:

$$Ann_{R\text{-}\mathbf{pr}}(M) = [\underline{0}, rad].$$

Definition 8.3. Let M be an R-module. We define

$$Cog(M) = \{N \in R\text{-Mod} \mid \text{there exists a set } X \text{ and a monomorphism } f\colon N \to M^X\}.$$

Remark 8.4. Let M be an R-module. Then

$$Ann_{R\textbf{-pr}}(M) = [0, \omega_0^M] \text{ and } Cog(M) = \mathbb{F}_{\omega_0^M}.$$

The following definition is inspired by Definition 2.5.

Definition 8.5. We say that a nonzero R-module M is strongly second if

$$Ann_{R\textbf{-pr}}(M) = Ann_{R\textbf{-pr}}(M/N) \text{ for all } N \lneq M.$$

Example 8.6. Let $p \in \mathbb{Z}$ be a prime, $N \lneq {}_{\mathbb{Z}}\mathbb{Z}_{p^\infty}$ and $\sigma \in \mathbb{Z}\textbf{-pr}$. We have that $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}_{p^\infty}/N$, so $\sigma(\mathbb{Z}_{p^\infty}) \cong \sigma(\mathbb{Z}_{p^\infty}/N)$. Thus $\sigma(\mathbb{Z}_{p^\infty}) = 0$ if and only if $\sigma(\mathbb{Z}_{p^\infty}/N) = 0$, so $Ann_{\mathbb{Z}\textbf{-pr}}(\mathbb{Z}_{p^\infty}) = Ann_{\mathbb{Z}\textbf{-pr}}(\mathbb{Z}_{p^\infty}/N)$. Therefore $\mathbb{Z}_{p^\infty}$ is a strongly second module but by Proposition 5.1, $\mathbb{Z}_{p^\infty}$ is not a $\mathbb{Z}$**-pr**-second module as it is not a divisible torsion-free module or a semisimple homogeneous module. Therefore, a module ${}_RM$ can be strongly second without being R**-pr**-second.

By Remark 8.4, $Ann_{R\textbf{-pr}}(M) = Ann_{R\textbf{-pr}}(K)$ if and only if $\omega_0^M = \omega_0^K$, so we can describe a strongly second module M as a module such that $\mathbb{F}_{\omega_0^M} = Cog(M) = \mathbb{F}_{\omega_0^{M/N}} = Cog(M/N)$ for every $N \lneq M$, connecting Definition 8.1 with the concept of cogenerated, this idea is summarized in the following proposition that characterizes the strongly second modules:

Proposition 8.7. *For a left R-module ${}_RM$, the following statements are equivalent:*

(1) *M is strongly second.*
(2) *$Cog(M) = Cog(M/N)$ for all $N \lneq M$.*

Now, in the following theorem, we characterize all rings which are strongly second over themselves.

Theorem 8.8. *Let R be a ring. The following statements are equivalent:*

(1) *R is strongly second.*
(2) *$Cog(R) = Cog(R/I)$ for all $I \lneq R$.*
(3) *R is left local, left Kasch, left semiartinian, and simple.*
(4) *R is left semisimple and left local.*
(5) *Each nonzero R-module M is strongly second.*

Proof. (1) $\Rightarrow$ (2) By Proposition 8.7.

(2) $\Rightarrow$ (3) First let us see that R is left local. Let $S, K \in R\text{-}simp$ and take M, M' maximal left ideals of R such that $S \cong R/M$ and $K \cong R/M'$. By hypothesis, we have that $Cog(R/M) = Cog(R) = Cog(R/M')$, and besides we have that $Cog(S) = Cog(R/M)$ and $Cog(K) = Cog(R/M')$. Thus $Cog(S) = Cog(K) = Cog(R)$. Since

$S \in Cog(K)$, we have that S embeds in a product of copies of K, which implies that $S \cong K$. Therefore, R is left local.

Now, let $I \lneq R$. Then $Cog(R) = Cog(R/I)$ and $Cog(R) = Cog(S)$, so S embeds in a product of copies of R/I, which implies that S embeds in R/I. Therefore R is a left semiartinian left local ring, which implies that R is left Kasch. Finally, if J is a proper two-sided ideal of R, in particular J is a left ideal, so $Cog(R) = Cog(R/J)$, then R is embedded in a product of copies of R/J, which implies that $J = JR = 0$. Therefore R is simple.

(3) $\Rightarrow$ (4) Since R is left semiartinian, we have that $0 \neq soc(M)$, and as R is simple, then $soc(R) = R$. Hence R is semisimple. Moreover, R is left local, so only one type of simple module exists. Therefore R is semisimple homogeneous.

(4) $\Rightarrow$ (5) Since R is semisimple homogeneous, we have that R-**pr** $= \{\underline{0}, \underline{1}\}$, so

$$Ann_{R\text{-}\mathbf{pr}}(M) = \{\underline{0}\},$$

for any nonzero $M \in R$-*Mod*, from which it immediately follows that any R-module M is strongly second.

(5) $\Rightarrow$ (1) This is clear. □

Remark 8.9. Let S be the ring described in Example 6.3. As S is a simple ring, S is an S-**pr**-second module by Corollary 4.9. On the other hand, S is not a left semisimple ring, then S is not a strongly second S-module, by Theorem 8.8. Therefore, the fact that a module ${}_RM$ is R-**pr**-second does not necessarily means that it is strongly second.

Proposition 8.10. *For a left R-module ${}_RM$, if M is strongly second, then M is R-rad-second.*

Proof. Let $\beta \in R$-*rad*. Then $\beta(M/\beta(M)) = 0$, so $\beta \in Ann_{R\text{-}\mathbf{pr}}(M)$. Now, if $\beta(M) \neq M$, then $Ann_{R\text{-}\mathbf{pr}}(M) = Ann_{R\text{-}\mathbf{pr}}(M/\beta(M))$, so $\beta \in Ann_{R\text{-}\mathbf{pr}}(M)$ which implies that $\beta(M) = 0$. Therefore M is strongly second. □

Remark 8.11. By Example 4.10, we have that $\mathbb{Q}$ is $\mathbb{Z}$-**pr**-second and $\mathbb{Z}$-*rad* embeds in $\mathbb{Z}$-**pr**, then $\mathbb{Q}$ is $\mathbb{Z}$-*rad*-second by Lemma 3.9. On the other hand, $\mathbb{Q}$ is a torsion-free, then $tr_{\mathbb{Z}_2}(\mathbb{Q}) = 0$ but $f : \mathbb{Z}_2 \to \mathbb{Q}/\mathbb{Z}$, given by $f(a + 2\mathbb{Z}) = \frac{a}{2} + \mathbb{Z}$, is a nonzero $\mathbb{Z}$-morphism, so $tr_{\mathbb{Z}_2} \in Ann_{\mathbb{Z}\text{-}\mathbf{pr}}(\mathbb{Q})$ but $tr_{\mathbb{Z}_2} \notin Ann_{\mathbb{Z}\text{-}\mathbf{pr}}(\mathbb{Q}/\mathbb{Z})$. Thus $\mathbb{Q}$ is not a strongly second $\mathbb{Z}$-module.

Therefore, the converse of Proposition 8.10 doesn't hold.

Proposition 8.12. *For a left R-module ${}_RM$, if M is strongly second, then M is second.*

Proof. It follows from the fact of there is a lattice isomorphism between $t-Rad$ and $Lat(R)$, the fact of $t-rad$ is a sublattice of R-*rad*, Proposition 8.10 and Corollary 3.10. $\square$

Remark 8.13. By Example 8.11, the converse of Proposition 8.12 is not generally true.

Remark 8.14. Note that the same example as in Remark 8.11 is another example that, a module ${}_RM$ can be R-**pr**-second without being strongly second.

9. Characterizations of rings by classes of $\mathscr{A}$-second modules

In this section we obtain characterizations of the following kinds of left local rings. In Theorems 9.2 and 9.3, we characterize the left semisimple left local, and in Theorem 9.4, we characterize left local left perfect rings.

Proposition 9.1. *$\mathbb{S}_\sigma = R$-Mod if and only if $\sigma = \underline{1}$ or $\sigma = \underline{0}$.*

Proof. It is clear that if $\sigma = \underline{1}$ or $\sigma = \underline{0}$, then $\mathbb{S}_\sigma = R$-*Mod*.

On the other hand, suppose that $\sigma \neq \underline{0}$. Then there exists $M \in R$-*Mod* such that $\sigma(M) \neq 0$. Now, let $0 \neq N \in R$-*Mod*. We have that $\sigma(N \oplus M) = \sigma(N) \oplus \sigma(M) \neq 0$ and $N \oplus M \in \mathbb{S}_\sigma$, so $\sigma(N \oplus M) = N \oplus M$, then $\sigma(N) = N$. Therefore $\sigma = \underline{1}$. $\square$

Theorem 9.2. *For a ring R, the following statements are equivalent:*

1. *R is a left semisimple left local ring.*
2. *$\mathbb{S}_{R\text{-}id} = R$-Mod.*
3. *$\mathbb{S}_{R\text{-}rad} = R$-Mod.*

Proof. (1) $\Rightarrow$ (2) If R is a left semisimple left local ring, then R-**pr** $= \{\underline{0}, \underline{1}\}$, in particular, R-*id* $= \{\underline{0}, \underline{1}\}$. Therefore $\mathbb{S}_{R\text{-}id} = R$-*Mod*.

(2) $\Rightarrow$ (1) Suppose that $\mathbb{S}_{R\text{-}id} = R$-*Mod*. Let $S \in R$-*simp*. Then $\alpha_S^S \in R$-*id*, so $\mathbb{S}_{\alpha_S^S} = R$-*Mod*. Moreover, $\alpha_S^S \neq \underline{0}$, then $\alpha_S^S = \underline{1}$ by Proposition 9.1. It follows that R is left semisimple left local.

(1) $\Rightarrow$ (3) If R is a left semisimple left local ring, then R-**pr** $= \{\underline{0}, \underline{1}\}$, in particular, R-*rad* $= \{\underline{0}, \underline{1}\}$. Therefore, $\mathbb{S}_{R\text{-}rad} = R$-*Mod*.

(3) $\Rightarrow$ (1) Suppose that $\mathbb{S}_{R\text{-}rad} = R$-*Mod*. Let $0 \neq N \in R$-*Mod*. Then $\omega_0^N \in R$-*rad*, so $\mathbb{S}_{\omega_0^N} = R$-*Mod*. Moreover, $\omega_0^N \neq \underline{1}$, then $\omega_0^N = \underline{0}$ by Proposition 9.1. Therefore every nonzero module $N \in R$-*Mod* cogenerates R-Mod.

Now, let $S, S' \in R$-*simp*. As S is cogenerated by R and S' is cogenerated by S, then S embeds in R and $S \cong S'$. Therefore, R is left local and $0 \neq soc(R)$.

On the other hand, if $soc(R) \lneq R$, then R is cogenerated by $R/soc(R)$, so there exists a set X such that R embeds in $(R/soc(R))^X$. Then $soc(R) = soc(R)R$ embeds in $soc(R)((R/soc(R))^X) = 0$, so $soc(R) = 0$, which is a contradiction. Then $soc(R) = R$.

Therefore, R is a left semiartinian left local ring. □

Theorem 9.3. *For a ring R, the following statements are equivalent:*

(1) *R is a left semiartinian left local V-ring.*
(2) *R is a semisimple left local ring.*
(3) *$\mathbb{S}_{R\text{-}pr} = R$-Mod.*

Proof. (2) $\Rightarrow$ (1) We already know that R is left local, and every R-module is injective and projective since R is semisimple. In particular, every R-simple module is injective, so R is a V-ring.

Now, since R is left local semiartinian, for all nonzero module M, there exists a nonempty set I such that $M \cong S^I$ where S is simple, which implies that S embeds in M, so R is semiartinian.

(1) $\Rightarrow$ (2) Since R is left semiartinian, there exists a simple submodule S of R. Furthermore, since R is a V-ring, then S is a direct summand of R, so it is projective. Suppose $soc(R) \lneq R$. Then there exists a maximal ideal M of R such that $soc(R) \lneq M$. Moreover, since R is left local, we have that $R/M \cong S$, so the following sequence splits:

$$0 \longrightarrow M \longrightarrow R \longrightarrow R/M \longrightarrow 0.$$

Then there exists a submodule K of R such that $K \cap M = \{0\}$, and $K \cong R/M$, so $K \cap soc(R) = \{0\}$ and K is simple, which is a contradiction. Therefore $R = soc(R)$ implies R is semisimple and by hypothesis, left local.

(2) $\Rightarrow$ (3) It follows from Lemma 4.4.

(3) $\Rightarrow$ (2) Let S be a simple R-module, as $\mathbb{S}_{R\text{-}\mathbf{pr}} = R$-Mod we have that $\mathbb{S}_{\alpha^S_S} = R$-Mod. Now, by Proposition 9.1, $\alpha^S_S = \underline{1}$. Therefore, R is a semisimple left local ring. □

By [2, Lemma 28.3], $rad(R)$ is left T-nilpotent if and only if $rad(R)M \neq M$ for each ${}_RM$.

Theorem 9.4. *Let R be a ring such that $rad(R)$ is left T-nilpotent. Then the following statements are equivalent:*

(1) *$\mathbb{S}_{R\text{-}pr} = \mathbb{T}_{tr_S}$ for some $S \in R$-simp.*
(2) *R is a left perfect and left local ring.*

Proof. (2) $\Rightarrow$ (1) This follows from Proposition 5.2.

(1) $\Rightarrow$ (2) Let $K \in R\text{-}simp$. Then $K \in \mathbb{S}_{R\text{-}\mathbf{pr}}$ by Lemma 4.3, so $tr_S(K) = K$, which implies that $S \cong K$. Therefore R is a left local ring.

If $R/rad(R)$ is not semisimple, then it is not homogeneous semisimple. So by hypothesis, $R/rad(R)$ is not an R-**pr**-second module. Then there exists a two-sided ideal I such that $0 \neq I/rad(R) \lneq R/rad(R)$. Let us take a maximal left ideal J containing I. Then $R/J \cong S$ and there is a monomorphism $R/rad(S) \rightarrowtail (R/J)^X$ for some set X. Moreover $I(R/J) = 0$ and $I \cdot \_ \in R$-**pr**, so $I((R/J)^X) = 0$ and the following diagram commutes

$$\begin{tikzcd} R/rad(R) \arrow[r, tail] & (R/J)^X \\ I(R/rad(R)) \arrow[r, tail] \arrow[u, hook] & I((R/J)^X) = 0. \arrow[u, hook] \end{tikzcd}$$

Then $I/rad(R) = I(R/rad(R)) = 0$, and thus $I = rad(R)$, a contradiction. Hence $R/rad(R)$ is a semisimple R-module.

Therefore, by [2, Theorem 28.4], R is left perfect and left local. $\square$

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Luis Fernando García Mora and **Hugo Alberto Rincón-Mejía** (Corresponding Author)
Departamento de Matemáticas
Facultad de Ciencias
Universidad Nacional Autónoma de México
Mexico City, Mexico
e-mails: lu1sgarc1agm1995@gmail.com (L. F. García Mora)
hurincon@gmail.com (H. A. Rincón-Mejía)