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# FINITE ELEMENT ANALYSIS OF LINEARLY EXTRAPOLATED BLENDED BACKWARD DIFFERENCE FORMULA (BLEBDF) FOR THE NATURAL CONVECTION FLOWS

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Abstract. In this paper, we study the stability and convergence of fully discrete finite element method with grad-div stabilization for the incompressible non-isothermal fluid flows. The proposed scheme uses finite element discretization in space and linearly extrapolated blended Backward Differentiation Formula (BLEBDF) in time. We prove the unconditional stability over finite time interval and optimally convergence of the scheme. We also present numerical experiments to verify our theoretical convergence rates and show the reliability of the scheme.

## 1. Introduction

Most of practical engineering problems including insulating in windows, solar collectors, cooling in electronics are modelled by natural convection flows. In the dimensionless form, the equations governed by these flows are given on the domain $\Omega \subset \mathbb{R}^d (d = 2 \text{ or } 3)$ and a time interval $(0, t^*]$, $t^* < \infty$, as follows

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = RiT\boldsymbol{\xi} + \mathbf{f}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

$$T_t + (\mathbf{u} \cdot \nabla)T - \kappa\Delta T = g, \tag{1.3}$$

where $\mathbf{u}$ is the velocity filed, $p$ the pressure, $T$ the temperature and $\mathbf{f}$ and $g$ are the external forcing and thermal source. $\boldsymbol{\xi}$ is the unit vector in the direction of the gravitational acceleration, $\nu$ is the dimensionless kinematic viscosity which is inversely proportional to the Reynolds number, i.e. $\nu = Re^{-1}$, $\kappa$ the thermal conductivity defined as $\kappa = Re^{-1}Pr^{-1}$ where $Pr$ is the Prandtl number and $Ri$ the Richardson number, and Rayleigh number is defined by $Ra = RiRe^2Pr$. The system is complemented with the appropriate initial and boundary conditions.

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This system is well posed under some restriction on the Rayleigh and Prandtl numbers [21]. Simulations with standard Galerkin finite element method of (1.1)-(1.3) for high Rayleigh number leads to severe computational problems and can exhibit global spurious oscillations, [21, 6, 18]. One remedy to overcome this issue is to use the grad-div stabilization. This type of stabilization adds the penalization term $\gamma\nabla(\nabla\cdot\mathbf{u})$ to the momentum equation which leads to $\gamma(\nabla\cdot\mathbf{u}_h, \nabla\cdot\mathbf{v}_h)$ in the discretization. It was originally proposed in [3] and since then it has studied from both theoretical and computational points of view. The studies on grad-div stabilization show that this stabilization improves mass conservation, leads to much more accurate approximate solutions for the Stokes/Navier-Stokes and related coupled multiphysics flow problems, [5, 12, 14, 15, 19, 20].
The aim of this study is to use this advantage of grad-div stabilized finite element for approximating of the natural convection flows. For the temporal discretization, a new second order time stepping scheme called an blended three step Backward Differentiation Formula (BDF) is used. This selection is due to the fact that such scheme is of second order accuracy with a smaller constant in truncation error term, $A$-stable and is more accurate than two-step BDF scheme, [22, 16, 2, 10].
The remaining of the paper is organized as follows. Section 2 presents some mathematical preliminaries necessary for the finite element analysis. Section 3 introduces the numerical scheme. Section 4 and 5 provides theoretical results of the stability and convergence. We show that approximate solutions are unconditionally stable over finite time interval and converge both in time and space quadratically. Section 6 provides two numerical experiments. The first one verifies the second order convergence in space and time. The second one, on the other hand, tests the reliability and efficiency of the algorithm. For this aim, we compare the solutions of the scheme with BLEBDF (without the stabilization) on Marsigli's experiment. The results show that our method captures very well the flow pattern at each time level.

## 2. Mathematical Preliminaries

We consider the domain $\Omega\subset\mathbb{R}^d, d=2,3$ to be a convex polygon or polyhedra. The $L^2$-inner product and its induced norm will be denoted as $(\cdot,\cdot)$ and $\|\cdot\|$, the $H^k$-norm by $\|\cdot\|_k$, and the $L^\infty$-norm by $\|\cdot\|_{L^\infty}$, [1]. Continuous velocity, pressure and temperature spaces are given by respectively:

$$\mathbf{X} := \mathbf{H}_0^1(\Omega) = \{\mathbf{v}\in(L^2(\Omega))^d : \nabla\mathbf{v}\in L^2(\Omega)^{d\times d},\ \mathbf{v}=0 \ on\ \partial\Omega\},$$
$$Q := L_0^2(\Omega) = \{q\in L^2(\Omega) : \int_\Omega q\,d\mathbf{x}=0\},$$
$$Y := H_0^1(\Omega).$$

Further, we define the space $\mathbf{V}\subset\mathbf{X}$ to be the divergence free subset of $\mathbf{X}$. The dual space of $\mathbf{X}$ is denoted by $\boldsymbol{H}^{-1}$ with the norm

$$\|\mathbf{f}\|_{-1} := \sup_{0\neq\mathbf{v}\in\mathbf{X}} \frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla\mathbf{v}\|}.$$

We frequently use the Poincaré-Friedrich's inequality, [11]: there exists a constant $C_P$ such that

$$\|\mathbf{v}\| \le C_P\|\nabla\mathbf{v}\|,\ \ \forall\mathbf{v}\in\mathbf{X}.$$

Define the skew symmetrized trilinear form for the non-linear terms to ensure stability of the numerical method

$$b_1(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}\left[(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - (\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})\right], \quad \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X},$$

$$b_2(\mathbf{u},\theta,\Phi) := \frac{1}{2}\left[(\mathbf{u}\cdot\nabla\theta,\Phi) - (\mathbf{u}\cdot\nabla\Phi,\theta)\right], \quad \mathbf{u}\in\mathbf{X},\theta,\Phi\in Y.$$

**Lemma 2.1.** *For $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}$ and $\mathbf{v},\nabla\mathbf{v}\in L^\infty$, the skew symmetrized trilinear form $b(\cdot,\cdot,\cdot)$ is bounded as follows,* [13]

$$b_1(\mathbf{u},\mathbf{v},\mathbf{w}) \le C\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|, \tag{2.1}$$

$$b_1(\mathbf{u},\mathbf{v},\mathbf{w}) \le C\|\mathbf{u}\|\left(\|\nabla\mathbf{v}\|_{L^3} + \|\mathbf{v}\|_{L^\infty}\right)\|\nabla\mathbf{w}\|. \tag{2.2}$$

We assume that $\tau_h$ is a regular, conforming mesh with a maximum diameter $h$, and $\mathbf{X}_h\subset\mathbf{X}$, $Q_h\subset Q$, $Y_h\subset Y$ be conforming finite element spaces which satisfy approximation properties of piece-wise polynomials of local degree $k, k-1$, and $k$ with $k\ge 1$ respectively, [7]

$$\inf_{\mathbf{v}_h\in\mathbf{X}_h}\left\{\|\mathbf{u}-\mathbf{v}_h\| + h\|\nabla(\mathbf{u}-\mathbf{v}_h)\|\right\} \le Ch^{k+1}\|\mathbf{u}\|_{k+1},$$

$$\inf_{q_h\in Q_h}\|p-q_h\| \le Ch^k\|p\|_k.$$

$$\inf_{T_h\in Y_h}\left\{\|T-T_h\| + h\|\nabla(T-T_h)\|\right\} \le Ch^{k+1}\|T\|_{k+1}.$$

We also assume that the velocity-pressure finite element pair, $(\mathbf{X}_h, Q_h)$, satisfy the discrete inf-sup condition:

$$\inf_{q_h\in Q_h}\sup_{\mathbf{v}_h\in\mathbf{X}_h}\frac{(q_h,\,\nabla\cdot\mathbf{v}_h)}{||\nabla\mathbf{v}_h\,||\,||\,q_h\,||} \ge \beta > 0.$$

We denote the discretely divergence-free space by $\mathbf{V}_h$ and defined as

$$\mathbf{V}_h := \{\mathbf{v}_h\in\mathbf{X}_h :\ (\nabla\cdot\mathbf{v}_h, q_h) = 0,\ \forall q_h\in Q_h\}.$$

We also introduce the following norms on time interval $[0,t^*]$: $1\le p<\infty$

$$\|\phi\|_{p,k} := \left(\int_0^{t^*}\|\phi(t,\cdot)\|_k^p dt\right)^{1/p}, \quad \|\phi\|_{\infty,k} := \sup_{0\le t\le t^*}\|\phi(t,\cdot)\|_\infty$$

and discrete norms

$$|||\phi|||_{p,k} := \left(\Delta t\sum_{n=0}^{N-1}\|\phi(t^n,\cdot)\|_k^p\right)^{1/p}, \quad |||\phi|||_{\infty,k} := \max_{0\le n\le N}\|\phi(t^n,\cdot)\|_k,$$

where $t^n = n\Delta t,\ \ n = 0,1,2,...,N = t^*/\Delta t$.
We also introduce the following notations for the stability and convergence analysis

$$\delta[\phi^{n+1}] := \frac{5}{3}\phi^{n+1} - \frac{5}{2}\phi^n + \phi^{n-1} - \frac{1}{6}\phi^{n-2}, \tag{2.3}$$

$$E[\phi^{n+1}] := \phi^{n+1} - 3\phi^n + 3\phi^{n-1} - \phi^{n-2}, \tag{2.4}$$

and estimates which are the conclusion of the use of Taylor's Theorem with integral remainder term, [10]:

$$\|\phi_t^{n+1} - \frac{\delta[\phi^{n+1}]}{\Delta t}\|^2 \leq \frac{7}{3}(\Delta t)^3 \int_{t^{n-2}}^{t^{n+1}} \|\phi_{ttt}\|^2 dt, \tag{2.5}$$

$$\|E[\phi^{n+1}]\|^2 \leq 9\Delta t^5 \int_{t^{n-2}}^{t^{n+1}} \|\phi_{ttt}\|^2 dt. \tag{2.6}$$

To simplify our finite element analysis, we use the G-stability framework as in [8]. For third order backward differentiation, the positive definite matrix G-matrix and the associated norm are given as

$$G = \frac{1}{12}\begin{pmatrix} 19 & -12 & 3 \\ -12 & 10 & -3 \\ 3 & -3 & 1 \end{pmatrix}, \ \|\mathcal{U}\|_G^2 = (\mathcal{U}, G\mathcal{U}), \ \mathcal{U} \in L^2(\Omega).$$

For $\mathcal{U}^{n+1} := \begin{bmatrix} u^{n+1} & u^n & u^{n-1}\end{bmatrix}^T, \ \forall u^i \in L^2(\Omega)$, the following identity holds, [2]

$$\left(\delta[u^{n+1}], u^{n+1}\right) = \|\mathcal{U}^{n+1}\|_G^2 - \|\mathcal{U}^n\|_G^2 + \frac{1}{12}\|E[u^{n+1}]\|^2. \tag{2.7}$$

The G- and $L^2$-norms are equivalent in the sense that: there exist $C_l, C_u > 0$ positive constants such that

$$C_l\|\mathcal{U}\|_G^2 \leq \|\mathcal{U}\|^2 \leq C_u\|\mathcal{U}\|_G^2. \tag{2.8}$$

We also use Young's inequality and the discrete version of Gronwall Lemma.

**Lemma 2.2.** *Let $a, b$ be non-negative real numbers. Then for any $\varepsilon > 0$,*

$$ab \leq \frac{\varepsilon}{p}a^p + \frac{\varepsilon^{-q/p}}{q}b^q; \quad \frac{1}{p} + \frac{1}{q} = 1, \ \text{and} \ \ 1 \leq p, q \leq \infty.$$

**Lemma 2.3** (Gronwall Lemma)**.** *Let $\Delta t, H$ and $a_n, b_n, c_n, d_n$ be non-negative numbers such that*

$$a_N + \Delta t \sum_{n=0}^{N} b_n \leq \Delta t \sum_{n=0}^{N-1} d_n a_n + \Delta t \sum_{n=0}^{N-1} c_n + H, \ \text{for } N \geq 0.$$

*Then for all $\Delta t > 0$*

$$a_N + \Delta t \sum_{n=0}^{N} b_n \leq \exp\left(\Delta t \sum_{n=0}^{N-1} d_n\right)\left(\Delta t \sum_{n=0}^{N-1} c_n + H\right).$$

## 3. Numerical Scheme

The proposed numerical scheme uses three-step backward differentiation in time and finite element in space.

**Algorithm 3.1.** *Let forcing terms $\mathbf{f} \in L^2(0, t^*; \mathbf{H}^{-1}(\Omega))$, $g \in L^2(0, t^*; H^{-1}(\Omega))$ Choose an end time $t^*$ and a time step $\Delta t$ such that $t^* = N\,\Delta t$. Denote the discrete solutions at time levels $t^n := n\Delta t$ by*

$$\mathbf{u}_h^n := \mathbf{u}_h(t^n), \ \ p_h^n := p_h(t^n), \ \ T_h^n := T_h(t^n), \quad n = 0, 1, 2, ..., N.$$

*Let initial conditions* $\mathbf{u}_h^{-2}, \mathbf{u}_h^{-1}, \mathbf{u}_h^0 \in \boldsymbol{L}^2(\Omega)$ *and* $T_h^{-2}, T_h^{-1}, T_h^0 \in L^2(\Omega)$ *be given. For* $n = 0, ..., N-1$, *find* $(\mathbf{u}_h^{n+1},\ p_h^{n+1}, T_h^{n+1})$ *such that the equations below are satisfied:* $\forall(\mathbf{v}_h, q_h, \chi_h) \in (\mathbf{X}_h, Q_h, Y_h)$

$$\left(\frac{\delta[\mathbf{u}_h^{n+1}]}{\Delta t}, \mathbf{v}_h\right) + b_1\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}_h^{n+1}, \mathbf{v}_h\right) + \nu\left(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h\right)$$
$$+ \gamma\left(\nabla \cdot \mathbf{u}_h^{n+1}, \nabla \cdot \mathbf{v}_h\right) - \left(p_h^{n+1}, \nabla \cdot \mathbf{v}_h\right) = Ri(\left(3T_h^n - 3T_h^{n-1} + T_h^{n-2}\right)\boldsymbol{\xi}, \mathbf{v}_h)$$
$$+ \left(\mathbf{f}^{n+1}, \mathbf{v}_h\right), \tag{3.1}$$
$$\left(\nabla \cdot \mathbf{u}_h^{n+1},\ q_h\right) = 0, \tag{3.2}$$
$$\left(\frac{\delta[T_h^{n+1}]}{\Delta t},\ \chi_h\right) + b_2\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, T_h^{n+1}, \chi_h\right) + \kappa\left(\nabla T_h^{n+1}, \nabla \chi_h\right)$$
$$= \left(g^{n+1}, \chi_h\right). \tag{3.3}$$

## 4. Stability Analysis

This section is devoted to proving the stability of Algorithm 3.1. We show that the proposed algorithm's solutions are stable over finite time interval without any time step restriction.

**Lemma 4.1.** *The solutions of Algorithm 3.1 are unconditionally stable over* $(0, t^*)$ *and they satisfy the bound: for any* $\Delta t > 0$

$$\|\mathcal{T}_h^N\|_G^2 + \frac{1}{2}\kappa\Delta t \sum_{n=0}^{N-1} \|T_h^{n+1}\|^2 \leq \|\mathcal{T}_h^0\|_G^2 + \frac{1}{2}\kappa^{-1}\|g\|^2_{L^2(0,t^*;H^{-1}(\Omega))} =: K_T, \tag{4.1}$$

*and*

$$\|\mathcal{U}_h^N\|_G^2 + \frac{1}{2}\nu\Delta t \sum_{n=0}^{N-1} \|\nabla \mathbf{u}_h^{n+1}\|^2 + \gamma\Delta t \sum_{n=0}^{N-1} \|\nabla \cdot \mathbf{u}_h^{n+1}\|^2$$
$$\leq \|\mathcal{U}_h^0\|_G^2 + 27\nu^{-1}Ri^2 C_P^2 C_u K_T |\boldsymbol{\xi}|^2 t^* + \frac{1}{2}\nu^{-1}\|\mathbf{f}\|^2_{L^2(0,t^*;H^{-1}(\Omega))} =: K_{\mathbf{u}}. \tag{4.2}$$

*Proof.* We will first obtain the bound on discrete temperature solution. Letting $\chi_h = \Delta t\, T_h^{n+1}$ in (3.3), which vanishes the non-linear term, and then using (2.7) followed by the Cauchy-Schwarz and Young's inequalities we obtain

$$\|\mathcal{T}_h^{n+1}\|_G^2 - \|\mathcal{T}_h^n\|_G^2 + \frac{1}{12}\|E[T_h^{n+1}]\|^2 + \kappa\Delta t\|\nabla T_h^{n+1}\|^2$$
$$= \Delta t\left(g^{n+1}, T_h^{n+1}\right)$$
$$\leq \Delta t\|g^{n+1}||_{H^{-1}}\|\nabla T_h^{n+1}\|$$
$$\leq \frac{1}{2}\kappa^{-1}\Delta t\|g^{n+1}||^2_{H^{-1}} + \frac{1}{2}\kappa\Delta t\|\nabla T_h^{n+1}||^2. \tag{4.3}$$

From which, reordering terms gives

$$||\mathcal{T}_h^{n+1}||_G^2 - ||\mathcal{T}_h^n||_G^2 + \frac{1}{12}||E[T_h^{n+1}]||^2 + \frac{1}{2}\kappa\Delta t\|\nabla T_h^{n+1}||^2 \leq \frac{1}{2}\kappa^{-1}\Delta t\|g^{n+1}||^2_{H^{-1}}.$$

Dropping the third left hand side term and summing over time steps gives the bound on temperature solution. To get the stability bound on velocity, set $q_h = p_h^{n+1}$ in

(3.2) and $\mathbf{v}_h = \Delta t \mathbf{u}_h^{n+1}$ in (3.1). The pressure and the non-linear terms vanishes, and using the same identity produces

$$\|\mathcal{U}_h^{n+1}\|_G^2 - \|\mathcal{U}_h^n\|_G^2 + \frac{1}{12}\|E[\mathbf{u}_h^{n+1}]\|^2 + \nu\Delta t\|\nabla \mathbf{u}_h^{n+1}\|^2 + \gamma\Delta t\|\nabla\cdot\mathbf{u}_h^{n+1}\|^2$$
$$= \Delta t Ri\big(\big(3T_h^n - 3T_h^{n-1} + T_h^{n-2}\big)\boldsymbol{\xi}, \mathbf{u}_h^{n+1}\big) + \Delta t\big(\mathbf{f}^{n+1}, \mathbf{u}_h^{n+1}\big). \quad (4.4)$$

Using Cauchy-Schwarz, Young's and the Poincaré-Friedrich's inequalities together with (2.8) and (4.1) on the first right hand side term, we have

$$\begin{aligned}
&\Delta t Ri\big(\big(3T_h^n - 3T_h^{n-1} + T_h^{n-2}\big)\boldsymbol{\xi}, \mathbf{u}_h^{n+1}\big)\\
&\leq \Delta t Ri\ \big[3\|T_h^n\| + 3\|T_h^{n-1}\| + \|T_h^{n-2}\|\big]\,|\boldsymbol{\xi}| C_P\|\nabla\mathbf{u}_h^{n+1}\|\\
&\leq 27\nu^{-1}\Delta t Ri^2 C_P^2\big[\|T_h^n\|^2 + \|T_h^{n-1}\|^2 + \|T_h^{n-2}\|^2\big]\,|\boldsymbol{\xi}|^2 + \frac{1}{4}\nu\Delta t\|\nabla\mathbf{u}_h^{n+1}\|^2\\
&\leq 27\nu^{-1}\Delta t Ri^2 C_P^2 C_u\|\mathcal{T}_h^n\|_G^2|\boldsymbol{\xi}|^2 + \frac{1}{4}\nu\Delta t\|\nabla\mathbf{u}_h^{n+1}\|^2\\
&\leq 27\nu^{-1}\Delta t Ri^2 C_P^2 C_u K_T|\boldsymbol{\xi}|^2 + \frac{1}{4}\nu\Delta t\|\nabla\mathbf{u}_h^{n+1}\|^2,
\end{aligned}$$

and for the forcing term use Cauchy-Schwarz and Young's inequalities to get

$$\Delta t\big(\mathbf{f}^{n+1}, \mathbf{u}_h^{n+1}\big) \leq \nu^{-1}\Delta t\|\mathbf{f}^{n+1}\|_{H^{-1}}^2 + \frac{1}{4}\nu\Delta t\|\nabla\mathbf{u}_h^{n+1}\|^2.$$

Combining these estimates together with (4.4) gives

$$\|\mathcal{U}_h^{n+1}\|_G^2 - \|\mathcal{U}_h^n\|_G^2 + \frac{1}{12}\|E[\mathbf{u}_h^{n+1}]\|^2 + \frac{\nu\Delta t}{2}\|\nabla \mathbf{u}_h^{n+1}\|^2 + \gamma\Delta t\|\nabla\cdot\mathbf{u}_h^{n+1}\|^2$$
$$\leq 27\nu^{-1}\Delta t Ri^2 C_P^2 C_u K_T|\boldsymbol{\xi}|^2 + \nu^{-1}\Delta t\|\mathbf{f}^{n+1}\|_{H^{-1}}^2. \quad (4.5)$$

Summing over time steps, dropping the third left hand side term gives the desired stability bound on the velocity. □

## 5. Convergence analysis

This section is devoted to the finite element error analysis of Algorithm 3.1 We will show that the finite element solutions convergences to the true solutions quadratically both in time and space. In the analysis, we will use the following error notations: $\forall n = 0, 1, ..., N$

$$\mathbf{e}_{\mathbf{u}}^n := \mathbf{u}^n - \mathbf{u}_h^n, \quad e_T^n := T^n - T_h^n, \quad (5.1)$$

and error's decomposition

$$\begin{aligned}
&\mathbf{e}_{\mathbf{u}}^n := \eta_{\mathbf{u}}^n - \phi_{h,\mathbf{u}}^n, \quad \phi_{h,\mathbf{u}}^n := \mathbf{u}_h^n - \tilde{\mathbf{u}}_h^n, \quad \eta_{\mathbf{u}}^n := \mathbf{u}^n - \tilde{\mathbf{u}}_h^n,\\
&e_T^n := \eta_T^n - \phi_{h,T}^n, \quad \phi_{h,T}^n := T_h^n - \tilde{T}_h^n, \quad \eta_T^n := T^n - \tilde{T}_h^n
\end{aligned} \quad (5.2)$$

True solutions at $t^{n+1}$ satisfies the following equations:

$$\left(\frac{\delta[\mathbf{u}^{n+1}]}{\Delta t}, \mathbf{v}_h\right) + b_1(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, \mathbf{u}^{n+1}, \mathbf{v}_h) + \nu(\nabla \mathbf{u}^{n+1}, \nabla \mathbf{v}_h)$$
$$+ \gamma(\nabla\cdot\mathbf{u}^{n+1}, \nabla\cdot\mathbf{v}_h) - (p^{n+1}, \nabla\cdot\mathbf{v}_h) = Ri((3T^n - 3T^{n-1} + T^{n-2})\boldsymbol{\xi}, \mathbf{v}_h)$$
$$+ (\mathbf{f}^{n+1}, \mathbf{v}_h) + \Lambda_1(\mathbf{u}, T, \mathbf{v}_h), \tag{5.3}$$
$$(\nabla\cdot\mathbf{u}^{n+1}, q_h) = 0, \tag{5.4}$$
$$\left(\frac{\delta[T^{n+1}]}{\Delta t}, \chi_h\right) + b_2(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, T^{n+1}, \chi_h) + \kappa(\nabla T^{n+1}, \nabla \chi_h)$$
$$= (g^{n+1}, \chi_h) + \Lambda_2(\mathbf{u}, T, \chi_h), \tag{5.5}$$

where

$$\Lambda_1(\mathbf{u}, T, \mathbf{v}_h) := \left(\frac{\delta[\mathbf{u}^{n+1}]}{\Delta t} - \mathbf{u}_t^{n+1}, \mathbf{v}_h\right) - b_1(E[\mathbf{u}^{n+1}], \mathbf{u}^{n+1}, \mathbf{v}_h)$$
$$+ Ri(E[T^{n+1}]\boldsymbol{\xi}, \mathbf{v}_h), \tag{5.6}$$
$$\Lambda_2(\mathbf{u}, T, \chi_h) := \left(\frac{\delta[T^{n+1}]}{\Delta t} - T_t^{n+1}, \chi_h\right) - b_2(E[\mathbf{u}^{n+1}], T^{n+1}, \chi_h) \tag{5.7}$$

are consistency errors. We now give the bounds for the consistency errors.

**Lemma 5.1.**

$$|\Lambda_1(\mathbf{u}, T, \mathbf{v}_h)|$$
$$\leq \frac{9}{\varepsilon}\nu^{-1}\Delta t^5\left(C\|\nabla\mathbf{u}^{n+1}\|^2\|\nabla\mathbf{u}_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)} + Ri^2C_P^2|\boldsymbol{\xi}|^2\|T_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)}\right)$$
$$+ \frac{7}{6\varepsilon}\nu^{-1}(\Delta t)^3C_P^2\|\mathbf{u}_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)} + \varepsilon\nu\|\nabla\mathbf{v}_h\|^2, \tag{5.8}$$

*and*

$$|\Lambda_2(\mathbf{u}, T, \chi_h)|$$
$$\leq \frac{9}{2\varepsilon}C\,\kappa^{-1}\,\Delta t^5\|\nabla T^{n+1}\|^2\|\nabla\mathbf{u}_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)}$$
$$+ \frac{7}{6\varepsilon}\kappa^{-1}(\Delta t)^3C_P^2\|T_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)} + \varepsilon\kappa\|\nabla\chi_h\|^2. \tag{5.9}$$

*Proof.* Using the Cauchy-Schwarz, Young's and Poincaré-Friedrich's inequalities together with (2.1) produces the bounds. □

**Theorem 5.2.** *Assume that true solutions* $(\mathbf{u}, p, T)$ *satisfies the following regularity conditions*

$$\mathbf{u} \in L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega)), \quad T \in L^\infty(0,T;H^{k+1}(\Omega)), \quad p \in L^2(0,T;H^{k+1}(\Omega)),$$
$$\mathbf{u}_t \in L^2(0,T;\boldsymbol{H}^1(\Omega)), \quad T_t \in L^2(0,T;H^1(\Omega)), \quad T_{ttt} \in L^2(0,T;L^2(\Omega)),$$
$$\mathbf{u}_{ttt} \in L^2(0,T;\boldsymbol{L}^2(\Omega)\cap\boldsymbol{H}^1(\Omega)).$$

*Then the errors defined in* (5.1) *satisfy the following bound:*

$$\begin{aligned}&\|\mathbf{e}_{\mathbf{u}}^N\|_G^2 + \|e_T^N\|_G^2 + \frac{1}{12}\sum_{n=0}^{N-1}\left[\|E[\mathbf{e}_{\mathbf{u}}^{n+1}]\|^2 + \|E[e_T^{n+1}]\|^2\right]\\&+\frac{\Delta t}{2}\sum_{n=0}^{N-1}\left[\nu\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 + \kappa\|\nabla e_T^{n+1}\|^2\right] + \frac{\gamma\Delta t}{2}\sum_{n=0}^{N-1}\|\nabla\cdot\mathbf{e}_{\mathbf{u}}^{n+1}\|^2\\&\leq C(\Delta t^4 + h^{2k}),\end{aligned}$$

*where $C$ is the general constant independent of $h$ and $\Delta t$.*

*Proof.* We divide the error analysis into thee parts. In the first part, we will give the bounds for the velocity error equation and in the second part, the bounds for the temperature. In the third part, we will apply the Gronwall Lemma and triangle inequality to the error terms to finish the proof.

**Step 1 [*The error bound for the velocity*]**
Subtracting (3.1)-(3.2) from (5.3)-(5.4) and using error notations given in (5.1) produces: $\forall\, q_h \in Q_h$

$$\begin{aligned}&\left(\frac{\delta[\mathbf{e}_{\mathbf{u}}^{n+1}]}{\Delta t}, \mathbf{v}_h\right) + \nu(\nabla\mathbf{e}_{\mathbf{u}}^{n+1}, \nabla\mathbf{v}_h) + \gamma(\nabla\cdot\mathbf{e}_{\mathbf{u}}^{n+1}, \nabla\cdot\mathbf{v}_h) - (p^{n+1} - q_h, \nabla\cdot\mathbf{v}_h)\\&\quad + b_1(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, \mathbf{u}^{n+1}, \mathbf{v}_h) - b_1(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}_h^{n+1}, \mathbf{v}_h)\\&\qquad = Ri((3e_T^n - 3e_T^{n-1} + e_T^{n-2})\boldsymbol{\xi}, \mathbf{v}_h) + \Lambda_1(\mathbf{u}, T, \mathbf{v}_h), \quad (5.10)\\&\qquad (\nabla\cdot\mathbf{e}_{\mathbf{u}}^{n+1},\ q_h) = 0, \quad (5.11)\end{aligned}$$

Using error decomposition's in (5.2) and setting $\mathbf{v}_h = \Delta t\phi_{h,\mathbf{u}}^{n+1}$ and applying (2.7) gives

$$\begin{aligned}&\|\phi_{h,\mathbf{u}}^{n+1}\|_G^2 - \|\phi_{h,\mathbf{u}}^{n}\|_G^2 + \frac{1}{12}\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \nu\Delta t\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \gamma\Delta t\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2\\&= \left(\delta[\eta_{\mathbf{u}}^{n+1}], \phi_{h,\mathbf{u}}^{n+1}\right) + \nu\Delta t\left(\nabla\eta_{\mathbf{u}}^{n+1}, \nabla\phi_{h,\mathbf{u}}^{n+1}\right) + \gamma\Delta t\left(\nabla\cdot\eta_{\mathbf{u}}^{n+1}, \nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\right)\\&\quad - \Delta t\left(p^{n+1} - q_h, \nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\right) + \Delta t\, b_1\left(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}\right)\\&\quad - \Delta t b_1\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}_h^{n+1}, \phi_{h,\mathbf{u}}^{n+1}\right)\\&\quad - Ri\,\Delta t\left((3\eta_T^n - 3\eta_T^{n-1} + \eta_T^{n-2})\boldsymbol{\xi},\ \phi_{h,\mathbf{u}}^{n+1}\right)\\&\quad + Ri\,\Delta t\left((3\phi_{h,T}^n - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2})\boldsymbol{\xi},\ \phi_{h,\mathbf{u}}^{n+1}\right) - \Delta t\,\Lambda_1\left(\mathbf{u}, T, \phi_{h,\mathbf{u}}^{n+1}\right). \quad (5.12)\end{aligned}$$

We now bound below the right hand side terms of (5.12).The first term is zero due to the $L^2$-projection. The next three terms are bounded below by using the Cauchy-Schwarz and Young's inequalities as follows:

$$\begin{aligned}\nu\Delta t\left(\nabla\eta_{\mathbf{u}}^{n+1}, \nabla\phi_{h,\mathbf{u}}^{n+1}\right) &\leq \nu\Delta t\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \frac{\nu\Delta t}{4}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2,\\ \gamma\Delta t\left(\nabla\cdot\eta_{\mathbf{u}}^{n+1}, \nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\right) &\leq \gamma\Delta t\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \frac{\gamma\Delta t}{4}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2\\ \Delta t\left(p^{n+1} - q_h, \nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\right) &\leq \gamma^{-1}\Delta t\inf_{q_h\in Q_h}\|p^{n+1} - q_h\|^2 + \frac{\gamma\Delta t}{4}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2.\end{aligned}$$

For the non-linear terms, we add and subtract the terms below

$$b_1(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1})$$

to get

$$\begin{aligned} & b_1(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}) - b_1\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}_h^{n+1}, \phi_{h,\mathbf{u}}^{n+1}\right) \\ & = b_1(3\eta_{\mathbf{u}}^n - 3\eta_{\mathbf{u}}^{n-1} + \eta_{\mathbf{u}}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}) - b_1(3\phi_{h,\mathbf{u}}^n - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}) \\ & + b_1(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \eta_{\mathbf{u}}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}). \end{aligned}$$

For the first non-linear term, we apply (2.1) to get

$$\begin{aligned} & \Delta t\, b_1(3\eta_{\mathbf{u}}^n - 3\eta_{\mathbf{u}}^{n-1} + \eta_{\mathbf{u}}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}) \\ & \quad \leq C\,\Delta t\, \left(3\|\nabla\eta_{\mathbf{u}}^n\| + 3\|\nabla\eta_{\mathbf{u}}^{n-1}\| + \|\nabla\eta_{\mathbf{u}}^{n-2}\|\right) \|\nabla\mathbf{u}^{n+1}\|\|\nabla\phi_{\mathbf{u},h}^{n+1}\| \\ & \quad \leq 108C\,\Delta t\,\nu^{-1} \left(\|\nabla\eta_{\mathbf{u}}^n\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-2}\|^2\right) \|\nabla\mathbf{u}^{n+1}\|^2 + \frac{\nu\Delta t}{16}\|\nabla\phi_{\mathbf{u},h}^{n+1}\|^2. \end{aligned}$$

For the second term, use (2.2) together with Young's inequality which leads to

$$\begin{aligned} & \Delta t\, b_1\left(3\phi_{h,\mathbf{u}}^n - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}\right) \\ & \leq C\Delta t\|3\phi_{h,\mathbf{u}}^n - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}\| \left(\|\nabla\mathbf{u}^{n+1}\|_{L^3} + \|\mathbf{u}^{n+1}\|_{L^\infty}\right) \|\nabla\phi_{h,\mathbf{u}}^{n+1}\| \\ & \leq 16\,\Delta t\,C\,\nu^{-1}\left(\|\nabla\mathbf{u}^{n+1}\|_{L^3}^2 + \|\mathbf{u}^{n+1}\|_{L^\infty}^2\right) \|3\phi_{h,\mathbf{u}}^n - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}\|^2 + \frac{\nu\Delta t}{32}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2. \end{aligned}$$

For the last non-linear term, we apply (2.1) and Young's inequality to get

$$\begin{aligned} & \Delta t\, b_1(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \eta_{\mathbf{u}}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}) \\ & \qquad \leq C\,\Delta t\,\|\nabla\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\right)\|\|\nabla\eta_{\mathbf{u}}^{n+1}\|\|\nabla\phi_{h,\mathbf{u}}^{n+1}\| \\ & \qquad \leq 8\,C\,\nu^{-1}\Delta t\|\nabla\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\right)\|^2\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \frac{\nu\Delta t}{32}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2. \end{aligned}$$

The next two terms are estimated in a similar way:

$$\begin{aligned} & Ri\,\Delta t\left((3\phi_{h,T}^n - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2})\boldsymbol{\xi},\ \phi_{h,\mathbf{u}}^{n+1}\right) \\ & \qquad \leq Ri\,\Delta t\|3\phi_{h,T}^n - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\||\boldsymbol{\xi}|C_P\|\nabla\phi_{h,\mathbf{u}}^{n+1}\| \\ & \qquad \leq 16C_P^2\,Ri^2\nu^{-1}\Delta t\|3\phi_{h,T}^n - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2|\boldsymbol{\xi}|^2 + \frac{\nu\Delta t}{64}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2, \end{aligned}$$

and

$$\begin{aligned} & Ri\,\Delta t\left((3\eta_T^n - 3\eta_T^{n-1} + \eta_T^{n-2})\boldsymbol{\xi},\ \phi_{h,\mathbf{u}}^{n+1}\right) \\ & \qquad \leq Ri\,\Delta t\|3\eta_T^n - 3\eta_T^{n-1} + \eta_T^{n-2}\||\boldsymbol{\xi}|C_P\|\nabla\phi_{h,\mathbf{u}}^{n+1}\| \\ & \qquad \leq 16C_P^2\,Ri^2\nu^{-1}\Delta t\|3\eta_T^n - 3\eta_T^{n-1} + \eta_T^{n-2}\|^2|\boldsymbol{\xi}|^2 + \frac{\nu\Delta t}{64}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2. \end{aligned}$$

Now take $\mathbf{v}_h = \phi_{h,\mathbf{u}}^{n+1}$ in (5.8) with $\varepsilon = 3/32$. Then considering the resulting inequality and all these estimates above on the right hand side of (5.12) and combining like terms yields

$$\|\phi_{h,\mathbf{u}}^{n+1}\|_G^2 - \|\phi_{h,\mathbf{u}}^{n}\|_G^2 + \frac{1}{12}\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \frac{\nu\Delta t}{2}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \frac{\gamma\Delta t}{2}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2$$
$$\leq (\nu+\gamma)\Delta t\,\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \gamma^{-1}\Delta t \inf_{q_h\in Q_h}\|p^{n+1}-q_h\|^2$$
$$+ 108\,C\,\nu^{-1}\,\Delta t\,\left(\|\nabla\eta_{\mathbf{u}}^{n}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-2}\|^2\right)\|\nabla\mathbf{u}^{n+1}\|^2$$
$$+ 16C\nu^{-1}\left(\|\nabla\mathbf{u}^{n+1}\|_{L^3}^2 + \|\mathbf{u}^{n+1}\|_{L^\infty}^2\right)\|3\phi_{h,\mathbf{u}}^{n} - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}\|^2$$
$$+ 8\,C\,\nu^{-1}\Delta t\|\nabla\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\right)\|^2\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2$$
$$+ 16C_P^2 Ri^2\nu^{-1}\Delta t\|3\phi_{h,T}^{n} - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2|\boldsymbol{\xi}|^2$$
$$+ 16C_P^2\,Ri^2\nu^{-1}\Delta t\|3\eta_T^n - 3\eta_T^{n-1} + \eta_T^{n-2}\|^2|\boldsymbol{\xi}|^2$$
$$+ C\nu^{-1}(\Delta t)^4\left(\|T_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|\nabla\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2\right). \tag{5.13}$$

**Step 2** [***The error bound for the temperature***] First subtract (3.3) from (5.5) and consider error notations given in (5.1) to get

$$\left(\frac{\delta[e_T^{n+1}]}{\Delta t},\chi_h\right) + \kappa(\nabla e_T^{n+1},\nabla\chi_h) + b_2(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, T^{n+1},\chi_h)$$
$$- b_2(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, T_h^{n+1},\chi_h) = \Lambda_2(\mathbf{u},T,\chi_h). \tag{5.14}$$

Then using error decompositions, setting $\chi_h = \Delta t\phi_{h,T}^{n+1}$ and recalling (2.7) gives

$$\|\phi_{h,T}^{n+1}\|_G^2 - \|\phi_{h,T}^{n}\|_G^2 + \frac{1}{12}\|E[\phi_{h,T}^{n+1}]\|^2 + \kappa\Delta t\|\nabla\phi_{h,T}^{n+1}\|^2$$
$$= \left(\delta[\eta_T^{n+1}],\phi_{h,T}^{n+1}\right) + \kappa\Delta t\left(\nabla\eta_T^{n+1},\nabla\phi_{h,T}^{n+1}\right)$$
$$+ \Delta t b_2(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, T^{n+1},\phi_{h,T}^{n+1})$$
$$- \Delta t b_2(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, T_h^{n+1},\phi_{h,T}^{n+1}) - \Delta t\Lambda_2\left(\mathbf{u},T,\phi_{h,T}^{n+1}\right). \tag{5.15}$$

Proceeding in a similar way as in Step 1, we can bound the right hand side of (5.15) as follows

$$\|\phi_{h,T}^{n+1}\|_G^2 - \|\phi_{h,T}^{n}\|_G^2 + \frac{1}{12}\|E[\phi_{h,T}^{n+1}]\|^2 + \frac{\kappa\Delta t}{2}\|\nabla\phi_{h,T}^{n+1}\|^2$$
$$\leq \kappa\Delta t\,\|\nabla\eta_T^{n+1}\|^2 + 108\,C\,\kappa^{-1}\,\Delta t\,\left(\|\nabla\eta_{\mathbf{u}}^{n}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-2}\|^2\right)\|\nabla T^{n+1}\|^2$$
$$+ 16\,C\kappa^{-1}\left(\|\nabla T^{n+1}\|_{L^3}^2 + \|T^{n+1}\|_{L^\infty}^2\right)\|3\phi_{h,T}^{n} - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2$$
$$+ 8\,C\,\kappa^{-1}\Delta t\|\nabla\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\right)\|^2\|\nabla\eta_T^{n+1}\|^2$$
$$+ C\,\kappa^{-1}(\Delta t)^4\left(C_P^2\|T_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|\nabla\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2\right). \tag{5.16}$$

**Step 3** [***The application of the Gronwall Lemma***] Add (5.13) to (5.16) to get

$$
\begin{aligned}
&\left(\|\phi_{h,\mathbf{u}}^{n+1}\|_G^2 + \|\phi_{h,T}^{n+1}\|_G^2\right) - \left(\|\phi_{h,\mathbf{u}}^{n}\|_G^2 + \|\phi_{h,T}^{n}\|_G^2\right) + \frac{1}{12}\left(\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \|E[\phi_{h,T}^{n+1}]\|^2\right)\\
&+ \frac{\nu\Delta t}{2}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \frac{\kappa\Delta t}{2}\|\nabla\phi_{h,T}^{n+1}\|^2 + \frac{\gamma\Delta t}{2}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2\\
&\le (\nu+\gamma^{-1})\Delta t\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \kappa\Delta t\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \gamma^{-1}\Delta t\inf_{q_h\in Q_h}\|p^{n+1}-q_h\|^2\\
&+ 108\,C\Delta t\left[\nu^{-1}\|\nabla\mathbf{u}^{n+1}\|^2 + \kappa^{-1}\|\nabla T^{n+1}\|^2\right]\left[\|\nabla\eta_{\mathbf{u}}^{n}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-2}\|^2\right]\\
&+ 16C\nu^{-1}\Delta t\left[\|\nabla\mathbf{u}^{n+1}\|_{L^3}^2 + \|\mathbf{u}^{n+1}\|_{L^\infty}^2\right]\|3\phi_{h,\mathbf{u}}^{n} - 3\phi_{h,\mathbf{u}}^{n-1} + 3\phi_{h,\mathbf{u}}^{n-2}\|^2\\
&+ 16C\kappa^{-1}\Delta t\left[\|\nabla T^{n+1}\|_{L^3}^2 + \|T^{n+1}\|_{L^\infty}^2\right]\|3\phi_{h,\mathbf{u}}^{n} - 3\phi_{h,\mathbf{u}}^{n-1} + 3\phi_{h,\mathbf{u}}^{n-2}\|^2\\
&+ 8C\Delta t(\nu^{-1}\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \kappa^{-1}\|\nabla\eta_{T}^{n+1}\|^2)\|\nabla(3\mathbf{u}_h^{n} - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2})\|^2\\
&+ 16C_P^2 Ri^2\nu^{-1}\Delta t\|3\phi_{h,T}^{n} - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2|\boldsymbol{\xi}|^2\\
&+ 16C_P^2 Ri^2\nu^{-1}\Delta t\|3\eta_{T}^{n} - 3\eta_{T}^{n-1} + \eta_{T}^{n-2}\|^2|\boldsymbol{\xi}|^2\\
&+ C\nu^{-1}(\Delta t)^4\left(\|\nabla\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|T_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2\right)\\
&+ C\kappa^{-1}(\Delta t)^4\left(\|\nabla\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|T_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2\right).
\end{aligned}
$$

Then summing over time steps and using approximating properties produces

$$
\begin{aligned}
&\|\phi_{h,\mathbf{u}}^{N}\|_G^2 + \|\phi_{h,T}^{N}\|_G^2 + \frac{1}{12}\sum_{n=0}^{N-1}\left[\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \|E[\phi_{h,T}^{n+1}]\|^2\right]\\
&+ \frac{\Delta t}{2}\sum_{n=0}^{N-1}\left[\nu\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \kappa\|\nabla\phi_{h,T}^{n+1}\|^2\right] + \frac{\gamma\Delta t}{2}\sum_{n=0}^{N-1}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2\\
&\le C\Delta t\sum_{n=0}^{N-1}M^{n+1}\left[\|3\phi_{h,\mathbf{u}}^{n} - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}\|^2 + \|3\phi_{h,T}^{n} - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2\right]\\
&\quad + \Delta t\sum_{n=0}^{N-1}4C\left[\nu^{-1}\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \kappa^{-1}\|\nabla\eta_{T}^{n+1}\|^2\right]\|\nabla(3\mathbf{u}_h^{n} - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\|^2\\
&\quad + C(\nu^{-1}+\kappa^{-1})|||\eta_{\mathbf{u}}^{n+1}|||_{2,0}^2 + CRi^2\nu^{-1}|||\eta_{T}^{n+1}|||_{2,0}^2\\
&\quad + C\gamma^{-1}\inf_{q_h\in Q_h}|||p-q_h|||_{2,0}^2\\
&\quad + C\Delta t^4\left[(\nu^{-1}+\kappa^{-1})(\|\nabla\mathbf{u}_{ttt}\|_{2,0}^2 + \|T_{ttt}\|_{2,0}^2) + \nu^{-1}\|\mathbf{u}_{ttt}\|_{2,0}^2\right]\\
&\quad + \|\phi_{h,\mathbf{u}}^{0}\|_G^2 + \|\phi_{h,T}^{0}\|_G^2,
\end{aligned}
$$

where

$$
\begin{aligned}
M^{n+1} := \max\{&16C\nu^{-1}[\|\nabla\mathbf{u}^{n+1}\|_{L^3}^2 + \|\mathbf{u}^{n+1}\|_{L^\infty}^2], 16C\kappa^{-1}[\|\nabla T^{n+1}\|_{L^3}^2 + \|T^{n+1}\|_{L^\infty}^2],\\
&16C_P^2 Ri^2\nu^{-1}|\boldsymbol{\xi}|^2\}.
\end{aligned}
$$

Now apply the Gronwall's Lemma to get

$$
\begin{aligned}
&\|\phi_{h,\mathbf{u}}^N\|_G^2 + \|\phi_{h,T}^N\|_G^2 + \frac{1}{12}\sum_{n=0}^{N-1}\Big[\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \|E[\phi_{h,T}^{n+1}]\|^2\Big] \\
&+ \frac{\Delta t}{2}\sum_{n=0}^{N-1}\Big[\nu\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \kappa\|\nabla\phi_{h,T}^{n+1}\|^2\Big] + \frac{\gamma\Delta t}{2}\sum_{n=0}^{N-1}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2 \\
&\le \exp\left(\Delta t\sum_{n=0}^{N-1}M^{n+1}\right) \\
&\Bigg(\Delta t\sum_{n=0}^{N-1}C\left(\nu^{-1}+\kappa^{-1}\right)\|\nabla(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\|^2 + C(\nu^{-1}+\kappa^{-1})\||\eta_{\mathbf{u}}^{n+1}\||_{2,0}^2 \\
&\qquad + CRi^2\nu^{-1}\||\eta_T^{n+1}\||_{2,0}^2 + C\gamma^{-1}\inf_{q_h\in Q_h}\||p-q_h|\|_{2,0}^2 \\
&\qquad + C\Delta t^4\left[(\nu^{-1}+\kappa^{-1})(\|\nabla\mathbf{u}_{ttt}\|_{2,0}^2 + \|T_{ttt}\|_{2,0}^2) + \nu^{-1}\|\mathbf{u}_{ttt}\|_{2,0}^2\right]\Bigg)
\end{aligned}
$$

Using the stability result on the right hand side, drooping the third left hand side term and applying the triangle inequality to the error terms finishes the proof. □

## 6. Numerical Studies

In this section, we impose two numerical experiments. The first numerical experiment verify our convergence rates obtained in Theorem 5.2 while the second one reveals the effectiveness of the proposed algorithm. The numerical experiment was implemented using the software package FreeFem++, [9].

### 6.1. Convergence rate verification.

To verify theoretical findings, we pick true velocity, pressure and temperature solutions

$$
\begin{aligned}
\mathbf{u}(\mathbf{x},t) &= \begin{pmatrix}\cos(\pi(y-t)) \\ \sin(\pi(x+t))\end{pmatrix}\exp(t), \\
p(\mathbf{x},t) &= \sin(x+y)(1+t^2), \quad T(\mathbf{x},t) = \sin(\pi x) + y\exp(t),
\end{aligned}
$$

on region $\Omega = (0,1)\times(0,1)$ with $\nu = \kappa = Ri = \gamma = 1.0$. Forcing terms $\mathbf{f}$ and $g$ are calculated from (1.1)-(1.2). We impose the following boundary conditions:

$$
\mathbf{u}_h(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t), \quad T_h(\mathbf{x},t) = T(\mathbf{x},t) \text{ on } \partial\Omega.
$$

To verify the spatial convergence rates, fix end time $t^* = 0.001$ with a time step $\Delta t = t^*/8$, and use $(\mathbf{P}_2, P_1, P_2)$ finite elements Then we run Algorithm 3.1 on successively mesh refinements. The calculated rates show that the spatial convergence for both velocity and temperate are of second order, see Table 1 .

For the temporal convergence rate verification, we fix mesh size $h = 1/128$, and take end time $t^* = 1$. Then we calculate the solutions of Algorithm 3.1 for $\Delta t = 1/4, 1/8, 1/16, 1/32, 1/64$. The errors and rates obtained from these calculations are presented in Table 2 and verify our theoretical findings.

| $h$ | $\|\|\|\nabla(\mathbf{u}-\mathbf{u}_h)\|\|\|_{2,0}$ | Rate | $\|\|\|\nabla(T-T_h)\|\|\|_{2,0}$ | Rate |
|---|---|---|---|---|
| 1/4 | 8.8479e-5 | – | 6.2564e-5 | – |
| 1/8 | 2.2657e-5 | 1.9653 | 1.6021e-5 | 1.9978 |
| 1/16 | 5.6982e-6 | 1.9914 | 4.0292e-6 | 1.9654 |
| 1/32 | 1.4267e-6 | 1.9978 | 1.0088e-6 | 1.9978 |
| 1/64 | 3.5681e-7 | 1.9995 | 2.52230e-7 | 1.9994 |

Table 1. Spatial velocity and temporal errors and rates.

| $\Delta t$ | $\|\|\|\nabla(\mathbf{u}-\mathbf{u}_h)\|\|\|_{2,0}$ | Rate | $\|\|\|\nabla(T-T_h)\|\|\|_{2,0}$ | Rate |
|---|---|---|---|---|
| 1/4 | 2.0299e-3 | – | 1.4317e-3 | – |
| 1/8 | 5.1350e-4 | 1.9829 | 3.6024e-2 | 1.9840 |
| 1/16 | 1.2974e-4 | 1.9845 | 9.0220e-5 | 1.9829 |
| 1/32 | 3.2319e-5 | 2.0052 | 2.2566e-5 | 1.9993 |
| 1/64 | 8.0195e-6 | 2.0108 | 5.6420e-6 | 1.9999 |

Table 2. Temporal velocity and temperature errors and rates.

6.2. **Marsigli's experiment.** This numerical experiment tests and aims to show the effectiveness of Algorithm 3.1 on a physical situation which demonstrates that when two fluids with different densities meet, a motion driven by the gravitational force is created: the fluid with higher density rises over the lower one. Since the density differences can be modelled by the temperature differences with the help of the Boussinesq approximation, this physical problem is modelled by the incompressible Boussinesq system (1.1)-(1.3) studied herein. For the experiment's set-up, we follow the paper written by Johnston and co-workers, [17]. The domain is an insulated box, $\Omega = [0, 8] \times [0, 1]$ and dimensionless flow parameters are set to be

$$Re = 1.000,\ Ri = 4.0,\ Pr = 1.0,\ \gamma = 1.0.$$

We take time step $\Delta t = 0.2$ and use $(\boldsymbol{P}_2, P_1, P_2)$ finite element spaces for the velocity, the pressure and temperature. We run both our algorithm and BLEBDF without grad-div stabilization. The obtained results presented in Figure 1 and Figure 2 indicate that Algorithm 3.1 catches very well the flow pattern at each time level. However, BLEBDF without grad-div stabilization creates very poor solutions and builds significant oscillations as time progresses.

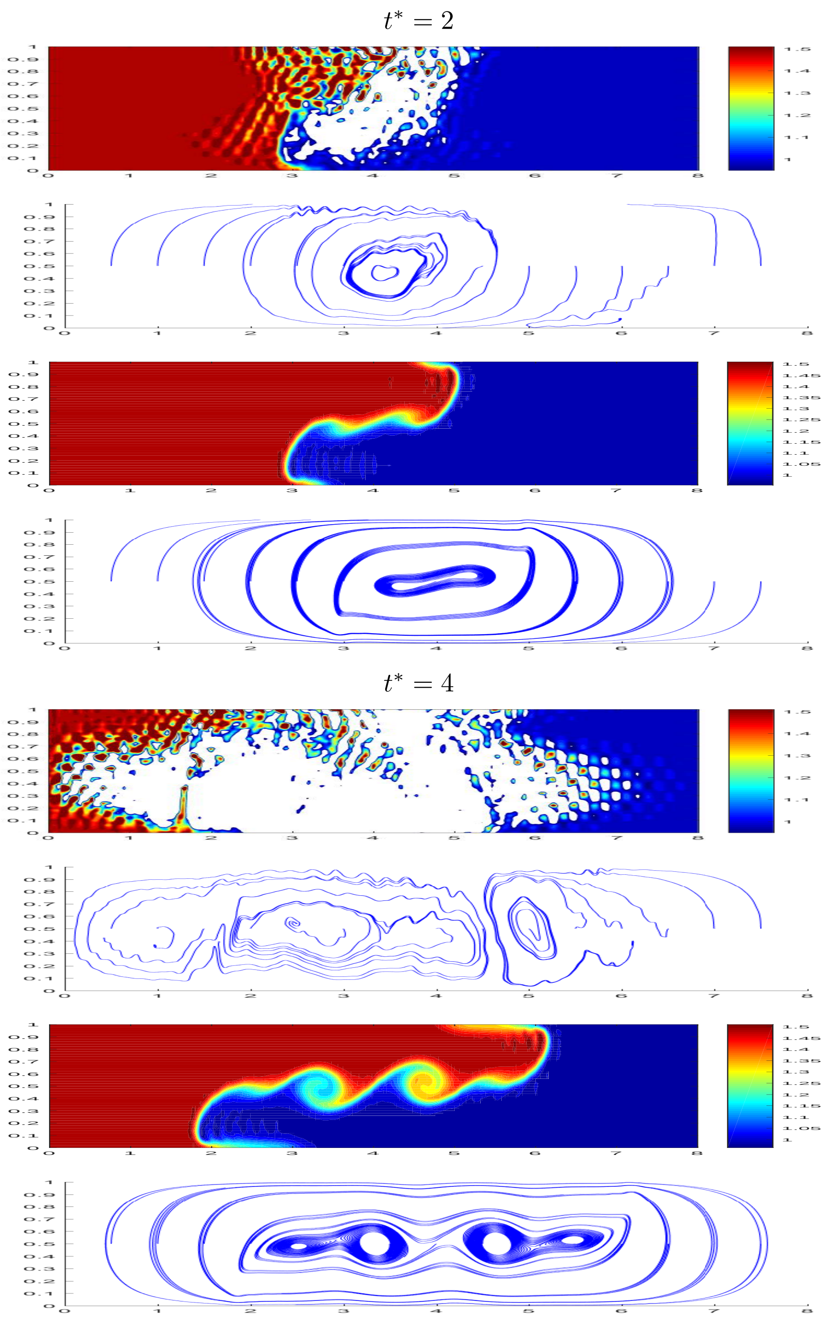


FIGURE 1. The temperature contours and velocity streamlines of BLEBDF without grad-div stabilization and of Algorithm 3.1, respectively, from a coarse mesh computation at $t^* = 2, 4$.

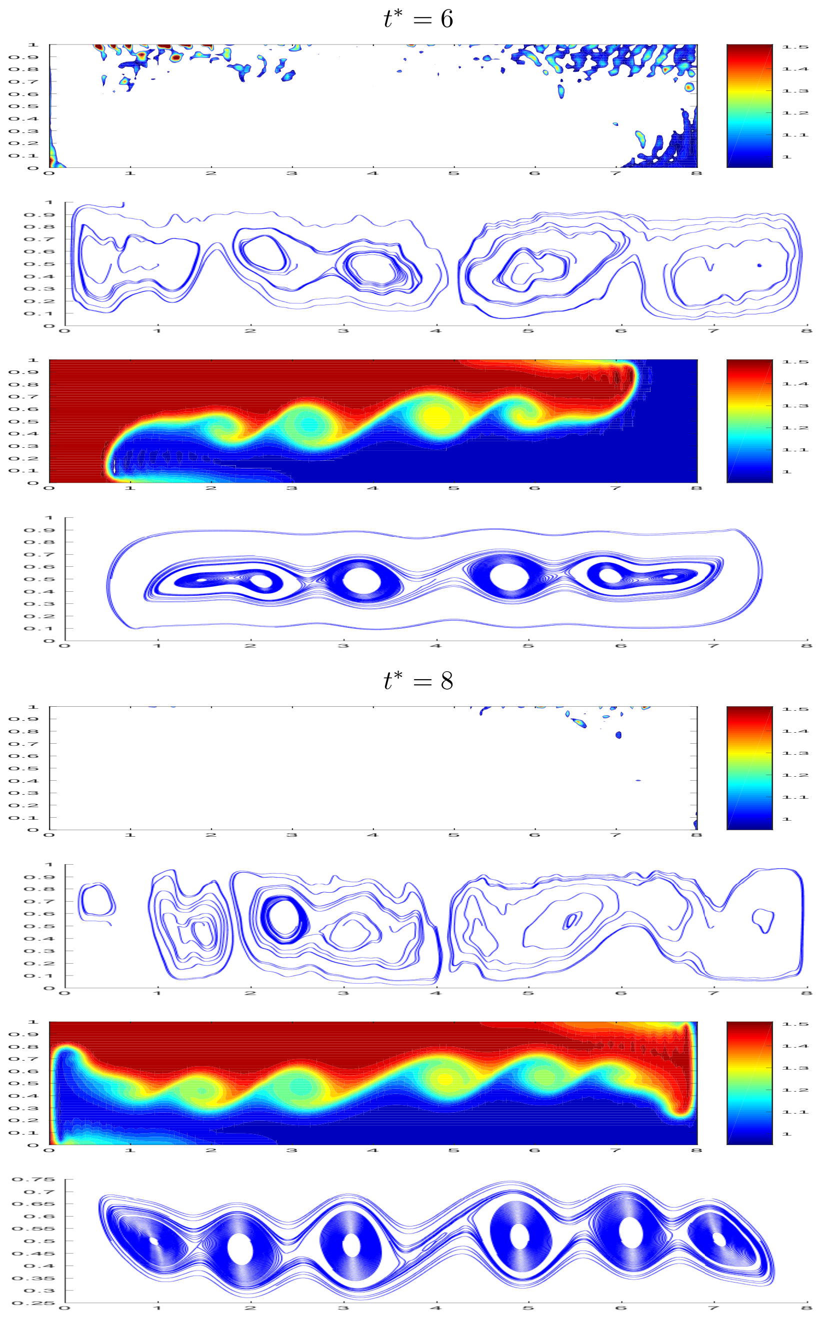


FIGURE 2. The temperature contours and velocity streamlines of BLEBDF without grad-div stabilization and of Algorithm 3.1, respectively, from a coarse mesh computation at $t^* = 6, 8$.

## 7. Conclusion

In this paper, we used grad-div stabilized finite element method for approximating natural convection flow problems. We applied a new class second order time stepping called linearized blended tree-step BDF in time. The proposed scheme is unconditionally stable over finite time interval and of second order convergent both in time and space. The numerical experiments presented here verifies theoretical convergence rates, and reveals the reliability of the proposed scheme.

## Conflict of interests

The authors declare that there is no conflict of interests.

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