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Variants of sets and functions with primals

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Abstract. Through this paper, we will discuss the limit points of a set and its complement set. To do this the authors will consider Acherjee et al.'s mathematical structure primal. These limit points via primal will further express the representation of nowhere dense sets in the literature. Expression of $\diamond$-local function and its associated set-valued set function will also be discussed here. Levine's semi-open sets will also be further represented by these limit points and will be decomposed.

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1. Introduction

The theory of topology encompasses a diverse spectrum of disciplines, including computer science, natural sciences, medical science, and social science. The creation of several new topological notions such as ideal, filter, grill, and primal has led to a wide range of new research areas, broadening the scope of the discipline and encouraging more in depth investigation of its theoretical and practical aspects. The notion of ideal attracted the attention of general topologists beginning with the works of Kuratowski [17] and Vaidyanathswamy [29]. Following that, ideal topological spaces and their applications have been examined in depth by Janković and Hamlett [15]. The behaviour of limit points of a set using ideal has elaborately being studied by a good number of mathematicians. The concept of "primal" in topology was introduced in the year 2022 by Acharjee et al. [2] and has gained significant popularity among general topologists. In this regard, Al-Omari et al. introduced a new type of operator, and topology suitable for primal in primal topological spaces, and discussed their various interesting outcomes in [4]. For further informations about primal, interested readers are referred to see the papers [3, 5–12, 26, 28]. Notion of limit points of set in aspect of primal is still not studied in literature.

The purpose of this research paper is to explore the concept of limit points of a set within the framework of primal topological spaces. The paper is structured into eight sections. Section 1 provides an introduction, while Section 2 introduces the basic notations and terminologies used throughout the study. In Section 3, we investigate limit points of a set in primal topological spaces. Section 4 examines limit points of the complement of a set in the primal context. Besides we present the notion of MDM space here and demonstrate several important characterizaions of MDM space. Section 5 delves into sets related to the $\psi_{\mathcal{P}}$-operator in primal topological spaces. Section 6 explores sets associated with mixed operators, specifically the $\psi_{\mathcal{P}}$ and $\diamond$-operators. Section 7 introduces the $\psi^{\diamond}_{\mathcal{P}}$ function and examines its various properties. Finally, Section 8 concludes the paper with a summary of findings and implications.

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2. Definitions and Notations

Throughout this manuscript, we denote 'iff' and **TS** as if and only if and topological space respectively.

- $(\overline{X}, l)$ or simply $\overline{X}$ be a **TS**.
- Throughout this manuscript, for a **TS** $\overline{X}$, we denote 'c' and 'i' as the closure operator and interior operator respectively, where as $i \circ c$ denoted as the interior(closure) operator. Similarly, $c \circ i \circ c$ is denoted as the closure(interior(closure)) operator.
- If $\overline{X}$ is a **TS**, then $I \subseteq \overline{X}$ is called semi-open [18] (respectivey, preopen [19], α-open [25], β-open [1]) if $I \subseteq c \circ i(I))$ (respectively, $I \subseteq i \circ c(I)$, $I \subseteq i \circ c \circ i(I)$, $I \subseteq c \circ i \circ c(I)$). The set of all semi-open (respectively, preopen, α-open, β-open) sets in a **TS** $(\overline{X}, l)$ is denoted as $\overline{so}(\overline{X})$ (respectively, $\overline{po}(\overline{X})$, l^{α}, $\overline{\beta o}(\overline{X})$), then the relations $l \subset l^{\alpha}$, $l \subseteq \overline{so}(\overline{X}) \subseteq \overline{\beta o}(\overline{X})$ and $l \subseteq \overline{po}(\overline{X}) \subseteq \overline{\beta o}(\overline{X})$ hold. A subset $I \subseteq \overline{X}$ is called δ set [13] if $i \circ c(I) \subseteq c \circ i(I)$ and the set of all δ sets in a **TS** $\overline{X}$ is denoted as l^{δ}. An open set O in a **TS** $\overline{X}$ is called regular open [27] if $O = i \circ c(O)$.
- Let $(\overline{X}, l)$ and $(\overline{Y}, k)$ be two topological spaces. A function $f : \overline{X} \to \overline{Y}$ is called semi-continuous [18] (respectively, β-continuous [1]) if and only if for each open set U in $\overline{Y}$, $f^{-1}(U)$ is semi-open (respectively, β-open) in $\overline{X}$.
- Let I be a subset of a **TS** $\overline{X}$. A point $p \in \overline{X}$ is called ω limit point [30] of I if every neighbourhood of p contains infinitely many points of I.
- Let I be a subset of a **TS** $\overline{X}$. A point $p \in \overline{X}$ is called a condensation point [30] of I if every neighbourhood of p contains uncountably many points of I.
- A subset I of a **TS** $\overline{X}$ is called a nowhere dense set [30] in $\overline{X}$ if $i \circ c(I) = \varnothing$.
- Ideal: According to Kuratowski [17], $\bar{\mathbf{I}} \subseteq 2^{\overline{X}}$ is called an ideal on $\overline{X}$ if $\bar{\mathbf{I}}$ is closed under hereditary property and finite additivity property. If $\overline{X} \notin \bar{\mathbf{I}}$, then $\bar{\mathbf{I}}$ is called proper ideal. A proper ideal $\bar{\mathbf{I}}$ is called an admissible ideal [16] if $\bar{\mathbf{I}}$ contains every singleton.
- Primal: According to Acharjee et. al [2], a collection $\mathcal{P}$ of subsets of a nonempty set $\overline{X}$ is called a primal on $\overline{X}$ if it satisfies: (i) $\overline{X} \notin \mathcal{P}$, (ii) $I \in \mathcal{P}$ and $S \subseteq I$ implies $S \in \mathcal{P}$ and (iii) $I \cap S \in \mathcal{P}$ implies $I \in \mathcal{P}$ or $S \in \mathcal{P}$. Equivalently, a collection $\mathcal{P}$ of subsets of a nonempty set $\overline{X}$ is called a primal on $\overline{X}$ if it satisfies: (i) $\overline{X} \notin \mathcal{P}$, (ii) $S \notin \mathcal{P}$ and $S \subseteq I$ implies $I \notin \mathcal{P}$ and (iii) $I \notin \mathcal{P}$ and $S \notin \mathcal{P}$ implies $I \cap S \notin \mathcal{P}$. A **TS** $(\overline{X}, l)$ with a primal $\mathcal{P}$ is called a primal topological space (or simply **PTS**) and it is written as $(\overline{X}, l, \mathcal{P})$. Due to the primal $\mathcal{P}$ on a **TS** $(\overline{X}, l)$, the operator $\diamond : 2^{\overline{X}} \to 2^{\overline{X}}$, given by $I^{\diamond}(\mathcal{P}) = \{p \in \overline{X} : \forall\ U_p \in l(p),\ I^c \cup {U_p}^c \in \mathcal{P}\}$ [2] where $l(p) = \{U \in l : p \in U\}$. In the study of ideal, this type of operator is called local function [17] and it is denoted as $()^{*}$. The map $c^{\diamond} : 2^{\overline{X}} \to 2^{\overline{X}}$ where $c^{\diamond}(I) = I \cup I^{\diamond}(\mathcal{P})$ [2] where I is any subset of $\overline{X}$, is a Kuratowski's closure operator and hence induces a topology $l^{\diamond}(= \{I \subseteq \overline{X} : \ c^{\diamond}(\overline{X} \setminus I) = \overline{X} \setminus I\})$ on $\overline{X}$ and it is strictly finer than l in general. In this respect, we have mentioned that local function also gives a topology, called $*$-topology [15] and it is denoted as l^{*}. It is also noticed that in a certain situation both topologies are synonyms. The open set of $l^{\diamond}$ is called $\diamond$-open set and the complement of $\diamond$-open set is called $\diamond$-closed set. A subset $I \subset \overline{X}$ is $\diamond$-open if $i^{\diamond}(I) = I$ where $i^{\diamond}(I)$ is denoted interior of I with respect to $\diamond$-topology and $I \subset \overline{X}$ is $\diamond$-closed if $c^{\diamond}(I) = I$ where $c^{\diamond}(I)$ is denoted closure of I with respect to $\diamond$-topology $l^{\diamond}$.

  Associated complementary set operator [22] of $\diamond$ is $\psi_{\mathcal{P}}$ [4] and it can be defined as an separated set function, $\psi_{\mathcal{P}}(I) = \overline{X} \setminus (\overline{X} \setminus I)^{\diamond}(\mathcal{P})$. It is learnt from [22], $()^{\diamond}(\mathcal{P}) \sim^{\overline{X}} \psi_{\mathcal{P}}$.
- A subset I of a topological space $(\overline{X}, l)$ with a primal $\mathcal{P}$ is called $\mathcal{P}$-dense if $I^{\diamond}(\mathcal{P}) = \overline{X}$. Equivalently, the set I is called $\mathcal{P}$-dense if $\psi_{\mathcal{P}}(\overline{X} \setminus I) = \varnothing$.
- A subset I of a **TS** $(\overline{X}, l)$ with a primal $\mathcal{P}$ is called $\diamond$-dense if $c^{\diamond}(I) = \overline{X}$. Every $\mathcal{P}$-dense subset of $\overline{X}$ is a $\diamond$-dense but reverse may not be true.
- Let $f : \overline{X} \to \overline{Y}$ be a function. If $\mathcal{P}$ is a primal on $\overline{X}$, then $f^{\leftarrow}(\mathcal{P}) = \{I : I \subset f^{-1}(P) \in f^{-1}(\mathcal{P})\}$ is not a primal on $\overline{X}$.
- Let $\overline{X} \neq \varnothing$ and $\mathcal{P} \subseteq 2^{\overline{X}}$. Then, $\mathcal{P}$ is a primal on $\overline{X}$ iff $\bar{\mathbf{I}}(\mathcal{P}) = \{I : \ I \in 2^{\overline{X}},\ \overline{X} \setminus I \notin \mathcal{P}\}$ is an ideal on $\overline{X}$ [20].

- Let $\overline{X} \neq \varnothing$ and $\bar{\mathbf{I}} \subseteq 2^{\overline{X}}$. Then, $\bar{\mathbf{I}}$ is an ideal on $\overline{X}$ iff $\mathcal{P}(\bar{\mathbf{I}}) = \{I : I \in 2^{\overline{X}}, \overline{X} \setminus I \notin \mathcal{I}\}$ is a primal on $\overline{X}$ [20].

  In this regards, an admissible primal can be considered from an admissible ideal $\bar{\mathbf{I}}$ by using the above notion.
- Let $\overline{X} \neq \varnothing$ and $\bar{\mathbf{I}}$ and $\mathcal{P}$ be any ideal and primal on $\overline{X}$, respectively. Then, $\mathcal{P}(\bar{\mathbf{I}}(\mathcal{P})) = \mathcal{P}$ [20].
- Throughout the manuscript, for a **TS** $(\overline{X}, l)$, we denote '$C(\overline{X})$' as the collection of all closed sets.
- Throughout the manuscript, for a subset I of a nonempty set $\overline{X}$, we denote 'I^c' as the complement of I, i.e., $I^c = \overline{X} \setminus I$.

3. Limit Points due to Primal

Limit points of a set via ideal is well known for us whereas limit points of set via filters and grills have been discussed in [21, 23]. In the present section, we will discuss limit points of a set via primals.

Lemma 1. *Let $\overline{X}$ be an infinite set. Then, $\mathcal{P} = \{I \subset \overline{X} : \overline{X} \setminus I$ is infinite$\}$ is a primal on $\overline{X}$.*

Lemma 2. *Let $\overline{X}$ be a uncountable set. Then, $\mathcal{P} = \{I \subset \overline{X} : \overline{X} \setminus I$ is uncountable$\}$ is a primal on $\overline{X}$.*

The dual ideals of the Lemma 1 and Lemma 2 are the ideals of finite subsets of $\overline{X}$ and the ideal of countable subsets of $\overline{X}$, respectively. These are denoted by $\mathcal{P}_{cf}$ and $\mathcal{P}_{cc}$, respectively.

From above Lemmas, we have:

Lemma 3. *Let $(\overline{X}, l, \mathcal{P})$ be a* **PTS**. *Then,*
(i) $I^\diamond(\mathcal{P})$ = the set of all condensation points of I, when $\mathcal{P} = \mathcal{P}_{cf}$.
(ii) $I^\diamond(\mathcal{P})$ = the set of all ω limit points of I, when $\mathcal{P} = \mathcal{P}_{cc}$.

We have that various limit points of a set can moderate through a primal. The limit points of a set may be calculated by the primal $2^{\overline{X}} \setminus \{\overline{X}\}$, where $2^{\overline{X}}$ denotes the power set of $\overline{X}$. That is for a subset I of a **PTS** $(\overline{X}, l, 2^{\overline{X}} \setminus \{\overline{X}\})$, $I^\diamond(2^{\overline{X}} \setminus \{\overline{X}\}) = c(I)$.

Lemma 4. *Let $(\overline{X}, l)$ be a **TS**. Then, $\mathcal{P} = \{I \subset \overline{X} : \overline{X} \setminus c \circ i(I) \neq \varnothing\}$ is a primal on $\overline{X}$.*

Proof. (i) Given that $\overline{X} \setminus c \circ i(\overline{X}) = \varnothing \implies \overline{X} \notin \mathcal{P}$.
(ii) Let $I \subset S \in \mathcal{P}$. Then $X \setminus c \circ i(S) \neq \varnothing \implies \overline{X} \setminus c \circ i(S) \subset X \setminus c \circ i(I) \neq \varnothing$.
(iii) Suppose $I, S \notin \mathcal{P}$. This implies $\overline{X} \setminus c \circ i(I) = \varnothing$ and $\overline{X} \setminus c \circ i(S) = \varnothing$. Thus, for every nonempty open sets U, $U \cap i(I) \neq \varnothing$ and $U \cap i(S) \neq \varnothing \implies U \cap i(I) \cap i(S) \neq \varnothing \implies c \circ i(I \cap S) = \overline{X}$. Therefore, $I \cap S \notin \mathcal{P}$. □

This primal is called 'anti nowhere dense sets primal' and it is simply denoted as $\mathcal{P}_{ap}$.

The dual ideal of the above primal is, $\bar{\mathbf{I}}(\mathcal{P}_{ap}) = \{I \subset \overline{X} : i \circ c(I) = \varnothing\}$. This is the well known ideal on the study of ideal **TS** and called it by ideal of nowhere dense sets.

From this ideal of nowhere dense sets $\bar{\mathbf{I}}_{nwd}$ in a **TS** $(\overline{X}, l)$, $I^*(\bar{\mathbf{I}}_{nwd}) = c \circ i \circ c(I)$ [15] and $\psi_{\bar{\mathbf{I}}_{nwd}}(I) = i \circ c \circ i(I)$. For the proof of the second relation one can take the help of $()^* \sim^{\overline{X}} \psi$ [22].

From these point of views, we have followings:

Corollary 1. *Let $(\overline{X}, l)$ be a **TS** and $\mathcal{P}_{ap}$ be the anti nowhere dense sets primal on $\overline{X}$. Then, $I^\diamond(\mathcal{P}_{ap}) = c \circ i \circ c(I)$ and $\psi_{\mathcal{P}_{ap}}(I) = i \circ c \circ i(I)$.*

Further more for the 'anti nowhere dense sets primal' is,

Remark 1. *For a* **PTS** *$(\overline{X}, l, \mathcal{P}_{ap})$, $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathcal{P}_{ap}$.*

However, it is not always necessary to hold the converse of the aforementioned note.

Example 1. *Consider a* **PTS** *$(\overline{X}, l, \mathcal{P})$ where $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1, t_2\}\}$ and $\mathcal{P} = \{\varnothing, \{t_1\}, \{t_3\}, \{t_1, t_3\}\}$. Then, $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathcal{P}$. But $\mathcal{P}_{ap} = \{\varnothing, \{t_1\}, \{t_2\}, \{t_3\}, \{t_1, t_3\}, \{t_2, t_3\}\}$. Thus, $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathcal{P}$ though $\mathcal{P}$ is not anti nowhere dense sets primal.*

Due to the different types of primal on a particular set we can obtained various type of limit point of set in a **TS**. We can minimize this through following discussion.

Let $\overline{X}$ be a nonempty set. The collection $\circlearrowright_{\overline{X}}$ of all primals on $\overline{X}$ forms a partial order set with respect

to $\subseteq$. Furthermore, every chain in $(\circlearrowleft_{\overline{X}}, \subseteq)$ has an upper bound. Thus, by Zorn's Lemma, $(\circlearrowleft_{\overline{X}}, \subseteq)$ has an maximal element. This maximal element is called ultraprimal. Through out the paper, we will denote $\mathbb{P}$ as an ultraprimal. Note that every primal contained in an ultraprimal. In this paper, we will denote $\mathbb{P}$ as an ultraprimal.
Further for a **TS** $\overline{X}$ and the primals $\mathcal{P}_1$ and $\mathcal{P}_2$ on $\overline{X}$, $I^{\diamond}(\mathcal{P}_1) \subseteq I^{\diamond}(\mathcal{P}_2)$, when $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $I \subseteq \overline{X}$. Therefore, we can mitigate the various limit points of a set in a single primal through following results:

Theorem 1. *Let $\overline{X}$ be a **TS** and $\mathbb{P}$ be an ultraprimal on $\overline{X}$ containing the primal $\mathcal{P}$. Then,*

(1) for $I \subseteq \overline{X}$, $I^{\diamond}(\mathcal{P}) \subseteq I^{\diamond}(\mathbb{P})$.
(2) for $I \subseteq \overline{X}$, $\psi_{\mathbb{P}}(I) \subseteq \psi_{\mathcal{P}}(I)$.
(3) for $I \subseteq \overline{X}$, $I \cup I^{\diamond}(\mathcal{P}) \subseteq I \cup I^{\diamond}(\mathbb{P})$.
(4) for $I \subseteq \overline{X}$, $I \cap \psi_{\mathbb{P}}(I) \subseteq I \cap \psi_{\mathcal{P}}(I)$.
(5) $l^{\diamond}(\mathbb{P}) \subseteq l^{\diamond}(P)$.
(6) $l^{\diamond}(\mathbb{P})$ is the smaller topology on $\overline{X}$ than the topologies on $\overline{X}$ due to the primal contained in the ultraprimal $\mathbb{P}$.

Consequently, we can say that $I \cup I^{\diamond}(\mathbb{P})$ (resp. $I \cap \psi_{\mathbb{P}}(I)$) is the closure (resp. interior) of I in the **TS** $(\overline{X}, l^{\diamond}(\mathbb{P}))$.

Following example shows that two ultraprimals induce two different topologies on a single set.

Example 2. *Let $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1, t_2\}\}$. Two ultraprimals, $\mathbb{P}_1 = \{\varnothing, \{t_2\}, \{t_3\}, \{t_2, t_3\}\}$ and $\mathbb{P}_2 = \{\varnothing, \{t_1\}, \{t_3\}, \{t_1, t_3\}\}$ on $\overline{X}$. Then, $l^{\diamond}(\mathbb{P}_1) = \{\varnothing, \{t_1\}, \{t_1, t_2\}, \overline{X}\}$ and $l^{\diamond}(\mathbb{P}_2) = \{\varnothing, \{t_2\}, \{t_1, t_2\}, \overline{X}\}$ are not comparable. Further, if we consider the ultraprimal $\mathbb{P}_3 = \{\varnothing, \{t_1\}, \{t_2\}, \{t_1, t_2\}\}$, then $C(\overline{X}) \setminus \overline{X} \nsubseteq \mathbb{P}_3$.*

Note that the intersection of two ultraprimals is not always a primal and it has been easily shown from the Example 2.

Lemma 5. *For a **PTS** $(\overline{X}, l, \mathcal{P})$,*
(i) if $\mathbb{P}$ is an ultraprimal and $P^c \notin \mathbb{P}$, then $I^{\diamond}(\mathbb{P}) = (I \setminus P)^{\diamond}(\mathbb{P})$.
(ii) if $\mathbb{P}$ is an ultraprimal and $P^c \notin \mathbb{P}$, then $\psi_{\mathbb{P}}(I) = \psi_{\mathbb{P}}(I \cup P)$.

Proof. The proof is obvious by Corollary 3.4 and Theorem 3.6 of [4], since every ultraprimal is primal. □

4. Limit Points for Complement of a Set via Primal

From definition of $\psi_{\mathcal{P}}$-operator, limit points of a set may be related as the associated set valued set function [22, 24]. Properties of limit points will be discussed through this section.

Lemma 6. *Let $(\overline{X}, l, \mathbb{P})$ be a **PTS**. Then, $\overline{X} = \overline{X}^{\diamond}(\mathbb{P})$ iff $l \cap \bar{\mathbf{I}}(\mathbb{P}) = \{\varnothing\}$.*

Proof. Proof is obvious from the fact that $\mathbb{P}(\bar{\mathbf{I}}(\mathbb{P})) = \mathbb{P}$. □

As we know the space $(\overline{X}, l, \bar{\mathbf{I}})$ is Hayashi-Samuel [14] iff $l \cap \bar{\mathbf{I}} = \{\varnothing\}$. Thus, its dual concept in **PTS** is:

Lemma 7. *Let $(\overline{X}, l, \mathcal{P})$ be a **PTS**. Then, $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathcal{P}$ iff $l \cap \bar{\mathbf{I}}(\mathcal{P}) = \{\varnothing\}$.*

Proof. Converse part: Suppose $l \cap \bar{\mathbf{I}}(\mathcal{P}) = \{\varnothing\}$ holds. Let $I \in l \cap \bar{\mathbf{I}}(\mathcal{P})$. Then, $I \in C(\overline{X})$ and $I \neq \overline{X}$, and hence $\overline{X} \setminus I \notin \bar{\mathbf{I}}(\mathcal{P})$. Thus, $I \in \mathcal{P}(\bar{\mathbf{I}}(\mathcal{P})) = \mathcal{P}$. □

If a space $(\overline{X}, l, \mathcal{P})$ satisfies the condition $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathcal{P}$ is called **Primal Hayashi-Samuel Space or simply MDM space**.
A space is not a Primal Hayashi-Samuel Space even if the primal is an ultraprimal. However, following hold as well: If $\mathbb{P}$ is an ultraprimal on the **TS** $(\overline{X}, l)$ containing the primal $\mathcal{P}$ and if $(\overline{X}, l, \mathcal{P})$ is a MDM space, then $(\overline{X}, l, \mathbb{P})$ is also a MDM space. Since every primal contained in an ultraprimal, thus this result is remarkable. It also mentionable that MDM space with respect to ultraprimal does not imply that it is a MDM space with respect to any primal.

Example 3. *Let $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1, t_3\}\}$. Consider the ultraprimal $\mathbb{P} = \{\varnothing, \{t_1\}, \{t_2\}, \{t_3\}, \{t_1, t_2\}, \{t_1, t_3\}\}$ containing the primal $\mathcal{P} = \{\varnothing, \{t_1\}, \{t_3\}, \{t_1, t_3\}\}$. Then, $C(\overline{X}) \setminus \overline{X} \subset \mathbb{P}$ but $C(\overline{X}) \setminus \overline{X} \nsubseteq \mathcal{P}$.*

Due to the above synonyms we get the following equivalent conditions:

Theorem 2. *Let* $(\overline{X}, l, \mathbb{P})$ *be a* **PTS**. *The statements below are identical:*
(*i*) $(\overline{X}, l, \mathbb{P})$ *is a MDM space;*
(*ii*) $\psi_{\mathbb{P}}(\varnothing) = \varnothing$;
(*iii*) *If* $I \subset \overline{X}$ *is closed, then* $\psi_{\mathbb{P}}(I) \setminus I = \varnothing$;
(*iv*) *If* $I \subset \overline{X}$, *then* $i \circ c(I) = \psi_{\mathbb{P}}(i \circ c(I))$;
(*v*) *If* $I \subset \overline{X}$ *is regular open, then* $I = \psi_{\mathbb{P}}(I)$;
(*vi*) *If* $U \in l$, *then* $\psi_{\mathbb{P}}(U) \subset i \circ c(I)) \subset U^{\diamond}(\mathbb{P})$;
(*vii*) *If* $P \in \mathbb{P}$, *then* $\psi_{\mathbb{P}}(P) = \varnothing$;
(*viii*) $\overline{X} = \overline{X}^{\diamond}(\mathbb{P})$.

Theorem 3. *Let* $(\overline{X}, l)$ *be a* ***TS*** *and* $I \subset \overline{X}$. *Then, for any ultraprimal* $\mathbb{P}$ *containing the primal* $\mathcal{P}$, $i(I) \subseteq \psi_{\mathbb{P}}(I) \subseteq \psi_{\mathcal{P}}(I)$.

Proof. Since for any $I \subset \overline{X}$, $i(I)$ is open, then $i(I) \subset \psi_{\mathcal{P}}(i(I))$. Again, $i(I) \subset I$ and hence $\psi_{\mathcal{P}}(i(I)) \subset \psi_{\mathcal{P}}(I)$. Thus, $i(I) \subset \psi_{\mathcal{P}}(I)$ for any $I \subset \overline{X}$. □

Theorem 4. *For MDM space* $(\overline{X}, l, \mathcal{P})$,
(*i*) *For any* $I \subset \overline{X}$, $\psi_{\mathcal{P}}(I) \subset I^{\diamond}(\mathcal{P})$.
(*ii*) *For any* $I \subset \overline{X}$, $\psi_{\mathcal{P}}(I) \subset c(I)$.
(*iii*) *For any* $I \subset \overline{X}$, *then* $\psi_{\mathcal{P}}(I) \subset i \circ c(I)$.

Proof. (*i*) If possible let $\psi_{\mathcal{P}}(I) \nsubseteq I^{\diamond}(\mathcal{P})$. Then, there exists $p \in \psi_{\mathcal{P}}(I)$ but not in $I^{\diamond}(\mathcal{P})$. Thus, there exists $p \in U_p \in l(p)$ such that ${U_p}^c \cup I^c(\mathcal{P}) \notin \mathcal{P}$. Since $p \in \psi_{\mathcal{P}}(I)$, then there exists $p \in V_p \in l(p)$ such that $(V_p \setminus I)^c \notin \mathcal{P}$. Now $U_p \cap V_p$ is an open set containing p. Since ${U_p}^c \cup I^c \notin \mathcal{P}$, then ${U_p}^c \cup I^c \cup {V_p}^c \notin \mathcal{P}$ by hereditary property of $\mathcal{P}$. Thus, $(U_p \cap V_p)^c \cup I^c \notin \mathcal{P}$. Also, $(V_p \setminus I)^c \notin \mathcal{P}$ implies ${V_p}^c \cup A \notin \mathcal{P}$ and hence ${V_p}^c \cup I \cup {U_p}^c \notin \mathcal{P}$ by hereditary property of $\mathcal{P}$. Thus, $(U_p \cap V_p)^c \cup I \notin \mathcal{P}$. Combining, $((U_p \cap V_p)^c \cup I) \cap ((U_p \cap V_p)^c \cup I^c) \notin \mathcal{P}$ and hence $(U_p \cap V_p)^c \cup (I \cap I^c) \notin \mathcal{P}$. This implies, $(U_p \cap V_p)^c \notin \mathcal{P}$ which contradicts the fact that $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathcal{P}$. Hence, for any $I \subset \overline{X}$, $\psi(I) \subset I^{\diamond}(\mathcal{P})$.

(*ii*) From the Theorem 4(i), $\psi_{\mathcal{P}}(I) \subset I^{\diamond}(\mathcal{P})$ and hence $\psi_{\mathcal{P}}(I) \subset I \cup I^{\diamond}(\mathcal{P})$. This implies, $\psi_{\mathcal{P}}(I) \subset c^{\diamond}(I)$. Also, since $l^{\diamond}$ is finer than l, then $c^{\diamond}(I) \subset c(I)$ for any subset I of $\overline{X}$. Thus, $\psi_{\mathcal{P}}(I) \subset c(I)$ for any $I \subset \overline{X}$.

(*iii*) From the Theorem 4(ii), $\psi_{\mathcal{P}}(I) \subset c(I)$ for any $I \subset \overline{X}$. This implies, $i(\psi_{\mathcal{P}}(I)) \subset i \circ c(I)$ and hence $\psi_{\mathcal{P}}(I) \subset i \circ c(I)$, since $\psi_{\mathcal{P}}(I)$ is open. □

Theorem 5. *Let* $(\overline{X}, l)$ *be a* ***TS*** *and a primal* $\mathcal{P}$ *on* $\overline{X}$ *contained in the ultraprimal* $\mathbb{P}$. *Then, for each* $p \in \overline{X}$,
(1) $\overline{X} \setminus \{p\}$ *is* $\mathcal{P}$*-dense if and only if* $\psi_{\mathcal{P}}(\{p\}) = \varnothing$.
(2) $\overline{X} \setminus \{p\}$ *is* $\mathcal{P}$*-dense* $\Longrightarrow \psi_{\mathbb{P}}(\{p\}) = \varnothing$.

Proof. Assume, for each $p \in \overline{X}$, $\psi_{\mathcal{P}}(\{p\}) = \varnothing \Longleftrightarrow \psi_{\overline{\mathbf{I}}(\mathcal{P})}(\{p\}) = \varnothing \Longleftrightarrow \overline{X} \setminus (\overline{X} \setminus \{p\})^*(\overline{\mathbf{I}}(\mathcal{P})) = \varnothing \Longleftrightarrow (\overline{X} \setminus \{p\})^*(\overline{\mathbf{I}}(\mathcal{P})) = \overline{X} \Longleftrightarrow \overline{X} \setminus (\overline{X} \setminus \{p\})^{\diamond}((\mathcal{P}(\overline{\mathbf{I}}(\mathcal{P}))) = \mathcal{P}) = \varnothing$. □

Corollary 2. *Let* $(\overline{X}, l)$ *be a* ***TS*** *and* $\mathbb{P}$ *an ultraprimal on* $\overline{X}$ *containing the primal* $\mathcal{P}$. *For each* $p \in \overline{X}$, *if* $\psi_{\mathcal{P}}(\{p\}) = \varnothing$, *then*
(*i*) $\overline{X} \setminus \{p\}$ *is* $\diamond$*-dense with respect to the* $l^{\diamond}(\mathcal{P})$ *topology.*
(*ii*) $\overline{X} \setminus \{p\}$ *is* $\diamond$*-dense with respect to the* $l^{\diamond}(\mathbb{P})$ *topology.*

By the following example, we conclude that the reverse may not be true in general:

Example 4. *Consider a* **PTS** $(\overline{X}, l, \mathcal{P})$, *where* $\overline{X} = \{t_1, t_2, t_3\}$, $\tau = \{\varnothing, \{t_1\}, \{t_2\}, \{t_1, t_2\}, \overline{X}\}$ *and* $\mathcal{P} = \{\varnothing, \{t_1\}, \{t_3\}, \{t_1, t_3\}\}$. *Take* $p = t_3$, *then* $(\overline{X} \setminus \{p\})^{\diamond}(\mathcal{P}) = \{t_1, t_2\}^{\diamond}(\mathcal{P}) = \{t_2, t_3\}$ *and hence* $c^{\diamond}(\overline{X} \setminus \{p\}) = \overline{X}$. *Thus,* $\overline{X} \setminus \{p\}$ *is a* $\diamond$*-dense but* $\psi_{\mathcal{P}}(\{p\}) = \{t_1\} \neq \varnothing$.

5. Sets Related to $\psi_{\mathcal{P}}$-Operator

In this section, we shall discuss some properties of $\psi_{\mathcal{P}}$-C set [9] on primal topological spaces:

Definition 1. *A subset* I *of a* **PTS** $(\overline{X}, l, \mathcal{P})$ *is called* $\psi_{\mathcal{P}}$*-C set [9] if* $I \subset c(\psi_{\mathcal{P}}(I))$.

The collection of all $\psi_{\mathcal{P}}$-C sets in a **PTS** $(\overline{X}, l, \mathcal{P})$ denoted as $\psi_{\mathcal{P}}(\overline{X}, l)$. Clearly, $\varnothing, \overline{X} \in \psi_{\mathcal{P}}(\overline{X}, l)$.

Theorem 6. *[9] For a* **PTS** $(\overline{X}, l, \mathcal{P})$, $l \subset \psi_{\mathcal{P}}(\overline{X}, l)$.

If $\mathbb{P}$ is an ultraprimal on a **TS** (X, l) containing the primal $\mathcal{P}$, then $\psi_{\mathbb{P}}$-C set implies $\psi_{\mathcal{P}}$-C but reverse inclusion may not be true in general.

Note the operator $\psi_{\mathcal{P}}$ neither the interior operator of the topology l on $\overline{X}$ nor the interior operator of the topology $l^{\diamond}(\mathcal{P})$ on $\overline{X}$. In this point of view the semi-open set and the $\psi_{\mathcal{P}}$ -C open set are different from each other. Moreover, if I is a member of the ultraprimal $\mathbb{P}$, then I is not a $\psi_{\mathbb{P}}$-C set. But it is not true that collection of all non $\psi_{\mathcal{P}}$-C set forms a ultraprimal.

Example 5. *Let* $\overline{X} = \{t_1, t_2, t_3, t_4\}$, $l = \{\varnothing, \overline{X}, \{t_3, t_4\}\}$ *and* $\mathcal{P} = 2^{\overline{X}} \setminus \{\{t_3, t_4\}, \{t_1, t_3, t_4\}, \{t_2, t_3, t_4\}, \overline{X}\}$. *Then,* $\psi_{\mathcal{P}}(\overline{X}, l) = \{\varnothing, \overline{X}, \{t_3, t_4\}, \{t_1, t_3, t_4\}, \{t_2, t_3, t_4\}\}$. *But* $\psi_{\mathcal{P}}(\overline{X}, l)$ *is not a primal as* $\{t_1, t_3\}$ *is not member of* $\psi_{\mathcal{P}}(\overline{X}, l)$ *and hence it is not an ultraprimal.*

Theorem 7. *[9] In a* **PTS** $(\overline{X}, l, \mathcal{P})$, *the arbitrary union of* $\psi_{\mathcal{P}}$*-C sets is also a* $\psi_{\mathcal{P}}$*-C set.*

We know that collection of Levine's semi-open sets is larger than the given topology whereas our collection $\psi_{\mathcal{P}}(\overline{X}, l)$ is more larger than the collection of Levine's semi-open sets.

Theorem 8. *Let* $(\overline{X}, l, \mathcal{P})$ *be a* **PTS**. *Then, for an ultraprimal* $\mathbb{P}$ *on* $\overline{X}$ *containing the primal* $\mathcal{P}$, $\overline{so}(\overline{X}) \subset \psi_{\mathbb{P}}(\overline{X}, l) \subset \psi_{\mathcal{P}}(\overline{X}, l)$.

Following gives the answer for the question of the reverse inclusion:

Example 6. *Consider a* **PTS** $(\overline{X}, l, \mathcal{P})$, *where* $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1\}, \{t_2\}, \{t_1, t_2\}\}$ *and* $\mathcal{P} = \{\varnothing, \{t_1\}, \{t_2\}, \{t_1, t_2\}\}$. *Take* $I = \{t_3\}$. *Then,* $c \circ i(A) = \varnothing$. *Thus,* $I \nsubseteq c \circ i(A)$ *and hence* $I \notin \overline{so}(\overline{X})$. *Now,* $\psi_{\mathcal{P}}(I) = \overline{X} \setminus \{t_1, t_2\}^{\diamond}(\mathcal{P}) = \overline{X}$ *and hence* $I \subset c(\psi_{\mathcal{P}}(I))$. *Thus,* $I \in \psi_{\mathcal{P}}(\overline{X}, l)$.

Theorem 9. *For a MDM space* $(\overline{X}, l, \mathcal{P})$, $\psi_{\mathcal{P}}(\overline{X}, l) \subset \overline{\beta o}(\overline{X})$.

Proof. Let $I \in \psi_{\mathcal{P}}(\overline{X}, l)$. Then, $I \subseteq c(\psi_{\mathcal{I}(\mathcal{P})}(I))$, and $\psi_{\mathcal{I}(\mathcal{P})}(I) \subset i \circ c(A)$ (by Theorem 4). Thus, we have, $\psi_{\mathcal{P}(\mathcal{I}(\mathcal{P}))}(I) \subset i \circ c(A)$ and hence $\psi_{\mathcal{P}}(I) \subset i \circ c(A) \Rightarrow I \subseteq c \circ i \circ c(A) \Rightarrow A \in \overline{\beta o}(\overline{X}) \Rightarrow \psi_{\mathcal{P}}(\overline{X}, l) \subset \overline{\beta o}(\overline{X})$. □

For the reverse part of the Theorem 9, we have the example below:

Example 7. *Consider a* **PTS** $(\overline{X}, l, \mathcal{P})$, *where* $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1, t_2\}\}$ *and* $\mathcal{P} = \{\varnothing, \{t_1\}, \{t_3\}, \{t_1, t_3\}\}$. *Take* $I = \{t_1\}$. *Then,* $c \circ i \circ c(A) = \overline{X}$. *Thus,* $I \subseteq c \circ i \circ c(A)$ *and hence* $I \in \overline{\beta o}(\overline{X})$. *Now,* $\psi_{\mathcal{P}}(I) = \overline{X} \setminus \{t_2, t_3\}^{\diamond}(\mathcal{P}) = \overline{X} \setminus \{t_1, t_2, t_3\} = \varnothing$ *and hence* $I \nsubseteq c(\psi_{\mathcal{P}}(I))$. *Thus,* $I \notin \psi_{\mathcal{P}}(\overline{X}, l)$ *though* $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathcal{P}$.

Sandwich Theorem for $\psi_{\mathcal{P}}$-C sets:

Theorem 10. *For a MDM space* $(\overline{X}, l, \mathcal{P})$, $\overline{so}(\overline{X}) \subset \psi_{\mathcal{P}}(\overline{X}, l) \subset \overline{\beta o}(\overline{X})$.

Corollary 3. *Consider an anti nowhere dense sets primal* $\mathcal{P}_{ap}$ *on the* ***TS*** $(\overline{X}, l)$. *Then,* $\overline{so}(\overline{X}) = \psi_{\mathcal{P}_{ap}}(\overline{X}, l)$.

Proof. It is obvious from that fact that for any $I \subset \overline{X}$, $\psi_{\mathcal{P}_{ap}}(I) = i \circ c \circ i(A)$. □

However, the anti nowhere dense sets primal $\mathcal{P}_{ap}$ does not give the guarantee to the equality of $\psi_{\mathcal{P}_{ap}}(\overline{X}, l)$ and $\overline{\beta o}(\overline{X})$.

Example 8. *Consider a* **PTS** $(\overline{X}, l, \mathcal{P})$, *where* $\overline{X} = \{t_1, t_2, t_3\}\}$, $l = \{\varnothing, \overline{X}, \{t_1, t_2\}\}$ *and* $\mathcal{P} = \mathcal{P}_{ap} = \{\varnothing, \{t_1\}, \{t_2\}, \{t_3\}, \{t_1, t_3\}, \{t_2, t_3\}\}$. *Take* $I = \{t_1\}$. *Then,* $c \circ i \circ c(I) = \overline{X}$. *Thus,* $I \subseteq c \circ i \circ c(I)$ *and hence* $I \in \overline{\beta o}(\overline{X})$. *Now,* $\psi_{\mathcal{P}}(I) = \overline{X} \setminus \{t_2, t_3\}^{\diamond}(\mathcal{P}) = \overline{X} \setminus \{t_1, t_2, t_3\} = \varnothing$ *and hence* $I \nsubseteq c(\psi_{\mathcal{P}}(I))$. *Thus,* $I \notin \psi_{\mathcal{P}}(\overline{X}, l)$ *though* $\mathcal{P} = \mathcal{P}_{ap}$ *is anti nowhere dense sets primal.*

Remark 2. *If* I *is a nonempty subset with* $I^c \notin \mathcal{P}$ *where* $(\overline{X}, l, \mathcal{P})$ *is a MDM space, then* $\psi_{\mathcal{P}}(I) = \varnothing$. *This implies,* $c(\psi_{\mathcal{P}}(I)) = \varnothing$ *and hence* $I \nsubseteq c(\psi_{\mathcal{P}}(I))$. *Thus,* $I \notin \psi_{\mathcal{P}}(\overline{X}, l)$.

Also recalling that $\psi_{\mathcal{P}}(I) = \overline{X} \setminus (\overline{X} \setminus I)^{\diamond}(\mathcal{P})$ and from the Definition of $\mathcal{P}$-dense set it follows that $\psi_{\mathcal{P}}(I) = \varnothing$ iff $(\overline{X} \setminus I)$ is $\mathcal{P}$-dense. Therefore, for a MDM space $(\overline{X}, l, \mathcal{P})$, $I \notin \psi_{\mathcal{P}}(\overline{X}, l)$ if $I^c \notin \mathcal{P}$ or $(\overline{X} \setminus I)$ is $\mathcal{P}$-dense.

For the next section and the power of MDM space, we discuss the following lemma.

Lemma 8. *Let $(\overline{X}, l, \mathcal{P})$ be a MDM space. Then, for each $G \in l^{\diamond\mathcal{P}}$, $c^{\diamond\mathcal{P}}(G) = c(G)$ ($c^{\diamond\mathcal{P}}$ denotes the closure operator of $l^{\diamond\mathcal{P}}$ topology).*

Proof. For each $G \in l^{\diamond}$, $G \subset \psi_{\mathcal{P}}(G) \subset [\psi_{\mathcal{P}}(G)]^{\diamond}(\mathcal{P}) \subset (G^{\diamond}(\mathcal{P}))^{\diamond}(\mathcal{P})$ by [4] and the Theorem 4. This implies, $c^{\diamond}(G) = c(G)$ [4]. □

6. Sets Related to Mixed Operator $\psi_{\mathcal{P}}$ and $\diamond$

The section will discuss about some sets related to mixed operators $()^{\diamond}$ and $\psi_{\mathcal{P}}$. Their properties and examples will also be discussed here.

Definition 2. *A subset I of a* **PTS** *$(\overline{X}, l, \mathcal{P})$ is called $\psi^{\diamond}_{\mathcal{P}}$ set if $I \subset [\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P})$.*

The collection of all $\psi^{\diamond}_{\mathcal{P}}$ sets in a **PTS** $(\overline{X}, l, \mathcal{P})$ denoted as $\psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. Clearly, $\varnothing \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. But $\overline{X} \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$ if the space is a MDM space.

As an application of the Lemma 8, we can consider following:

Let $\mathcal{P}$ be a primal on a **TS** $(\overline{X}, l)$ contained in the ultraprimal $\mathbb{P}$. Then, $\psi^{\diamond}_{\mathbb{P}}(\overline{X}, l) \subseteq \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$.

Following theorem gives the existence of the $\psi^{\diamond}_{\mathcal{P}}$ set.

Theorem 11. *Let $(\overline{X}, l, \mathbb{P})$ be a MDM space. Then,*

(i) For any regular open set I, $I \in \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$.

(ii) $\psi_{\mathbb{P}}(\overline{X}, l) \subset \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$.

(iii) $I \in \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$, when $I \in \overline{po}(\overline{X})$ and $I \in l^{\delta}$.

(iv) $\overline{so}(\overline{X}) \subset \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$.

Proof. (i) Suppose I be any regular open set. Since $(\overline{X}, l, \mathbb{P})$ is a MDM space, then by Theorem 2, $I = \psi_{\mathbb{P}}(I)$ and hence $I = \psi_{\mathbb{P}}(I) \subset [\psi_{\mathbb{P}}(I)]^{\diamond}$ by the Theorem 3.1 of [4]. This implies, $I \in \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$.

(ii) Let $I \in \psi_{\mathbb{P}}(\overline{X}, l)$. Then, $I \subset c(\psi_{\mathbb{P}}(I))$. Since $\psi_{\mathbb{P}}(I)$ is open and $(\overline{X}, l, \mathbb{P})$ is a MDM space, then $c(\psi_{\mathbb{P}}(I)) \subset [\psi_{\mathbb{P}}(I)]^{\diamond}$ and hence $I = \psi_{\mathbb{P}}(I) \subset [\psi_{\mathbb{P}}(I)]^{\diamond}$. Thus, $I \in \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$ and hence $\psi_{\mathbb{P}}(\overline{X}, l) \subset \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$.

(iii) Given that $I \in \overline{po}(\overline{X})$ and $I \in l^{\delta}$. Then, $I \subset i \circ c(I) \subset c \circ i(A)$. So, $I \subset c(\psi_{\mathbb{P}}(I))$ by Theorem 3 and hence $I \subset [\psi_{\mathbb{P}}(I)]^{\diamond}$. Thus, $I \in \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$.

(iv) Let $I \in \overline{so}(\overline{X})$. Then, $I \subset c \circ i(A)$. So, $I \subset c(\psi_{\mathbb{P}}(I))$ by Theorem 3 and hence $I \subset [\psi_{\mathbb{P}}(I)]^{\diamond}$. Thus, $I \in \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$ and consequently $\overline{so}(\overline{X}) \subset \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$. □

Following example shows that a $\psi^{\diamond}_{\mathcal{P}}$ set need not be a semi-open set in general though the space is a MDM space.

Example 9. *Consider a* **PTS** *$(\overline{X}, l, \mathcal{P})$, where $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1\}, \{t_2\}, \{t_1, t_2\}\}$ and $\mathcal{P} = \{\varnothing, \{t_1\}, \{t_2\}, \{t_3\}, \{t_1, t_2\}, \{t_2, t_3\}\{t_1, t_3\}\}$. Here, the space is MDM. Take $I = \{t_3\}$. Then, $c \circ i(A) = \varnothing$. Thus, $I \nsubseteq c \circ i(I)$ and hence $I \notin \overline{so}(\overline{X})$. Now, $\psi_{\mathcal{P}}(I) = \overline{X} \setminus \{t_1, t_2\}^{\diamond}(\mathcal{P}) = \overline{X} \setminus \{t_1, t_2\} = \{t_3\}$ and hence $[\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P}) = \{t_3\}$. Thus, $I \subset [\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P})$ and hence I is a $\psi^{\diamond}_{\mathcal{P}}$ set.*

Following example shows that the condition $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathbb{P}$ is a necessary condition for (iv) of the Theorem 11.

Example 10. *Consider a* **PTS** *$(\overline{X}, l, \mathcal{P})$, where $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1\}, \{t_2\}, \{t_1, t_2\}\}$ and $\mathbb{P} = \{\varnothing, \{t_1\}, \{t_2\}\{t_1, t_2\}\}$. Here, $C(\overline{X}) \setminus \{\overline{X}\} \nsubseteq \mathbb{P}$. Take $I = \{t_2, t_3\}$. Then, $c \circ i(A) = \{t_2, t_3\}$. Thus, $I \subset c \circ i(A)$ and hence $I \in \overline{so}(\overline{X})$. Now, $\psi_{\mathbb{P}}(I) = \overline{X} \setminus \{t_1\}^{\diamond}(\mathbb{P}) = \overline{X}$ and hence $[\psi_{\mathbb{P}}(I)]^{\diamond}(\mathbb{P}) = \{t_3\}$. Thus, $I \nsubseteq [\psi_{\mathbb{P}}(I)]^{\diamond}(\mathbb{P})$ and hence I is not $\psi^{\diamond}_{\mathbb{P}}$ set.*

Theorem 12. *For a MDM space $(\overline{X}, l, \mathcal{P})$,*

(i) $I \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l) \Longrightarrow I^{\diamond}(\mathcal{P})$ is closed.

(ii) $I \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l) \Longrightarrow I \in \overline{\beta o}(\overline{X})$.

Proof. (*i*) Since $I \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$, then $I \subset [\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P}) \implies I \subset [I^{\diamond}(\mathcal{P})]^{\diamond}(\mathcal{P})$ by Theorem 4 and hence $I \subset I^{\diamond}(\mathcal{P})$. Thus, $c(I^{\diamond}(\mathcal{P})) = I^{\diamond}(\mathcal{P})$ [4]. This shows that $I^{\diamond}(\mathcal{P})$ is closed.

(*ii*) Given that $I \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. Then, $I \subset [\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P})$ and hence $I \subset i \circ c(I)^{\diamond}(\mathcal{P})$ by Theorem 4 $\implies I \subset c \circ i \circ c(I)$ [4] $\implies I \in \overline{\beta o}(\overline{X})$. □

Corollary 4. *For a MDM space* $(\overline{X}, l, \mathcal{P})$,
(*i*) $l \subset l^{\diamond} \subset \psi_{\mathcal{P}}(\overline{X}, l) \subset \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l) \subset \overline{\beta o}(\overline{X})$.
(*ii*) $l \subset l^{\diamond} \subset \psi_{\mathcal{P}}(\overline{X}, l) \subset \psi^{\diamond}_{\mathcal{P}}(\overline{X}, \tau) \subset C^{\diamond}(X)$, *where* $C^{\diamond}(\overline{X}) = \{I \subset \overline{X} : I^{\diamond}(\mathcal{P}) \text{ is closed}\}$.

Following examples show that the relations are not strictly hold.

Example 11. *Consider a* **PTS** $(\overline{X}, l, \mathcal{P})$, *where* $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1\}, \{t_2\}, \{t_1, t_2\}\}$ *and* $\mathcal{P} = C(\overline{X}) \setminus \{\overline{X}\}$. *Take* $I = \{t_3\}$. *Then,* $I \notin \overline{so}(\overline{X})$ *but* $I \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$.

Example 12. *Consider a* **PTS** $(\overline{X}, l, \mathcal{P})$, *where* $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1, t_2\}\}$ *and* $\mathcal{P} = \{\varnothing, \{t_1\}, \{t_3\}, \{t_1, t_3\}\}$. *Take* $I = \{t_1\}$. *Then,* $I \in \overline{\beta o}(\overline{X})$ *but* $I \notin \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$.

For the converse inclusion, we consider the Theorem 13:

Theorem 13. *For a MDM space* $(\overline{X}, l, \mathbb{P})$, $I \in \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$ *if* $I \in \overline{\beta o}(\overline{X})$ *and* $I \in C(\overline{X})$.

Proof. Suppose that $I \in \overline{\beta o}(\overline{X})$ and $I \in C(\overline{X})$. Then, $I \subset c \circ i \circ c(I)$. This implies, $I \subset c(\psi_{\mathbb{P}}(i \circ c(I)))$ as the space is MDM. This implies, $I \subset [\psi_{\mathbb{P}}(i \circ c(I))]^{\diamond}(\mathbb{P})$ by [4] and hence $I \subset [\psi_{\mathbb{P}}(c(I))]^{\diamond}(\mathbb{P}) = [\psi_{\mathbb{P}}(I)]^{\diamond}(\mathbb{P})$ as $I \in C(\overline{X})$. Thus, $I \in \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$. □

Theorem 14. *Let* $(\overline{X}, l)$ *be a* ***TS*** *and* $\mathcal{P}$ *be a primal contained in the ultraprimal* $\mathbb{P}$. *Then,* $\psi^{\diamond}_{\mathbb{P}}(\overline{X}, l) \subset \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l) \subset \psi_{\mathcal{P}}(\overline{X}, l)$.

Proof. Let $I \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. Then, $I \subset (\psi_{\mathcal{P}}(I))^{\diamond}(\mathcal{P}) \subset c(\psi_{\mathcal{P}}(I))$. This implies, $I \in \psi_{\mathcal{P}}(\overline{X}, l)$ and hence $\psi^{\diamond}_{\mathcal{P}}(\overline{X}, l) \subset \psi_{\mathcal{P}}(\overline{X}, l)$. □

By the following example, we are to show that the converse of the Theorem 14 is not true:

Example 13. *Consider a* **PTS** $(\overline{X}, l, \mathcal{P})$, *where* $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1\}, \{t_2\}, \{t_1, t_2\}\}$ *and* $\mathcal{P} = \{\varnothing, \{t_1\}, \{t_2\}, \{t_1, t_2\}\}$. *Take* $I = \{t_2, t_3\}$. *Then,* $\psi_{\mathcal{P}}(I) = \overline{X}$ *and* $[\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P}) = \{t_1\}$. *Hence,* $I \subset c(\psi_{\mathcal{P}}(I))$ *but* $I \nsubseteq [\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P})$. *Thus,* I *is* $\psi_{\mathcal{P}}$*-C set but not* $\psi^{\diamond}_{\mathcal{P}}$ *set.*

Theorem 15. *A* **PTS** $(\overline{X}, l, \mathcal{P})$ *is a MDM space iff* $\psi^{\diamond}_{\mathcal{P}}(\overline{X}, l) = \psi_{\mathcal{P}}(\overline{X}, l)$.

Followings are the decompositions of $\psi^{\diamond}_{\mathcal{P}}$ sets:

Corollary 5. *Let* $(\overline{X}, l, \mathbb{P})$ *be a MDM space. Then,*
(*i*) $I \in \psi^{\diamond}_{\mathbb{P}}(\overline{X}, l)$ *iff* $I \in \overline{\beta o}(\overline{X})$ *and* $I \in C(\overline{X})$.
(*ii*) $I \in \psi_{\mathbb{P}}(\overline{X}, l)$ *iff* $I \in \overline{\beta o}(\overline{X})$ *and* $I \in C(\overline{X})$.

Corollary 6. *Let* $(\overline{X}, l, \mathbb{P})$ *be a* **PTS**. *Then, the followings are equivalent:*
(*i*) $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathbb{P}$;
(*ii*) $\psi_{\mathbb{P}}(\varnothing) = \varnothing$;
(*iii*) *If* $I \subset \overline{X}$ *is closed, then* $\psi_{\mathbb{P}}(I) \setminus I = \varnothing$;
(*iv*) *If* $I \subset \overline{X}$, *then* $i \circ c(I) = \psi_{\mathbb{P}}(i \circ c(I))$;
(*v*) *If* $I \subset \overline{X}$ *is regular open, then* $I = \psi_{\mathbb{P}}(I)$;
(*vi*) *If* $U \in l$, *then* $\psi_{\mathbb{P}}(U) \subset i \circ c(I) \subset U^{\diamond}(\mathbb{P})$;
(*vii*) *If* $P \notin \mathbb{P}$, *then* $\psi_{\mathbb{P}}(P) = \varnothing$;
(*viii*) $\psi^{\diamond}_{\mathbb{P}}(\overline{X}, l) = \psi_{\mathbb{P}}(\overline{X}, l)$;
(*ix*) $[\psi_{\mathbb{P}}(I)]^{\diamond}(\mathbb{P}) = c(\psi_{\mathbb{P}}(I))$ *for each* $I \subset \overline{X}$.

7. $\psi^{\diamond}_{\mathcal{P}}$ Function and Its Properties

In this section, we discuss more results related to $\psi^{\diamond}_{\mathcal{P}}$ set and define $\psi^{\diamond}_{\mathcal{P}}$ function and discuss its properties on primal topological spaces with examples:

Theorem 16. *Followings hold for a MDM space* $(\overline{X}, l, \mathcal{P})$.
(*i*) $\psi_{\mathcal{P}}(I) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$.
(*ii*) $I^{\diamond}(\mathcal{P}) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$.
(*iii*) $[\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P}) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$.

Proof. (*i*) For any subset $I \subset \overline{X}$, $\psi_{\mathcal{P}}(I) \subset \psi_{\mathcal{P}}(\psi_{\mathcal{P}}(I)) \subset [\psi_{\mathcal{P}}(\psi_{\mathcal{P}}(I))]^{\diamond}(\mathcal{P})$ since $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathcal{P}$ and $\psi_{\mathcal{P}}(\psi_{\mathcal{P}}(I)) \in l$. This implies, $\psi_{\mathcal{P}}(I) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$.

(*ii*) Since $C(\overline{X}) \setminus \{\overline{X}\} \subset \mathcal{P}$, then by Theorem 4, $\psi_{\mathcal{P}}(I) \subset I^{\diamond}(\mathcal{P})$. This implies $\psi_{\mathcal{P}}(\psi_{\mathcal{P}}(I)) \subset \psi_{\mathcal{P}}[I^{\diamond}(\mathcal{P})]$ and hence $\psi_{\mathcal{P}}(I) \subset \psi_{\mathcal{P}}(\psi_{\mathcal{P}}(I)) \subset \psi_{\mathcal{P}}[I^{\diamond}(\mathcal{P})]$. This implies $I \subset [\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P}) \subset [\psi_{\mathcal{P}}(I^{\diamond})(\mathcal{P})]^{\diamond}(\mathcal{P})$ since $A \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. Hence, $I^{\diamond}(\mathcal{P}) \subset [[\psi_{\mathcal{P}}(I^{\diamond}(\mathcal{P}))]^{\diamond}(\mathcal{P})]^{\diamond}(\mathcal{P}) \subset [\psi_{\mathcal{P}}(I^{\diamond}(\mathcal{P}))]^{\diamond}(\mathcal{P})$. This implies $I^{\diamond}(\mathcal{P}) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$.

(*iii*) Given that $I \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. This implies $\psi_{\mathcal{P}}(I) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$ by Theorem 16(*i*) and hence $[\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P}) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$ using the Theorem 16(*ii*). □

Definition 3. *Let* $(\overline{X}, l, \mathcal{P})$ *be a* **PTS** *and* $(\overline{Y}, \sigma)$ *be a* ***TS***. *A function* $f : \overline{X} \to \overline{Y}$ *is said to be* $\psi^{\diamond}_{\mathcal{P}}$*-function iff for each open set* U *in* $\overline{Y}$, $f^{-1}(U) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$.

For the example of $\psi^{\diamond}_{\mathcal{P}}$-function, we consider the followings:

Example 14. *Consider a MDM space* $(\overline{X}, l, \mathcal{P})$ *and a* ***TS*** $(\overline{Y}, \sigma)$. *Also, consider a continuous function* $f : \overline{X} \to \overline{Y}$. *Let* G *be any open set in* $\overline{Y}$, *then by continuity of* f, $f^{-1}(G)$ *is open set in* $\overline{X}$. *This implies* $f^{-1}(G) \subset \psi_{\mathcal{P}}(f^{-1}(G)) \subset [\psi_{\mathcal{P}}(f^{-1}(G))]^{\diamond}(\mathcal{P})$, *since the space is a MDM space. This implies* $f^{-1}(G) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. *Thus,* f *is* $\psi^{\diamond}_{\mathcal{P}}$*-function.*

Example 15. *Consider a MDM space* $(\overline{X}, l, \mathcal{P})$ *and a* ***TS*** $(\overline{Y}, \sigma)$. *Also, consider a semi-continuous function* $f : \overline{X} \to \overline{Y}$. *Let* G *be any open set in* $\overline{Y}$, *then by semi-continuity of* f, $f^{-1}(G)$ *is semi-open set in* $\overline{X}$. *This implies* $f^{-1}(G) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$ *by Theorem 11. Thus,* f *is* $\psi^{\diamond}_{\mathcal{P}}$*-function.*

Theorem 17. *Let* $(\overline{X}, l, \mathcal{P})$ *be a MDM space and* $(\overline{Y}, \sigma)$ *be a* ***TS***. *If* $f : \overline{X} \to \overline{Y}$ *be a* $\psi^{\diamond}_{\mathcal{P}}$*-function. Then,*
(*i*) $[f^{-1}(U)]^{\diamond}(\mathcal{P})$ *is a* $\psi^{\diamond}_{\mathcal{P}}$*-set for each open set* U *in* $\overline{Y}$.
(*ii*) $[f^{-1}(U)]^{\diamond}(\mathcal{P}) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$ *is a* $\psi^{\diamond}_{\mathcal{P}}$*-set for each open set* U *in* $\overline{Y}$.

Proof. (*i*) Since $f : \overline{X} \to \overline{X}$ be a $\psi^{\diamond}_{\mathcal{P}}$-function, then for each open set U in $\overline{Y}$, $f^{-1}(U) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. This implies, $[f^{-1}(U)]^{\diamond}(\mathcal{P}) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$ by Theorem 16. Hence the result.

(*ii*) Since $f : \overline{X} \to \overline{X}$ be a $\psi^{\diamond}_{\mathcal{P}}$-function, then for each open set U in $\overline{Y}$, $f^{-1}(U) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. This implies, $[\psi_{\mathcal{P}}(f^{-1}(U))]^{\diamond}(\mathcal{P}) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$ by Theorem 16. □

Example 16. *Consider a* **PTS** $(\overline{X}, l, \mathcal{P})$, *where* $\overline{X} = \{t_1, t_2, t_3\}$, $l = \{\varnothing, \overline{X}, \{t_1, t_2\}\}$ *and* $\mathcal{P} = \{\varnothing, \{t_2\}, \{t_3\}, \{t_2, t_3\}\}$. *Here,* $(\overline{X}, l, \mathcal{P})$ *is a MDM space. Define a function* $f : (\overline{X}, l, \mathcal{P}) \to (\overline{X}, l, \mathcal{P})$ *by* $f(t_1) = t_2$, $f(t_2) = t_3$, $f(t_3) = t_1$. *Take* $I = \{t_1, t_3\}$. *Then,* $\psi_{\mathcal{P}}(I) = \overline{X}$ *and hence* $[\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P}) = \overline{X}$. *Hence,* $A \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. *Again take* $S = \{t_2, t_3\}$, *then* $\psi_{\mathcal{P}}(S) = \varnothing$. *This implies* $[\psi_{\mathcal{P}}(I)]^{\diamond}(\mathcal{P}) = \varnothing$ *and hence* $B \notin \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. *Thus,* f *is a* $\psi^{\diamond}_{\mathcal{P}}$*-function as* $f^{-1}(\{t_1, t_2\}) = I \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. *But,* $f^{-1}(f^{-1}(\{t_1, t_2\})) = f^{-1}(\{t_1, t_3\}) = S \notin \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. *Hence,* $f \circ f$ *is not a* $\psi^{\diamond}_{\mathcal{P}}$*-function. Observed that though* f *is not a continuous function, it is a* $\psi^{\diamond}_{\mathcal{P}}$*-function.*

As a result, composition of two $\psi^{\diamond}_{\mathcal{P}}$-functions does not create again a $\psi^{\diamond}_{\mathcal{P}}$-function. However, the following is also true:

Theorem 18. *Let* $(\overline{X}, l, \mathcal{P})$ *be a* **PTS**. *If* $g : (\overline{X}, l, \mathcal{P}) \to (\overline{Y}, \sigma)$ *be a* $\psi^{\diamond}_{\mathcal{P}}$*-function and* $f : (\overline{Y}, \sigma) \to (\overline{Z}, \mu)$ *be a continuous function, then* $f \circ g : (\overline{X}, l, \mathcal{P}) \to (\overline{Z}, \mu)$ *is a* $\psi^{\diamond}_{\mathcal{P}}$*-function.*

Proof. Since $f : (\overline{Y}, \sigma) \to (\overline{Z}, \mu)$ is a continuous function, then for each open set W in $\overline{Z}$, $f^{-1}(W)$ is open in $\overline{Y}$. Also, since $g : (\overline{X}, l, \mathcal{P}) \to (\overline{Y}, \sigma)$ is a $\psi^{\diamond}_{\mathcal{P}}$-function, then $g^{-1}(f^{-1}(W))$ is a $\psi^{\diamond}_{\mathcal{P}}$-set. This implies $(f \circ g)^{-1}(W)$ is a $\psi^{\diamond}_{\mathcal{P}}$-set and hence $f \circ g$ is a $\psi^{\diamond}_{\mathcal{P}}$-function. □

Theorem 19. *Let* $(\overline{X}, l, \mathcal{P})$ *be a MDM space. If* $f : (\overline{X}, l, \mathcal{P}) \to (\overline{Y}, \sigma)$ *is a* $\psi^{\diamond}_{\mathcal{P}}$*-function, then* f *is* β*-continuous.*

Proof. Since $f : (\overline{X}, l, \mathcal{P}) \to (\overline{Y}, \sigma)$ is a $\psi^{\diamond}_{\mathcal{P}}$-function, then for each open set U in $\overline{Y}$, $f^{-1}(U) \in \psi^{\diamond}_{\mathcal{P}}(\overline{X}, l)$. Then, by Theorem 12, $f^{-1}(U) \in \overline{\beta o}(\overline{X})$ and hence, f is β-continuous. □

8. Conclusion

The authors conclude followings for this article: The ideal, grill and filter are not only mathematical structures to discuss the common limit points of a set. To discuss the common frame of limit points of a set, the mathematical structure primal is also a tool. Again, the mathematical structures ideal, grill, filter and primal are related to one another by certain set operations. So, one can shift the study of primal through the other mathematical structures like ideal, grill, filter etc. Furthermore, in the field of the each of the above mathematical structure, Zorn's Lemma can be applicable and the study will be more effective for the study of the limit points of a set.

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