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GENERALIZED π-HIRANO INVERSES OF THE SUM IN BANACH ALGEBRAS

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Abstract. In this paper, we investigate some additive results on gπ-Hirano invertibility in Banach algebras. By applying our results, some new results for operator matrices are obtained. This extends the main results of [H. Zou, T. Li and Y. Wei, arXiv:2302.06080v1].

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1. Introduction

The study of rings and algebras in which every element can be decomposed as a sum of elements with special properties, such as potents, nilpotents, quasinilpotents and units, is very interesting. Very recently Diesl [9] has investigated the sum of commuting potent and nilpotent elements. In [6], this subject was studied for periodic rings. Motivated by these works and recent work of Zou et al. [13], we study and investigate additive property for gπ-Hirano inverses of elements in a complex Banach algebra, specially for operator matrices. Throughout this paper, $\mathcal{A}$ will denote a Banach algebra with an identity 1. The commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in \mathcal{A} \mid xa = ax\}$. The study of generalized inverse of elements in rings and Banach algebras is very interesting for many authors. Since Moore-Penrose and Drazin inverses were introduced, there comes up several generalized inverses, some of them as subclasses and some, as generalizations of these inverses. We list several inverses related to Drazin inverse as follows. An element a in $\mathcal{A}$ has a generalized Drazin inverse (g-Drazin inverse in short) if there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a - a^2b \in \mathcal{A}^{qnil}.$$

Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + ax \in U(\mathcal{A})$ for every $x \in comm(a)\}$ and is the set of all quasinilpotent elements of $\mathcal{A}$. As it is well known, for a Banach algebra $\mathcal{A}$,

$$a \in \mathcal{A}^{qnil} \Longleftrightarrow \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0.$$

Such b, if exists, is unique, and is denoted by a^d, where a^d is called the g-Drazin inverse of a. Following Chen and Sheibani, an element a in a ring R has a Hirano inverse if there exists $b \in R$ such that

$$ab = ba, bab = b \text{ and } a^2 - ab \in N(R).$$

In [14], the authors studied the notion of n-strong Drazin invertibility and for a positive integer n, an element $a \in R$ is called n-strong Drazin invertible if $a - a^{n+1} \in N(R)$. Motivated by this paper, recently Ghaffari et al. [10] introduced the notion of π-Hirano inverse as follows. An element $a \in \mathcal{A}$ has π-Hirano inverse if and only if $a - a^{n+1} \in N(\mathcal{A})$ for some positive integer n. By applying the π-Hirano and GD-invertibility, a new kind of generalized inverse was introduced in [13]. An element $a \in \mathcal{A}$ has a generalized π-Hirano inverse (gπ-Hirano inverse for short) if and only if $a - a^{n+1} \in \mathcal{A}^{qnil}$, for some positive integer n. It is obvious that if $a \in \mathcal{A}$ is n-strong Drazin invertible, then it has generalized π-Hirano inverse. The generalized invertibility of the sum of two elements is attractive. Several authors have studied such problems from many different views, e.g. [4,5,7,8,11,12,13]. In this paper we investigate the gπ-Hirano invertibility of the sum of two elements under some conditions. In [13] this problem was studied with some conditions. This paper is organized as follows.

In Section 2, we have some elementary lemmas which will be useful in proving our main results. Then we are concern with gπ-Hirano invertibility of the sum of two elements under several conditions. Let $a, b \in \mathcal{A}$ have gπ-Hirano inverses. If $aba^2 = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, we prove that $a + b \in \mathcal{A}^{g\pi H}$. In Section 3, we consider the gπ-Hirano invertibility of a 2×2 operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ (*)$$

where $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have gπ-Hirano inverses and X, Y are complex Banach spaces. Here, M is a bounded operator on $X \oplus Y$. We present the gπ-Hirano inverse for a 2×2 operator matrix M under a number of different conditions.

We always use $\mathcal{A}^{g\pi H}$ to denote the set of all gπ-Hirano invertible elements $a \in \mathcal{A}$. Let $a \in \mathcal{A}$ have a gπ-Hirano inverse $a^{g\pi H}$, the element $p = 1 - aa^{g\pi H}$ is called the spectral idempotent of a. In this paper we use $U(\mathcal{A})$, $N(\mathcal{A})$ to denote the set of all units and all nilpotents in $\mathcal{A}$, respectively and $\mathbb{N}$ stands for the set of all natural numbers.

2. Main results

The aim of this section is to present new additive properties of gπ-Hirano invertibility of elements in Banach algebras. We start by the following lemma.

Lemma 2.1. [13, Theorem 2.2] *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{g\pi H}$.
(2) $a - a^{n+1} \in \mathcal{A}^{qnil}$ *for some* $n \in \mathbb{N}$.
(3) $a^n - a^m \in \mathcal{A}^{qnil}$ *for some* $m, n \in \mathbb{N}$ *such that* $m \neq n$.

Lemma 2.2. [1, Lemma 2.1] *Let $a, b \in \mathcal{A}^{qnil}$. If $ab = 0$, then $a + b \in \mathcal{A}^{qnil}$.*

Lemma 2.3. [2, Corollary 2.3] *Let $a, b \in \mathcal{A}^{qnil}$. If $aba^2 = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^{qnil}$.*

Lemma 2.4. [13, Lemma 4.5] *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$. If $a, b \in \mathcal{A}$ have gπ-Hirano inverses, then $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in M_2(\mathcal{A})$ has a gπ-Hirano inverse.*

Theorem 2.5. *Let $a, b \in \mathcal{A}^{g\pi H}$. If $aba^2 = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^{g\pi H}$ and*

$$(a+b)^{g\pi H} = (a + bab + b^2)M^{g\pi H}\begin{pmatrix} a \\ 1 \end{pmatrix}, M^{g\pi H} = F^{g\pi H} + (F^{g\pi H})^2 G,$$

$$F^{g\pi H} = B^{g\pi H} + (B^{g\pi H})^2 A, A^2 = 0,$$

$$B^{g\pi H} = (I - KK^{g\pi H})\left[\sum_{n=0}^{\infty} K^n (H^{g\pi H})^n\right] H^{g\pi H}$$

$$+K^{g\pi H}\left[\sum_{n=0}^{\infty} (K^{g\pi H})^n H^n\right](I - HH^{g\pi H}),$$

$$H^{g\pi H} = \begin{pmatrix} (a^{g\pi H})^3 & 0 \\ (a^{g\pi H})^4 + b(a^{g\pi H})^5 & 0 \end{pmatrix},$$

$$K^{g\pi H} = \begin{pmatrix} 0 & 0 \\ (b^{g\pi H})^4 & (b^{g\pi H})^3 \end{pmatrix}.$$

Proof. Set
$$M=\begin{pmatrix} a^3+a^2b+aba+ab^2 & a^3b+a^2b^2+abab+ab^3\\ a^2+ab+ba+b^2 & a^2b+ab^2+bab+b^3\end{pmatrix}.$$
Then
$$M=\begin{pmatrix} a^2b+aba+ab^2 & a^3b+a^2b^2\\ 0 & a^2b+ab^2+bab\end{pmatrix}+\begin{pmatrix} a^3 & 0\\ a^2+ab+ba+b^2 & b^3\end{pmatrix}$$
$:=G+F$.

It is clear by computing that $G^2=0$ and $GF=0$. Since $a,b\in\mathcal{A}^{g\pi H}$, by [13, Corollary 2.3], $a^3,b^3\in\mathcal{A}$ have gπ-Hirano inverses. By virtue of Lemma 2.4, F has a gπ-Hirano inverse. According to [13, Theorem 3.2], M has a gπ-Hirano inverse. Obviously,
$$(a+b)^3=\left(\begin{pmatrix}1 & b\end{pmatrix}\begin{pmatrix}a\\ 1\end{pmatrix}\right)^3.$$
Since $M=\left(\begin{pmatrix}a\\ 1\end{pmatrix}\begin{pmatrix}1 & b\end{pmatrix}\right)^3$, by Cline's formula, $(a+b)^3$ has a gπ-Hirano inverse (see [13, Lemma 4.4]). In light of [3, Lemma 3.1], $a+b\in\mathcal{A}^{g\pi H}$. By applying [13, Lemma 4.4],
$$\begin{aligned}(a+b)^{g\pi H}&=\left(\begin{pmatrix}1 & b\end{pmatrix}\begin{pmatrix}a\\ 1\end{pmatrix}\right)^{g\pi H}\\ &=\begin{pmatrix}1 & b\end{pmatrix}\begin{pmatrix}a\\ 1\end{pmatrix}\begin{pmatrix}1 & b\end{pmatrix}M^{g\pi H}\begin{pmatrix}a\\ 1\end{pmatrix}\\ &=(a+bab+b^2)M^{g\pi H}\begin{pmatrix}a\\ 1\end{pmatrix}.\end{aligned}$$
Clearly, $F=A+B$, where
$$A=\begin{pmatrix}0 & 0\\ ab & 0\end{pmatrix},\quad B=\begin{pmatrix}a^3 & 0\\ a^2+ba+b^2 & b^3\end{pmatrix}.$$
Since $AB=0$ and $A^2=0$, we see that $F^{g\pi H}=B^{g\pi H}+(B^{g\pi H})^2A$. Moreover, we have
$$B=\begin{pmatrix}a^3 & 0\\ a^2+ba & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ b^2 & b^3\end{pmatrix}:=H+K.$$
One easily checks that
$$H=\begin{pmatrix}a^3 & 0\\ a^2+ba & 0\end{pmatrix}=\begin{pmatrix}a^2\\ a+b\end{pmatrix}\begin{pmatrix}a & 0\end{pmatrix}.$$

Since $\begin{pmatrix} a & 0 \end{pmatrix}\begin{pmatrix} a^2 \\ a+b \end{pmatrix} = a^3 \in \mathcal{A}^{g\pi H}$, it follows by Cline's formula that

$$H^{g\pi H} = \begin{pmatrix} a^2 \\ a+b \end{pmatrix}\left((a^3)^{g\pi H}\right)^2\begin{pmatrix} a & 0 \end{pmatrix} = \begin{pmatrix} a^2 \\ a+b \end{pmatrix}\left(a^{g\pi H}\right)^6\begin{pmatrix} a & 0 \end{pmatrix}$$
$$= \begin{pmatrix} (a^{g\pi H})^3 & 0 \\ (a^{g\pi H})^4 + b(a^{g\pi H})^5 & 0 \end{pmatrix}.$$

Likewise, we have

$$K^{g\pi H} = \begin{pmatrix} 0 \\ b^2 \end{pmatrix}(b^{g\pi H})^6\begin{pmatrix} 1 & b \end{pmatrix} = \begin{pmatrix} 0 & o \\ (b^{g\pi H})^4 & (b^{g\pi H})^3 \end{pmatrix}.$$

Clearly, $HK = 0$. So

$$B^{g\pi H} = (I - KK^{g\pi H})\left[\sum_{n=0}^{\infty} K^n (H^{g\pi H})^n\right]H^{g\pi H}$$
$$+K^{g\pi s}\left[\sum_{n=0}^{\infty}(K^{g\pi H})^n(H^n)\right](I - HH^{g\pi H}),$$

then $M^{g\pi H} = F^{g\pi H} + (F^{g\pi H})^2 G$, as desired. □

As an immediate consequence, we derive

Corollary 2.6. *Let $a, b \in \mathcal{A}^{g\pi H}$. If $aba = 0, ab^2 = 0$, then $a + b \in \mathcal{A}^{g\pi H}$.*

Example 2.7. Let $a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in M_4(\mathbb{C})$. Then $a + b \in M_4(\mathbb{C})^{g\pi H}$.

Proof. This is obvious by Lemma 2.4 that $a, b \in M_4(\mathbb{C})^{g\pi H}$. One easily checks that $abab = 0, aba^2 = 0, b^2 = 0$, in this case, by Corollary 2.6, $a+b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \in M_4(\mathbb{C})^{g\pi H}$ and $ab, ba \neq 0$. □

3. Applications

Let $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have gπ-Hirano inverses and M be given by $(*)$. The aim of this section is to consider the gπ-Hirano inverse of a 2×2 operator matrix M. In fact the explicit gπ-Hirano inverse of M could be computed by the formula in Theorem 2.5. Here M is a bounded linear operator on $X \oplus Y$. This problem is quite complicated. We apply our results to establish new conditions under which M has gπ-Hirano inverse.

Theorem 3.1. *If $BCA = 0, BCB = 0, ABD = 0$ and $CBD = 0$, then M is $g\pi$-Hirano invertible.*

Proof. Write
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q,$$
where
$$P = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} := P + Q.$$
By Lemma 2.4, P, Q have gπ-Hirano inverses. Also $Q^2 = 0, PQP^2 = 0$ and $PQPQ = 0$. Therefore we complete the proof by Theorem 2.5. □

Corollary 3.2. *If $BC = 0$ and $BD = 0$, then M has a $g\pi$-Hirano inverse.*

Proof. This is obvious by Theorem 3.1. □

Theorem 3.3. *If $BCA = 0, DCA = 0, CBC = 0$ and $CBD = 0$, then M has $g\pi$-Hirano inverse.*

Proof. Write
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q,$$
where
$$P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} := P + Q.$$
By Lemma 2.4, P, Q have gπ-Hirano inverses. As $Q^2 = 0, PQP^2 = 0$ and $PQPQ = 0$, the proof induced by Corollary 2.5. □

Corollary 3.4. *If $CA = 0$ and $CB = 0$, then M is $g\pi$-Hirano invertible.*

Example 3.5. Let
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$
Then $BCA = 0, DCA = 0, CBC = 0$ and $CBD = 0$, by using Theorem 3.3 M has a gπ-Hirano inverse. But $CB \neq 0$ and $CA \neq 0$.

Let M be an operator matrix given by $(*)$. It is of interest to consider the gπ-Hirano inverse of M under generalized Schur condition $D = CA^{g\pi H}B$ (see [8]). We now investigate various perturbation conditions with spectral idempotent under which M has a gπ-Hirano inverse.

Theorem 3.6. *Let* $A \in \mathcal{L}(X)^{qnil}, D \in \mathcal{L}(Y)^{g\pi H}$ *and* M *be given by* $(*)$. *If* $CA^{\pi}BC = 0, BCA^{\pi}AB = 0, BCA^{\pi}A^2 = 0, ABCA^{g\pi H} = BCAA^{g\pi H}$ *and* $D = CA^{g\pi H}B$, *then* $M \in \mathcal{L}(X \oplus Y)^{g\pi H}$.

Proof. Clearly, we have

$$M = \begin{pmatrix} A & B \\ C & CA^{g\pi H}B \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} A & B \\ CAA^{g\pi H} & CA^{g\pi H}B \end{pmatrix}.$$

By assumption, we verify that $P^2 = 0, QPQ^2 = 0, QPQP = 0$. Clearly, P is nilpotent, and then it has a gπ-Hirano inverse. Moreover, we see that

$$Q = Q_1 + Q_2, Q_1 = \begin{pmatrix} A^2A^{g\pi H} & AA^{g\pi H}B \\ CAA^{g\pi H} & CA^{g\pi H}B \end{pmatrix}, \quad Q_2 = \begin{pmatrix} AA^{\pi} & A^{\pi}B \\ 0 & 0 \end{pmatrix}$$

and $Q_1Q_2 = 0$. Since $AA^{\pi} \in \mathcal{L}(X \oplus Y)^{qnil}$, it follows by Lemma 2.4 that $Q_2 \in \mathcal{L}(X \oplus Y)^{g\pi H}$. Moreover, we have

$$Q_1 = \begin{pmatrix} AA^{g\pi H} \\ CA^{g\pi H} \end{pmatrix} \begin{pmatrix} A & AA^{g\pi H}B \end{pmatrix}.$$

Clearly, we see that

$$\begin{pmatrix} A & AA^{g\pi H}B \end{pmatrix} \begin{pmatrix} AA^{g\pi H} \\ CA^{g\pi H} \end{pmatrix} = A^2A^{g\pi H} + AA^{g\pi H}BCA^{g\pi H}.$$

Since $A^2A^{g\pi H} + AA^{g\pi H}BCA^{g\pi H} = AA^{g\pi H}[A + BCA^{g\pi H}]$, we will suffice to prove $[A + BCA^{g\pi H}]AA^{g\pi H} = A^2A^{g\pi H} + BCA^{g\pi H}$ has a gπ-Hirano inverse, by means of Cline's formula. Since $D = CA^{g\pi H}B$ has a gπ-Hirano inverse, it follows by [13, Lemma 4.4], that $BCA^{g\pi H}$ has a gπ-Hirano inverse. In view of [13, Lemma 4.4], $A^2A^{g\pi} = A(AA^{g\pi})$ has a gπ-Hirano inverse. Since $ABCA^{g\pi H} = BCAA^{g\pi H}$, we have

$$\begin{aligned} (A^2A^{g\pi H})(BCA^{g\pi H}) &= A(AA^{g\pi H}BCA^{g\pi H}) \\ &= ABCA^{g\pi H} \\ &= BCAA^{g\pi H} \\ &= (BCA^{g\pi H})(A^2A^{g\pi H}). \end{aligned}$$

In view of [13, Lemma 3.9], so does $A^2A^{g\pi H} + BCA^{g\pi H}$. By using Cline's formula, Q_1 has a gπ-Hirano inverse. Therefore Q has a gπ-Hirano inverse by [13, Theorem 3.2]. According to Corollary 2.6, M has a gπ-Hirano inverse, as required. □

Corollary 3.7. *Let $A \in \mathcal{L}(X)^{qnil}, D \in \mathcal{L}(Y)^{g\pi H}$ and M be given by $(*)$. If $CA^{\pi}BC = 0, BCA^{\pi}AB = 0, BCA^{\pi}A^2 = 0, ABCA = BCA^2$ and $D = CA^{g\pi H}B$, then $M \in \mathcal{L}(X \oplus Y)^{g\pi H}$.*

Proof. Since $A(BCA) = (BCA)A$, we have

$$ABCA^{g\pi H} = ABCA(A^{g\pi H})^2 = (BCA)A(A^{g\pi H})^2 = BCAA^{g\pi H}.$$

Therefore we obtain the result by Theorem 3.6. □

We are ready to prove the following result.

Theorem 3.8. *Let $A \in \mathcal{L}(X)^{qnil}, D \in \mathcal{L}(Y)^{g\pi H}$ and M be given by $(*)$. If $A^{\pi}ABCA = 0, CA^{\pi}BCA = 0, BCA^{\pi}BC = 0, ABCA^{g\pi H} = BCAA^{g\pi H}$ and $D = CA^{g\pi H}B$, then $M \in \mathcal{L}(X \oplus Y)^{g\pi H}$.*

Proof. Clearly, we have

$$M = \begin{pmatrix} A & B \\ C & CA^{g\pi H}B \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} A & AA^{g\pi H}B \\ C & CA^{g\pi H}B \end{pmatrix}.$$

By assumption, we verify that $P^2 = 0, QPQ^2 = 0, QPQP = 0$. Clearly, P is nilpotent, and then it has a gπ-Hirano inverse. Moreover, we see that

$$Q = Q_1 + Q_2, Q_1 = \begin{pmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} A^2A^{g\pi H} & AA^{g\pi H}B \\ CAA^{g\pi H} & CA^{g\pi H}B \end{pmatrix}$$

and $Q_1Q_2 = 0$. Clearly, Q_1 has a gπ-Hirano inverse by Lemma 2.4. Moreover, we have $Q_2 = \begin{pmatrix} AA^{g\pi H} \\ CA^{g\pi} \end{pmatrix} \begin{pmatrix} A & AA^{g\pi H}B \end{pmatrix}$. We see that

$$\begin{pmatrix} A & AA^{g\pi H}B \end{pmatrix} \begin{pmatrix} AA^{g\pi H} \\ CA^{g\pi H} \end{pmatrix} = A^2A^{g\pi H} + AA^{g\pi H}BCA^{g\pi H}.$$

Since $A^{\pi}BCA^2 = 0$ and $ABCA^{g\pi H} = BCAA^{g\pi H}$, as in the proof in Theorem 3.6, we see that Q_1 has a gπ-Hirano inverse. Therefore Q has a gπ-Hirano inverse. By using Corollary 2.6 again, M has a gπ-Hirano inverse, as required. □

Corollary 3.9. *Let $A \in \mathcal{L}(X)^{qnil}, D \in \mathcal{L}(Y)^{g\pi H}$ and M be given by $(*)$. If $A^{\pi}ABCA = 0, CA^{\pi}BCA = 0, BCA^{\pi}BC = 0, A^2BCA = ABCA^2$ and $D = CA^{g\pi H}B$, then $M \in \mathcal{L}(X \oplus Y)^{g\pi H}$.*

Proof. As in the proof of Corollary 3.7, $BCA^{g\pi H} = AA^{g\pi H}BCA^{g\pi H}$. Similarly to the proof in Corollary 3.7, it follows from $A^2BCA = ABCA^2$ that $ABCA^{g\pi H} = BCAA^{g\pi H}$. Therefore we complete the proof by Theorem 3.8. □

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