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New Sequence Spaces Derived by the Composition of Binomial and Quadruple Band Matrices

Kübra TOPAL*

[*]Çay Vocational and Technical Anatolian High School, 53020, Rize, Türkiye

[*]Sorumlu Yazar/Corresponding Author
E-mail: kubra_topal22@erdogan.edu.tr

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ABSTRACT

In this work, we construct the sequence spaces $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ and $b_\infty^{a,b}(Q)$, derived from the combination of binomial and quadruple band matrices, where Q is a quadruple band matrix. The study is divided into four main sections. The first section provides some fundamental definitions and equations that will be used later. In the second section, detailed discussions are made on some works previously completed by various authors using the domain of the binomial matrix. Then, the three new sequence spaces $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ and $b_\infty^{a,b}(Q)$ are defined. It is then shown that these obtained sequence spaces are BK-spaces. Following this, it is demonstrated that $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ and $b_\infty^{a,b}(Q)$ sequence spaces are linearly isomorphic to the spaces c_0, c and ℓ_∞, in turn, followed by some inclusion relations. In the third section, the Schauder bases of the new sequence spaces $b_0^{a,b}(Q)$ and $b_c^{a,b}(Q)$ are provided, and the $\alpha-$, $\beta-$ and $\gamma-$ duals of $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ and $b_\infty^{a,b}(Q)$ sequence spaces are determined. The fourth section characterizes some matrix classes. As a result, it is observed that the new matrix obtained from the combination of binomial and quadruple band matrices provides more general and comprehensive results than those obtained previously.

Keywords: Matrix transformations, Matrix domain, Schauder basis, α-, β- and γ- duals, Matrix classes.

Binom ve Dörtlü Band Matrisinin Kompozisyonu ile Türetilmiş Yeni Dizi Uzayları

ÖZ

Bu çalışmada, Q dörtlü band matrisi olmak üzere, binom ve dörtlü band matrisinin birleşiminden türetilmiş olan $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ ve $b_\infty^{a,b}(Q)$ dizi uzaylarını oluşturuyoruz. Çalışma dört ana bölüm olarak hazırlandı. İlk bölümde, daha sonra kullanacağımız bazı temel tanım ve eşitlikler verildi. İkinci bölümde, daha önce çeşitli yazarlar tarafından binom matrisinin etki alanı kullanılarak yapılmış çalışmalardan bazılarına ayrıntılı olarak değinildi. Ardından, üç yeni dizi uzayı olan $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ ve $b_\infty^{a,b}(Q)$ tanımlandı. Daha sonra elde edilen bu dizi uzaylarının BK-uzayı olduğu gösterildi. Bundan sonra $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ ve $b_\infty^{a,b}(Q)$ dizi uzaylarının sırasıyla c_0, c ve ℓ_∞ dizi uzaylarına lineer olarak izomorf oldukları gösterildi. Ardından bazı içerme bağınları verildi. Üçüncü bölümde elde ettiğimiz yeni dizi uzaylarından $b_0^{a,b}(Q)$ ve $b_c^{a,b}(Q)$ nin Schauder bazları verildi ve $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ ve $b_\infty^{a,b}(Q)$ dizi uzaylarının $\alpha -$, $\beta -$ ve $\gamma -$ dualleri belirlendi. Dördüncü bölümde bazı matris sınıflarını karakterize edildi. Sonuç olarak binom ve dörtlü band matrisinin birleşiminden elde edilen yeni matrisin daha önce elde edilenlerden daha genel ve kapsamlı sonuçlar verdiği görüldü.

Anahtar Kelimeler: Matris dönüşümleri, Matris etki alanı, Schauder bazı, α-, β- ve γ- dualleri, Matris sınıfları.

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1. Introduction

The set of all real (or complex) valued sequences is denoted by w. w is a vector space under scalar multiplication and pointwise addition operations, and each of its subspaces is called a sequence space. By c , c_0 and ℓ_∞ are symbols of convergent, null and all bounded sequence spaces, respectively.

Let the transformation $p_n: Z \to \mathbb{C}$ be defined as $p_n(z) = z_n$. If each of these transformations is continuous for $\forall n \in \mathbb{N}$, then a Banach space is defined as a BK space. (Choudhary and Nanda 1989). Taking into account this information, we can say that c, c_0 and ℓ_∞ are BK-spaces under the norm defined as:

$$\|z\|_\infty = \sup_{i \in \mathbb{N}} |z_i|.$$

For an infinite matrix $A = (a_{kj})$ with real (or complex) terms, the transformation A of a sequence $z = (z_j)$ is defined as:

$$(Az)_k = \sum_j a_{kj} z_j. \qquad (1)$$

Here, it is assumed that this transformation is convergent for each $k \in \mathbb{N}$. In the remaining of the paper, the summation without limits runs from 0 to ∞.

Then, by $(Z:Y)$, we denote the class of all matrices from Z into Y and define $(Z:Y)$ as follows:

$$(Z:Y) = \{A = (a_{kj}) : Az \in Y \text{ for all } z \in Z\}.$$

Furthermore, the matrix domain of A, denoted by Z_A, is a sequence space defined as:

$$Z_A = \{z = (z_j) \in w : Az \in Z\} \qquad (2)$$

(Wilansky, 1984).

Now, we write, the sets of convergent and all bounded series are defined as $cs = c_s$ and $bs = (\ell_\infty)_s$, respectively, where by $S = (s_{kj})$ we mean of the summation matrix and as follows:

$$s_{kj} = \begin{cases} 1, & 0 \le j \le k \\ 0, & j > k \end{cases}$$

for all $k, j \in \mathbb{N}$.

An infinite matrix $A = (a_{kj})$ called triangle matrix to be if $a_{kj} = 0$ for $j > k$ and $a_{kk} \neq 0$ for all $k, j \in \mathbb{N}$. The inverse of a triangle matrix is always present and is unique a triangle.

Recently, a lot of authors have used the method to establish new sequence spaces with the matrix domain of an infinite matrix. This are e_0^a and e_c^a in (Altay and Başar, 2005), e_p^a and e_∞^a in (Altay et al., 2006), $b_0^{a,b}$, $b_c^{a,b}$, $b_\infty^{a,b}$ and $b_p^{a,b}$ in (Bişgin, 2016a, 2016b), $\mathcal{B}_p^{r,s}$ in (Demiriz and Erdem, 2020a), $\mathcal{B}_\infty^{r,s}, \mathcal{B}^{r,s}, \mathcal{B}_{bp}^{r,s}$ and $\mathcal{B}_{reg}^{r,s}$ in (Demiriz and Erdem, 2020b), $\mathcal{B}_f^{r,s}$ and $\mathcal{B}_{f_0}^{r,s}$ in (Erdem and Demiriz, 2020), $\mathcal{B}_{[f]}^{r,s}$ and $\mathcal{B}_{[f_0]}^{r,s}$ in (Erdem and Demiriz, 2021), $[l_p]_{e.r}$ in (Bilgin Ellidokuzoğlu and Demiriz, 2021), $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ (Kızmaz, 1981), $a_p^r(\Delta)$ in (Demiriz and Çakan, 2011), $c_0(\hat{L}, \Delta)$ and $c(\hat{L}, \Delta)$ (Bilgin Ellidokuzoğlu and Demiriz, 2018), $c_0(\Delta^2)$, $c(\Delta^2)$ and $\ell_\infty(\Delta^2)$ in (Et, 1993), $c_0(B)$, $c(B)$, $\ell_\infty(B)$ and $\ell_p(B)$ in (Sönmez, 2011), $b_0^{a,b}(\nabla)$, $b_c^{a,b}(\nabla)$ and $b_\infty^{a,b}(\nabla)$ in (Meng and Song, 2017), $b_0^{a,b}(\mathrm{G})$, $b_c^{a,b}(\mathrm{G})$ and $b_\infty^{a,b}(\mathrm{G})$ in (Sönmez, 2019), $b_p^{a,b}(\mathrm{G})$ in (Bişgin, 2017a), $b_0^{a,b}(\mathrm{D})$, $b_c^{a,b}(\mathrm{D})$ and $b_\infty^{a,b}(\mathrm{D})$ in (Bişgin, 2017b), $b_p^{a,b}(\mathrm{D})$ in (Sönmez, 2020), $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$ in (Bişgin, 2023).

2. Some New Sequence Spaces

In this section, we briefly give previous studies which are made by using the matrix domain of binomial and Euler matrices. Then we define three new sequence spaces $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ and $b_\infty^{a,b}(Q)$. Moreover, we indicate that the sequence spaces $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ and $b_\infty^{a,b}(Q)$ are linearly isomorphic to the sequence spaces null, convergent and bounded, respectively, and satisfy some inclusions.

The Euler sequence spaces e_0^a, e_c^a and e_∞^a were first constructed by Altay, Başar and Mursaleen in (Altay and Başar, 2005; Altay et al., 2006) by using the matrix domain of the Euler matrix as follows;

$$e_0^a = \left\{ z = (z_j) \in w \colon \lim_{k\to\infty} \sum_{j=0}^{k} \binom{k}{j} (1-a)^{k-j} a^j z_j = 0 \right\},$$

$$e_c^a = \left\{ z = (z_j) \in w \colon \lim_{k\to\infty} \sum_{j=0}^{k} \binom{k}{j} (1-a)^{k-j} a^j z_j \text{ exists} \right\}$$

and

$$e_\infty^a = \left\{ z = (z_j) \in w \colon \sup_{k\in\mathbb{N}} \left| \sum_{j=0}^{k} \binom{k}{j} (1-a)^{k-j} a^j z_j \right| < \infty \right\}.$$

Therefore, Bişgin defined the Binomial sequence spaces $b_0^{a,b}$, $b_c^{a,b}$ and $b_\infty^{a,b}$ in (Bişgin, 2016a, 2016b), and has generalized Altay, Başar and Mursaleen's work as follows:

$$b_0^{a,b} = \left\{ z = (z_j) \in w \colon \lim_{k\to\infty} \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j z_j = 0 \right\},$$

$$b_c^{a,b} = \left\{ z = (z_j) \in w \colon \lim_{k\to\infty} \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j z_j \text{ exists} \right\}$$

and

$$b_\infty^{a,b} = \left\{ z = (z_j) \in w \colon \sup_{k\in\mathbb{N}} \left| \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j z_j \right| < \infty \right\}.$$

Binomial matrix $B^{a,b} = \left(b_{kj}^{a,b}\right)$ is defined with all $k, j \in \mathbb{N}$, $a, b \in \mathbb{R}$ and $ab > 0$ as follows;

$$b_{kj}^{a,b} = \begin{cases} \frac{1}{(a+b)^k} \binom{k}{j} b^{k-j} a^k, & 0 \le j \le k \\ 0, & j > k \end{cases}.$$

Where, if we take $a + b = 1$ we get the a order of the Euler matrix. So, Binomial matrix is more general the Euler matrix.

Subsequently, using the composition of binomial and difference matrices, Meng and Song generalized Bişgin (2016a, 2016b)'s work. The sequence spaces $b_0^{a,b}(\nabla)$, $b_c^{a,b}(\nabla)$ and $b_\infty^{a,b}(\nabla)$ have been defined by Meng and Song in (Meng and Song, 2017);

$$b_0^{a,b}(\nabla) = \left\{ z = (z_j) \in w : \lim_{k\to\infty} \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j (z_j - z_{j-1}) = 0 \right\},$$

$$b_c^{a,b}(\nabla) = \left\{ z = (z_j) \in w : \lim_{k\to\infty} \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j (z_j - z_{j-1}) \text{ exists} \right\}$$

and

$$b_\infty^{a,b}(\nabla) = \left\{ z = (z_j) \in w : \sup_{k\in\mathbb{N}} \left| \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j (z_j - z_{j-1}) \right| < \infty \right\}.$$

Aftermore, Sönmez taking Meng and Song's work forward defined the three new sequence spaces in (Sönmez, 2019) following;

$$b_0^{a,b}(\mathrm{G}) = \left\{ z = (z_j) \in w : \lim_{k\to\infty} \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j (t z_j + u z_{j-1}) = 0 \right\},$$

$$b_c^{a,b}(G) = \left\{ z = (z_j) \in w : \lim_{k\to\infty} \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j (t z_j + u z_{j-1}) \text{ exists} \right\}$$

and

$$b_\infty^{a,b}(\mathrm{G}) = \left\{ z = (z_j) \in w : \sup_{k\in\mathbb{N}} \left| \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j (t z_j + u z_{j-1}) \right| < \infty \right\},$$

where $G = (g_{kj})$ is called the generalized difference (or double band) matrix and is defined with all $k, j \in \mathbb{N}$ and $t, u \in \mathbb{R} \setminus \{0\}$ as follows:

$$g_{kj} = \begin{cases} t, & j = k \\ u, & j = k-1 \\ 0, & \text{otherwise} \end{cases}$$

Where, if given $t = 1$ and $u = -1$ obtain the difference matrix. Hence, the $b_0^{a,b}(\mathrm{G})$, $b_c^{a,b}(\mathrm{G})$ and $b_\infty^{a,b}(\mathrm{G})$ sequence spaces are more general than the $b_0^{a,b}(\nabla)$, $b_c^{a,b}(\nabla)$ and $b_\infty^{a,b}(\nabla)$ sequence spaces.

Subsequently, the sequence spaces $b_0^{a,b}(\mathrm{D})$, $b_c^{a,b}(\mathrm{D})$ and $b_\infty^{a,b}(\mathrm{D})$ defined by Bişgin in (Bişgin, 2017b) considering the triple band matrix and binomial matrix together such that,

$$b_0^{a,b}(\mathrm{D}) = \left\{ z = (z_j) \in w : \lim_{k\to\infty} \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j (t z_j + u z_{j-1} + v z_{j-2}) = 0 \right\},$$

$$b_c^{a,b}(D) = \left\{ z = (z_j) \in w : \lim_{k\to\infty} \frac{1}{(a+b)^k} \sum_{j=0}^{k} \binom{k}{j} b^{k-j} a^j (t z_j + u z_{j-1} + v z_{j-2}) \text{ exists} \right\}$$

and

$$b_\infty^{a,b}(\mathrm{D})=\left\{z=(z_j)\in w:\sup_{k\in\mathbb{N}}\left|\frac{1}{(a+b)^k}\sum_{j=0}^{k}\binom{k}{j}b^{k-j}a^j(tz_j+uz_{j-1}+vz_{j-2})\right|<\infty\right\},$$

where triple band matrix $D=(d_{kj})$ is defined with all $k,j\in\mathbb{N}$ and $t,u,v\in\mathbb{R}\setminus\{0\}$ as follows;

$$d_{kj}=\begin{cases}t, & j=k\\ u, & j=k-1\\ v, & j=k-2\\ 0, & \text{otherwise}\end{cases}$$

Now, we define the sequence spaces $b_0^{a,b}(\mathrm{Q})$, $b_c^{a,b}(\mathrm{Q})$ and $b_\infty^{a,b}(\mathrm{Q})$ by considering the binomial matrix and quadruple band matrix, as follows;

$$b_0^{a,b}(\mathrm{Q})=\left\{z=(z_j)\in w:\lim_{k\to\infty}\frac{1}{(a+b)^k}\sum_{j=0}^{k}\binom{k}{j}b^{k-j}a^j(oz_j+tz_{j-1}+uz_{j-2}+vz_{j-3})=0\right\},$$

$$b_c^{a,b}(Q)=\left\{z=(z_j)\in w:\lim_{k\to\infty}\frac{1}{(a+b)^k}\sum_{j=0}^{k}\binom{k}{j}b^{k-j}a^j(oz_j+tz_{j-1}+uz_{j-2}+vz_{j-3})\text{ exists}\right\}$$

and

$$b_\infty^{a,b}(\mathrm{Q})=\left\{z=(z_j)\in w:\sup_{k\in\mathbb{N}}\left|\frac{1}{(a+b)^k}\sum_{j=0}^{k}\binom{k}{j}b^{k-j}a^j(oz_j+tz_{j-1}+uz_{j-2}+vz_{j-3})\right|<\infty\right\},$$

where quadruple band matrix $Q=Q(o,t,u,v)=\left(q_{kj}(o,t,u,v)\right)$ is defined with all $k,j\in\mathbb{N}$ and $o,t,u,v\in\mathbb{R}\setminus\{0\}$ as follows;

$$q_{kj}(o,t,u,v)=\begin{cases}o, & j=k\\ t, & j=k-1\\ u, & j=k-2\\ v, & j=k-3\\ 0, & \text{otherwise}\end{cases}$$

Here, if we take $o=1$, $t=-3$, $u=3$ and $v=-1$ we get the third order difference matrix Δ^3, we take $v=0$ we get the triple band matrix $Q(o,t,u,0)=D$ in (Bişgin, 2017b), if we take $o=1$, $t=-2$, $u=1$ and $v=0$ we get the second order difference matrix Δ^2, if we take $u=v=0$ we get the double band matrix $Q(o,t,0,0)=G$ in (Sönmez, 2019), if we take $o=1$, $t=-1$ and $u=v=0$ we get the difference matrix Δ.

So, the sequence space Q is more general than the previous studies; see (Bişgin, 2023). Hence we generalize the sequence spaces $b_0^{a,b}(\mathrm{D})$, $b_c^{a,b}(\mathrm{D})$ and $b_\infty^{a,b}(\mathrm{D})$.

Therefore, we take into account (2) we can write the sequence spaces $b_0^{a,b}(\mathrm{Q})$, $b_c^{a,b}(\mathrm{Q})$ and $b_\infty^{a,b}(\mathrm{Q})$ as follows;

$$b_0^{a,b}(Q) = \left(b_0^{a,b}\right)_Q,\ b_c^{a,b}(Q) = \left(b_c^{a,b}\right)_Q \text{ and } b_\infty^{a,b}(Q) = \left(b_\infty^{a,b}\right)_Q \tag{3}$$

Also, by constructing a matrix $F^{a,b} = \left(f_{nk}^{a,b}\right) = B^{a,b}(Q)$ so that,

$$f_{nk}^{a,b} = \begin{cases} \dfrac{b^{n-k-3}a^k}{(a+b)^n}\left[ob^3\binom{n}{k} + tb^2a\binom{n}{k+1} + uba^2\binom{n}{k+2} + va^3\binom{n}{k+3}\right] & , 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, we over again define the sequence spaces $b_0^{a,b}$(Q), $b_c^{a,b}$(Q) and $b_\infty^{a,b}$(Q) via the matrix $F^{a,b} = \left(f_{nk}^{a,b}\right)$ as follows:

$$b_0^{a,b}(Q) = (c_0)_{F^{a,b}},\ b_c^{a,b}(Q) = (c)_{F^{a,b}} \text{ and } b_\infty^{a,b}(Q) = (\ell_\infty)_{F^{a,b}}. \tag{4}$$

Thus, for given arbitrary $x = (x_k) \in w$, the $F^{a,b}$-transform of x is defined by:

$$y_k = \left(F^{a,b}x\right)_k = \frac{1}{(a+b)^k}\sum_{i=0}^{k}\binom{k}{i}b^{k-i}a^i(ox_i + tx_{i-1} + ux_{i-2} + vx_{i-3}) \tag{5}$$

or

$$y_k = (F^{a,b}x)_k = \frac{1}{(a+b)^k}\sum_{i=0}^{k}\left[ob^3\binom{k}{i} + tb^2a\binom{k}{i+1} + uba^2\binom{k}{i+2} + va^3\binom{k}{i+3}\right]b^{k-i-3}a^ix_i \tag{6}$$

for all $k \in \mathbb{N}$.

Theorem 2.1. The sequence spaces $b_0^{a,b}$(Q), $b_c^{a,b}$(Q) and $b_\infty^{a,b}$(Q) are BK-spaces with respect to their norms as follows:

$$\|x\|_{b_0^{a,b}(Q)} = \|x\|_{b_c^{a,b}(Q)} = \|x\|_{b_\infty^{a,b}(Q)} = \left\|F^{a,b}x\right\|_\infty = \sup_{k\in\mathbb{N}}\left|\left(F^{a,b}x\right)_k\right|$$

Proof: It is clear that the spaces c_0, c and ℓ_∞ are BK-spaces with the norm $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$. Also, the situation (4) holds. Where, $F^{a,b} = \left(f_{nk}^{a,b}\right)$ is triangle matrix. If we considering all of these and Wilansky's Theorem 4.3.12 (Wilansky, 1984), we obtain the sequence spaces $b_0^{a,b}$(Q), $b_c^{a,b}$(Q) and $b_\infty^{a,b}$(Q) are BK-spaces.

Theorem 2.2. The sets $b_0^{a,b}$(Q), $b_c^{a,b}$(Q) and $b_\infty^{a,b}$(Q) are each linear isomorphic to the sequence spaces c_0, c and ℓ_∞, in turn.

Proof: Similarly, we only prove the theorem for the space $b_0^{a,b}$(Q). Let's define L be a transformation such that $L: b_0^{a,b}(Q) \to c_0$, $L(x) = F^{a,b}x$. The linearity of L and the fact that $x = \theta$ whenever $Lx = \theta$ are evident. For this reason, L is injective.

For an any sequence $y = (y_k) \in c_0$ and we define a sequence $x = (x_n)$ by,

$$x_n = \frac{1}{o}\sum_{k=0}^{n}\sum_{i=k}^{n}\sum_{j=0}^{n-i}\sum_{v=0}^{n-i-j}\binom{i}{k}\mu_1^{n-i-j-v}\mu_2^v\mu_3^j(-b)^{i-k}(a+b)^k a^{-i}y_k \quad (\forall k \in \mathbb{N}) \tag{7}$$

Where, μ_1, μ_2 and μ_3 are random three roots of the equation $ox^3 + tx^2 + ux + v = 0$.

Besides, by using a simple calculation, we have (Bişgin, 2023)

$$\mu_1+\mu_2+\mu_3=-\frac{t}{o}$$

$$\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3=\frac{u}{o}$$

$$\mu_1\mu_2\mu_3=-\frac{v}{o}$$

$$\mu_1^3+\frac{t}{o}\mu_1^2+\frac{u}{o}\mu_1+\frac{v}{o}=0 \qquad (\mathbf{8})$$

$$\mu_1^2+\mu_2^2+\frac{t}{o}(\mu_1+\mu_2)+\mu_1\mu_2+\frac{u}{o}=0 \qquad (\mathbf{9})$$

$$\mu_1^2+\mu_2^2+\mu_3^2+\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3+\frac{t}{o}(\mu_1+\mu_2+\mu_3)+\frac{u}{o}=0 \qquad (\mathbf{10})$$

$$\mu_1+\mu_2+\mu_3+\frac{t}{o}=0 \qquad (\mathbf{11})$$

Hence, by considering the equations (8)-(11), we have

$$\begin{aligned}
(Qx)_k &= ox_k+tx_{k-1}+ux_{k-2}+vx_{k-3}\\
&=\sum_{l=0}^{k}\sum_{i=l}^{k}\sum_{j=0}^{k-i}\sum_{v=0}^{k-i-j}\binom{i}{l}\mu_1^{k-i-j-v}\mu_2^{v}\mu_3^{j}(-b)^{i-l}(a+b)^l a^{-i}y_l\\
&+\frac{t}{o}\sum_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{k-i-1}\sum_{v=0}^{k-i-j-1}\binom{i}{l}\mu_1^{k-i-j-v-1}\mu_2^{v}\mu_3^{j}(-b)^{i-l}(a+b)^l a^{-i}y_l\\
&+\frac{u}{o}\sum_{l=0}^{k-2}\sum_{i=l}^{k-2}\sum_{j=0}^{k-i-2}\sum_{v=0}^{k-i-j-2}\binom{i}{l}\mu_1^{k-i-j-v-2}\mu_2^{v}\mu_3^{j}(-b)^{i-l}(a+b)^l a^{-i}y_l\\
&+\frac{v}{o}\sum_{l=0}^{k-3}\sum_{i=l}^{k-3}\sum_{j=0}^{k-i-3}\sum_{v=0}^{k-i-j-3}\binom{i}{l}\mu_1^{k-i-j-v-3}\mu_2^{v}\mu_3^{j}(-b)^{i-l}(a+b)^l a^{-i}y_l\\
&=\sum_{l=0}^{k-3}\left[\sum_{i=l}^{k-3}\left[\sum_{j=0}^{k-i-3}\binom{i}{l}(-b)^{i-l}(a+b)^l a^{-i}\left[\sum_{v=0}^{k-i-j-3}\mu_3^{j}\mu_2^{v}\mu_1^{k-i-j-v-3}\left(\mu_1^3+\frac{t}{o}\mu_1^2+\frac{u}{o}\mu_1+\frac{v}{o}\right)\right.\right.\right.\\
&\left.+\mu_3^{j}\mu_2^{k-i-j-v-2}\left(\mu_1^2+\mu_2^2+\frac{t}{o}(\mu_1+\mu_2)+\mu_1\mu_2+\frac{u}{o}\right)\right]\\
&+\mu_3^{k-i-2}\left(\mu_1^2+\mu_2^2+\mu_3^2+\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3+\frac{t}{o}(\mu_1+\mu_2+\mu_3)+\frac{u}{o}\right)\\
&+\binom{k-2}{l}(-b)^{k-l-2}(a+b)^l a^{2-k}(\mu_1^2+\mu_2^2+\mu_3^2+\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3\\
&+\frac{t}{o}(\mu_1+\mu_2+\mu_3)+\frac{u}{o})\\
&+\binom{k-1}{l}(-b)^{k-l-1}(a+b)^l a^{1-k}\left(\mu_1+\mu_2+\mu_3+\frac{t}{o}\right)\\
&\left.+\binom{k}{l}(-b)^{k-l}(a+b)^l a^{-k}\right]y_l
\end{aligned}$$

$$
\begin{aligned}
&+y_{k-2}(a+b)^{k-2}\left[a^{2-k}\left(\mu_1^2+\mu_2^2+\mu_3^2+\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3+\frac{t}{o}(\mu_1+\mu_2+\mu_3)+\frac{u}{o}\right)\right.\\
&\left.+a^{1-k}\binom{k-1}{k-2}(-b)\left(\mu_1+\mu_2+\mu_3+\frac{t}{o}\right)+\binom{k}{k-2}(-b)^2a^{-k}\right]\\
&+y_{k-1}(a+b)^{k-1}\left[a^{1-k}\left(\mu_1+\mu_2+\mu_3+\frac{t}{o}\right)+\binom{k}{k-1}(-b)a^{-k}\right]+y_k\left(\frac{a+b}{a}\right)^k\\
&=\sum_{l=0}^{k-3}\binom{k}{l}(-b)^{k-l}(a+b)^l a^{-k}\,y_l+y_{k-2}(a+b)^{k-2}\binom{k}{k-2}(-b)^2a^{-k}\\
&+y_{k-1}(a+b)^{k-1}\binom{k}{k-1}(-b)a^{-k}+y_k(a+b)^k a^{-k}\\
&=\sum_{l=0}^{k}\binom{k}{l}(-b)^{k-l}(a+b)^l a^{-k}\,y_l
\end{aligned}
$$

then we obtain,

$$
\begin{aligned}
\lim_{n\to\infty}\left(F^{a,b}x\right)_n&=\lim_{n\to\infty}\frac{1}{(a+b)^n}\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k(ox_k+tx_{k-1}+ux_{k-2}+vx_{k-3})\\
&=\lim_{n\to\infty}\frac{1}{(a+b)^n}\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\sum_{l=0}^{k}\binom{k}{l}(-b)^{k-l}(a+b)^l a^{-k}\,y_l\\
&=\lim_{n\to\infty}y_n\\
&=0
\end{aligned}
$$

which implies that $x\in b_0^{a,b}(Q)$ and $L(x)=y$. As a result, L is surjective. Also, L is norm preserving from the Theorem 2.1. Consequently, $b_0^{a,b}(Q)\cong c_0$. Thus, the proof is complete.

Theorem 2.3. The subsumption $b_0^{a,b}(Q)\supset c$ strictly holds, whenever $o+t+u+v=0$.

Proof: Suppose that $o+t+u+v=0$ and $x=(x_k)\in c$, namely $\lim_{k\to\infty}x_k=\nu$. Then, we have $\lim_{k\to\infty}(Qx)_k=(o+t+u+v)\nu=0$. Also, the binomial matrix is regular for $ab>0$. If we consider three facts, we obtain that $B^{a,b}Qx\in c_0$ whenever $x\in c$, namely $x\in b_0^{a,b}(Q)$ whenever $x\in c$. So, the subsumption $b_0^{a,b}(Q)\supset c$ holds.

Now, we show that the subsumption of $b_0^{a,b}(Q)\supset c$ is strict. We define a sequence $x=(x_k)$ such that $x_k=\ln(k+4)$ for all $k\in\mathbb{N}$. It is clear that the sequence $x=(x_k)\notin c$, but

$$
\begin{aligned}
F^{a,b}x&=\frac{1}{(a+b)^k}\sum_{i=0}^{k}\binom{k}{i}b^{k-i}a^i\left[t\ln\left(\frac{k+3}{k+4}\right)+u\ln\left(\frac{k+2}{k+4}\right)+v\ln\left(\frac{k+1}{k+4}\right)\right]\\
&=t\ln\left(\frac{k+3}{k+4}\right)+u\ln\left(\frac{k+2}{k+4}\right)+v\ln\left(\frac{k+1}{k+4}\right).
\end{aligned}
$$

Then, $\lim_{k\to\infty}F^{a,b}x=0$. That is $x=(x_k)\in b_0^{a,b}(Q)$. Consequently, the subsumption $b_0^{a,b}(Q)\supset c$ is strict.

Theorem 2.4. The inclusions $b_0^{a,b}(Q)\subset b_c^{a,b}(Q)\subset b_\infty^{a,b}(Q)$ strictly hold.

Proof: It is well known that the inclusions $c_0 \subset c \subset \ell_\infty$. So, the inclusions $b_0^{a,b}(Q) \subset b_c^{a,b}(Q) \subset b_\infty^{a,b}(Q)$ hold.

Now, we define two sequences $x = (x_k)$ and $y = (y_k)$ as follows;

$$x_k = \frac{1}{o}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i} \mu_1^{k-j-i-v}\mu_2^v\mu_3^i$$

and

$$y_k = \frac{1}{o}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i} \mu_1^{k-j-i-v}\mu_2^v\mu_3^i\left(-\frac{2b+a}{a}\right)^j$$

for $\forall k \in \mathbb{N}$.

Then we can observe that $F^{a,b}x = (1,1,1,\dots) \in c\backslash c_0$ and $F^{a,b}y = \left((-1)^k\right) \in \ell_\infty\backslash c$, namely $x = (x_k) \in b_c^{a,b}(Q)\backslash b_0^{a,b}(Q)$ and $y = (y_k) \in b_\infty^{a,b}(Q)\backslash b_c^{a,b}(Q)$. These two facts show that the inclusions $b_0^{a,b}(Q) \subset b_c^{a,b}(Q) \subset b_\infty^{a,b}(Q)$ are strict.

2. The Schauder Basis and $\alpha-, \beta-, \gamma-$ Duals

In this part, the Schauder basis of the sequence spaces $b_0^{a,b}(Q)$ and $b_c^{a,b}(Q)$ has been given. Then, the $\alpha-, \beta-$ and $\gamma-$ duals of the sequence spaces $b_0^{a,b}(Q)$, $b_c^{a,b}(Q)$ and $b_\infty^{a,b}(Q)$ are determined.

For a normed space $(X, \|.\|)$ and a set $\{y_k : y_k \in X\}_{k\in\mathbb{N}}$, the sequence (y_k) is called a Schauder basis of X if for every $x \in X$ there exists a unique sequence $\lambda = (\lambda_k)$ of scalars such that:

$$\lim_{n\to\infty}\left\| x - \sum_{k=0}^{n}\lambda_k y_k \right\| = 0$$

Theorem 3.1. For $\forall k \in \mathbb{N}$, let $\xi_k = \left(F^{a,b}x\right)_k$, and suppose $\lim_{k\to\infty}\left(F^{a,b}x\right)_k = \nu$. We define the two sequences $\rho = (\rho_k)$ and $\rho^{(k)}(a,b) = \left\{\rho_n^{(k)}(a,b)\right\}_{n\in\mathbb{N}}$ as follows:

$$\rho_k = \frac{1}{o}\sum_{l=0}^{k}\sum_{i=l}^{k}\sum_{j=0}^{k-i}\sum_{v=0}^{k-i-j}\binom{i}{l}\mu_1^{k-i-j-v}\mu_2^v\mu_3^j(-b)^{i-l}(a+b)^l a^{-i}$$

and

$$\rho_n^{(k)}(a,b) = \begin{cases} 0 & , \quad 0 \le n < k \\ \dfrac{1}{o}\displaystyle\sum_{i=k}^{n}\sum_{j=0}^{n-i}\sum_{v=0}^{n-i-j}\binom{i}{k}\mu_1^{n-i-j-v}\mu_2^v\mu_3^j(-b)^{i-k}(a+b)^k a^{-i} & , \quad k \le n \end{cases}$$

for all $n, k \in \mathbb{N}$. In this case, the followings hold:

(a) The Schauder basis of the sequence space $b_0^{a,b}(Q)$ is the set $\left\{\rho^{(k)}(a,b)\right\}_{k\in\mathbb{N}}$ } and each $x = (x_k) \in b_0^{a,b}(Q)$ has a unique represantation defined by:

$$x = \sum_k \xi_k \rho^{(k)}(a,b)$$

(b) The set $\{\rho, \rho^{(0)}(a,b), \rho^{(1)}(a,b), \dots\}$ is a Schauder basis for the sequence space $b_c^{a,b}(Q)$ and all $x = (x_k) \in b_c^{a,b}(Q)$ has a unique representation defined by:

$$x = \nu\rho + \sum_k [\xi_k - \nu]\rho^{(k)}(a,b)$$

Proof: (a) The sequence $e^{(k)}$ is a sequence with 1 in the k-th place and zeros elsewhere. Obviously, $F^{a,b}\rho^{(k)}(a,b) = e^{(k)} \in c_0$ for all $k \in \mathbb{N}$. Then we conclude that the inclusion $\{\rho^{(k)}(a,b)\} \subset b_0^{a,b}(Q)$ holds.

Let $x = (x_k) \in b_0^{a,b}(Q)$. We demonstrate;

$$x^{[m]} = \sum_{k=0}^{m} \xi_k \rho^{(k)}(a,b)$$

for all $m \in \mathbb{N}$. Moreover, if we apply the $F^{a,b} = (f_{nk}^{a,b})$ to $x^{[m]}$, we obtain

$$F^{a,b}x^{[m]} = \sum_{k=0}^{m} \xi_k F^{a,b}\rho^{(k)}(a,b) = \sum_{k=0}^{m} (F^{a,b}x)_k e^{(k)}$$

and

$$\{F^{a,b}(x - x^{[m]})\}_n = \begin{cases} 0 & , \ 0 \le n \le m \\ (F^{a,b}x)_n & , \ n > m \end{cases}$$

for all $n, m \in \mathbb{N}$.

Now, given for each $\epsilon > 0$ there exist $m_0 = m_0^{(\epsilon)} \in \mathbb{N}$ such that

$$\left|(F^{a,b}x)_m\right| < \frac{\epsilon}{2}$$

for all $m_0 \le m$. Furthermore,

$$\|x - x^{[m]}\|_{b_0^{a,b}(Q)} = \sup_{m \le n} \left|(F^{a,b}x)_n\right| \le \sup_{m_0 \le n} \left|(F^{a,b}x)_n\right| \le \frac{\epsilon}{2} < \epsilon$$

for all $m_0 \le m$, which implies shows us;

$$x = \sum_k \xi_k \rho^{(k)}(a,b)$$

Now, let x has another representation as follows,

$$x = \sum_k \lambda_k \rho^{(k)}(a,b)$$

Considering the continuousness of the L-transformation defined in the proof of Theorem 2.2, the following equation can be written;

$$\left(F^{a,b}x\right)_n = \sum_k \lambda_k \left[F^{a,b}\rho^{(k)}(a,b)\right]_n = \sum_k \lambda_k\, e_n^{(k)} = \lambda_n$$

for $\forall n \in \mathbb{N}$.

This consequence contradicts with the $\left(F^{a,b}x\right)_n = \xi_n$ for all $n \in \mathbb{N}$. Therefore, all $x = (x_k) \in$

$b_0^{a,b}(Q)$ has a unique representation.

(b) It is clear that $\left\{\rho^{(k)}(a,b)\right\} \subset b_0^{a,b}(Q)$ and $F^{a,b}\rho = e \in c$. Therefore, the inclusion $\left\{\rho, \rho^{(k)}(a,b)\right\} \subset b_c^{a,b}(Q)$ clearly holds. For $\lim_{k\to\infty} \xi_k = \nu$, we create a sequence $y = (y_k)$ such that $y = x - \nu d$, where $x = (x_k) \in b_c^{a,b}(Q)$ is an arbitrary sequence space. Then it is clear that $y = (y_k) \in b_0^{a,b}(Q)$ and by the part (a) $y = (y_k)$ has a unique representation. Namely, every $x = (x_k) \in b_c^{a,b}(Q)$ has a unique representation as follows;

$$x = \nu\rho + \sum_k [\xi_k - \nu]\rho^{(k)}(a,b)$$

Corollary 3.2. The sequence spaces $b_0^{a,b}(Q)$ and $b_c^{a,b}(Q)$ are separable.

The set $M(X,Y)$ is named as multiplier space of X and Y. Also this set is defined as

$$M(X,Y) = \{a = (a_k) \in w: ax = (a_k x_k) \in Y \text{ for all } x = (x_k) \in X\}$$

where, X and Y are any sequence spaces. Then, the $\alpha -$, $\beta -$ and $\gamma -$ duals of the sequence space X are defined by

$$X^\alpha = M(X,\ell_1), \qquad X^\beta = M(X,cs) \qquad \text{and} \qquad X^\gamma = M(X,bs)$$

respectively.

Now, to use in the next lemma we write the following:

$$\sup_{\kappa\in\mathcal{F}} \sum_n \left|\sum_{k\in\kappa} a_{nk}\right| < \infty \tag{12}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty \tag{13}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k \left|\lim_{n\to\infty} a_{nk}\right| \tag{14}$$

$$\lim_{n\to\infty} a_{nk} = \xi_k \text{ for all } k \in \mathbb{N} \tag{15}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \xi \tag{16}$$

where $\mathcal{F}$ represents the set of all finite subsets of $\mathbb{N}$.

Lemma 3.3. (Stieglitz and Tietz, 1977)

Let an infinite matrix $A = (a_{nk})$, in which case the following statements hold:

(i) $A = (a_{nk}) \in (c_0 : \ell_1) = (c : \ell_1) \Leftrightarrow (12)$ holds,
(ii) $A = (a_{nk}) \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty) \Leftrightarrow (13)$ holds,
(iii) $A = (a_{nk}) \in (c_0 : c) \Leftrightarrow (13)$ and (15) hold,
(iv) $A = (a_{nk}) \in (c : c) \Leftrightarrow (13),\ (15)$ and (16) hold,
(v) $A = (a_{nk}) \in (\ell_\infty : c) \Leftrightarrow (14)$ and (15) hold,
(vi) $A = (a_{nk}) \in (c_0 : c_0) \Leftrightarrow (13)$ and (15) hold with $\xi_k = 0,\ \forall k \in \mathbb{N}$.

Theorem 3.4. The α −dual of the sequence spaces $b_0^{a,b}(Q)$ and $b_c^{a,b}(Q)$ is the set

$$\eta_1^{a,b}(Q) = \left\{ t = (t_k) \in w : \sup_{\kappa \in \mathcal{F}} \sum_n \left| \frac{1}{o} \sum_{k \in \kappa} \sum_{i=k}^{n} \sum_{j=0}^{n-i} \sum_{v=0}^{n-i-j} \binom{i}{k} \mu_1^{n-i-j-v} \mu_2^v \mu_3^j (-b)^{i-k} (a+b)^k a^{-i} t_n \right| < \infty \right\}$$

Proof: Let $t = (t_k) \in w$ and $x = (x_n)$ be defined by (7), we obtain

$$\begin{aligned} t_n x_n &= \sum_{k=0}^{n} \left[\frac{1}{o} \sum_{i=k}^{n} \sum_{j=0}^{n-i} \sum_{v=0}^{n-i-j} \binom{i}{k} \mu_1^{n-i-j-v} \mu_2^v \mu_3^j (-b)^{i-k} (a+b)^k a^{-i} t_n \right] y_k \\ &= \sum_{k=0}^{n} g_{nk}^{r,s} y_k \\ &= (G^{r,s} y)_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $tx = (t_n x_n) \in \ell_1$ whenever $x = (x_k) \in b_0^{a,b}(Q)$ or $b_c^{a,b}(Q)$ if and only if $G^{a,b} y \in \ell_1$ whenever $y = (y_k) \in c_0$ or c, which implies that $t = (t_n) \in \left[b_0^{a,b}(Q)\right]^\alpha = \left[b_c^{a,b}(Q)\right]^\alpha$ if and only if $G^{a,b} \in (c_0 : \ell_1) = (c : \ell_1)$. We are combining this consequence and Lemma 3.3 (i), we get

$$t = (t_n) \in \left\{b_0^{a,b}(Q)\right\}^\alpha \Leftrightarrow \sup_{\kappa \in \mathcal{F}} \sum_n \left| \frac{1}{o} \sum_{k \in \kappa} \sum_{i=k}^{n} \sum_{j=0}^{n-i} \sum_{v=0}^{n-i-j} \binom{i}{k} \mu_1^{n-i-j-v} \mu_2^v \mu_3^j (-b)^{i-k} (a+b)^k a^{-i} t_n \right| < \infty$$

Thus, we have $\left[b_0^{a,b}(Q)\right]^\alpha = \left[b_c^{a,b}(Q)\right]^\alpha = \eta_1^{a,b}(Q)$.

Theorem 3.5. Define the sets $\eta_2^{a,b}(Q), \eta_3^{a,b}(Q), \eta_4^{a,b}(Q)$ and $\eta_5^{a,b}(Q)$ by

$$\eta_2^{a,b}(Q) = \left\{ t = (t_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| z_{nk}^{a,b} \right| < \infty \right\},$$

$$\eta_3^{a,b}(Q) = \left\{ t = (t_k) \in w : \lim_{n \to \infty} z_{nk} \text{ exists for } \forall k \in \mathbb{N} \right\},$$

$$\eta_4^{a,b}(Q) = \left\{ t = (t_k) \in w : \lim_{n \to \infty} \sum_k \left| z_{nk}^{a,b} \right| = \sum_k \left| \lim_{n \to \infty} z_{nk}^{a,b} \right| \right\}$$

and

$$\eta_5^{a,b}(Q) = \left\{ t = (t_k) \in w: \lim_{n\to\infty} \sum_k z_{nk}^{a,b} \text{ exists} \right\},$$

where the matrix $Z^{a,b} = \left(z_{nk}^{a,b}\right)$ defined as follows:

$$z_{nk}^{a,b} = \begin{cases} \frac{1}{o} \sum_{i=k}^{n} \sum_{l=k}^{i} \sum_{j=0}^{i-l} \sum_{v=0}^{i-l-j} \binom{l}{k} \mu_1^{i-l-j-v} \mu_2^{v} \mu_3^{j} (-b)^{l-k} (a+b)^k a^{-l} t_i \ , & 0 \le k \le n \\ 0 \ , & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Then, the following hold;

(i) $\left\{b_0^{a,b}(Q)\right\}^{\beta} = \eta_2^{a,b}(Q) \cap \eta_3^{a,b}(Q)$,

(ii) $\left\{b_c^{a,b}(Q)\right\}^{\beta} = \eta_2^{a,b}(Q) \cap \eta_3^{a,b}(Q) \cap \eta_5^{a,b}(Q)$,

(iii) $\left\{b_\infty^{a,b}(Q)\right\}^{\beta} = \eta_3^{a,b}(Q) \cap \eta_4^{a,b}(Q)$,

(iv) $\left\{b_0^{a,b}(Q)\right\}^{\gamma} = \left\{b_c^{a,b}(Q)\right\}^{\gamma} = \left\{b_\infty^{a,b}(Q)\right\}^{\gamma} = \eta_2^{a,b}(Q)$

Proof: Since the other parts are similar, we only give the proof of (i). For given $t = (t_n) \in w$, by considering the sequence $x = (x_k)$ defined in (7), we can write

$$\begin{aligned} \sum_{k=0}^{n} t_k x_k &= \sum_{k=0}^{n} \left[\frac{1}{o} \sum_{i=0}^{k} \sum_{l=i}^{k} \sum_{j=0}^{k-l} \sum_{v=0}^{k-l-j} \binom{l}{i} \mu_1^{k-l-j-v} \mu_2^{v} \mu_3^{j} (-b)^{l-i} (a+b)^i a^{-l} y_i \right] t_k \\ &= \sum_{k=0}^{n} \left[\frac{1}{o} \sum_{i=k}^{n} \sum_{l=k}^{i} \sum_{j=0}^{i-l} \sum_{v=0}^{i-l-j} \binom{l}{k} \mu_1^{i-l-j-v} \mu_2^{v} \mu_3^{j} (-b)^{l-k} (a+b)^k a^{-l} t_i \right] y_k \\ &= \left(Z^{a,b} y\right)_n \end{aligned}$$

for all $n, k \in \mathbb{N}$. Therefore, $tx = (t_n x_n) \in cs$ whenever $x = (x_k) \in b_0^{a,b}(Q)$ if and only if $Z^{a,b} y \in c$ whenever $y \in c_0$. This result show us that $t = (t_k) \in \left\{b_0^{a,b}(Q)\right\}^{\beta}$ if and only if $Z^{a,b} \in (c_0 : c)$. If we consider this result and Lemma 3.3 (iii) we obtain that $t = (t_k) \in \left\{b_0^{a,b}(Q)\right\}^{\beta}$ if and only if

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left|z_{nk}^{a,b}\right| < \infty$$

and

$$\lim_{n\to\infty} z_{nk}^{a,b} \text{ exists for } \forall k \in \mathbb{N}.$$

So, $\left\{b_0^{a,b}(Q)\right\}^{\beta} = \eta_2^{a,b}(Q) \cap \eta_3^{a,b}(Q)$. Thus, the proof is complete.

3. Some Matrix Classes

In this part, we examine certain classes of matrices of the new sequence space $b_c^{a,b}(Q)$ obtained.

We define $\rho_{nk}^{a,b}$ for simplicity of use as follows:

$$\rho_{nk}^{a,b}=\frac{1}{o}\sum_{i=k}^{\infty}\sum_{l=k}^{i}\sum_{j=0}^{i-l}\sum_{v=0}^{i-l-j}\binom{l}{k}\mu_1^{i-l-j-v}\mu_2^{v}\mu_3^{j}(-b)^{l-k}(a+b)^{k}a^{-l}t_{ni}$$

for all $n,k\in\mathbb{N}$.

Lemma 4.1. (Başar and Altay, 2003)

Assume that X and Y be any two sequence spaces and A be an infinite matrix. In that case, $A\in(X:Y_E)\Leftrightarrow EA\in(X:Y)$, where E is triangle matrix.

Theorem 4.2. $A\in\left(b_c^{a,b}(Q):\ell_\infty\right)$ if and only if

$$\sup_{n\in\mathbb{N}}\sum_{k}\left|\rho_{nk}^{a,b}\right|<\infty\,, \tag{17}$$

$$\rho_{nk}^{a,b}\quad\text{exist for all }\ n,k\in\mathbb{N}, \tag{18}$$

$$\sup_{m\in\mathbb{N}}\sum_{k}\left|\frac{1}{o}\sum_{i=k}^{m}\sum_{l=k}^{i}\sum_{j=0}^{i-l}\sum_{v=0}^{i-l-j}\binom{l}{k}\mu_1^{i-l-j-v}\mu_2^{v}\mu_3^{j}(-b)^{l-k}(a+b)^{k}a^{-l}t_{ni}\right|<\infty\quad(m\in\mathbb{N}) \tag{19}$$

and

$$\lim_{m\to\infty}\sum_{k}\frac{1}{o}\sum_{i=k}^{m}\sum_{j=k}^{i}\sum_{l=0}^{i-j}\sum_{v=0}^{i-j-l}\binom{j}{k}\mu_1^{i-j-l-v}\mu_2^{v}\mu_3^{l}(-b)^{j-k}(a+b)^{k}a^{-j}t_{ni}\quad\text{exists }\ \forall n\in\mathbb{N} \tag{20}$$

Proof: Let $A\in\left(b_c^{a,b}(Q):\ell_\infty\right)$. Obviously, Ax exists and $Ax\in\ell_\infty$ for every $x=(x_k)\in b_c^{a,b}(Q)$. So this leads us to $\{t_{nk}\}_{k\in\mathbb{N}}\in\left\{b_c^{a,b}(Q)\right\}^{\beta}$ for $\forall n\in\mathbb{N}$. By using this and Theorem 3.5 (ii), we deduce that the statements (18), (19) and (20) hold. Now, we define a sequence such that

$$x=\frac{1}{o}\left[\frac{\mu_1^2-\mu_1^{n+3}}{(1-\mu_1)(\mu_1-\mu_2)(\mu_1-\mu_3)}-\frac{\mu_2^2-\mu_2^{n+3}}{(1-\mu_2)(\mu_1-\mu_2)(\mu_2-\mu_3)}+\frac{\mu_3^2-\mu_3^{n+3}}{(1-\mu_3)(\mu_1-\mu_3)(\mu_2-\mu_3)}\right]$$

If we take account of $x\in b_c^{a,b}(Q)$ and $Ax\in\ell_\infty$ for all $x\in b_c^{a,b}(Q)$, we obtain that the statement (17) holds.

Conversely, suppose that the statements (17)-(20) hold. Let $x=(x_k)\in b_c^{a,b}(Q)$ be an arbitrary sequence space and consider the equality

$$\begin{aligned}\sum_{k=0}^{m}t_{nk}x_k&=\sum_{k=0}^{m}\left[\frac{1}{o}\sum_{l=0}^{k}\sum_{i=l}^{k}\sum_{j=0}^{k-i}\sum_{v=0}^{k-i-j}\binom{i}{l}\mu_1^{k-i-j-v}\mu_2^{v}\mu_3^{j}(-b)^{i-l}(a+b)^{l}a^{-i}y_l\right]t_{nk}\\&=\frac{1}{o}\sum_{k=0}^{m}\sum_{l=k}^{m}\left[\sum_{i=k}^{l}\sum_{j=0}^{l-i}\sum_{v=0}^{l-i-j}\binom{i}{k}\mu_1^{l-i-j-v}\mu_2^{v}\mu_3^{j}(-b)^{i-k}(a+b)^{k}a^{-i}\right]t_{nl}y_k\end{aligned} \tag{21}$$

for all $m,n\in\mathbb{N}$. If we get limit (21) side to side as $m\to\infty$ we get

$$\sum_k t_{nk}x_k = \sum_k \rho_{nk}^{a,b} y_k \tag{22}$$

for all $n \in \mathbb{N}$. Furthermore, by taking sup-norm (22) side, we get

$$\|Ax\|_\infty \leq \sup_{n\in\mathbb{N}} \sum_k \left|\rho_{nk}^{a,b}\right| |y_k| \leq \|y\|_\infty \sup_{n\in\mathbb{N}} \sum_k \left|\rho_{nk}^{a,b}\right| < \infty.$$

Therefore, $Ax \in \ell_\infty$, namely $A \in \left(b_c^{a,b}(Q): \ell_\infty\right)$. Completes the proof.

Theorem 4.3. $A \in \left(b_c^{a,b}(Q): c\right)$ iff circumstances (17)-(20) hold, and

$$\lim_{n\to\infty} \sum_k \rho_{nk}^{a,b} = \lambda \tag{23}$$

$$\lim_{n\to\infty} \rho_{nk}^{a,b} = \lambda_k \quad \text{for all } k \in \mathbb{N} \tag{24}$$

Proof: Assume that $A \in \left(b_c^{a,b}(Q): c\right)$. We know that $c \subset \ell_\infty$ holds. By combination the inclusion and Theorem 4.2, we conclude that the circumstances (17)-(20) hold. Furthermore, obviously Ax exists and $Ax \in c$ for all $x = (x_k) \in b_c^{a,b}(Q)$. Hence, if we choose two sequences

$$x = \frac{1}{o}\left[\frac{\mu_1^2 - \mu_1^{n+3}}{(1-\mu_1)(\mu_1-\mu_2)(\mu_1-\mu_3)} - \frac{\mu_2^2 - \mu_2^{n+3}}{(1-\mu_2)(\mu_1-\mu_2)(\mu_2-\mu_3)} + \frac{\mu_3^2 - \mu_3^{n+3}}{(1-\mu_3)(\mu_1-\mu_3)(\mu_2-\mu_3)}\right]$$

and $x = \rho^{(k)}(a,b)$ we deduce that the circumstances (23) and (24) hold, where, $x = \rho^{(k)}(a,b)$ is defined in Theorem 3.1.

Conversely, for a given $x = (x_k) \in b_c^{a,b}(Q)$, suppose that the circumstances (17)-(20), (23) and (24) hold. Afterwards considering Theorem 3.5 (ii) we can see that $\{t_{nk}\}_{k\in\mathbb{N}} \in \left\{b_c^{a,b}(Q)\right\}^\beta$ for $\forall n \in \mathbb{N}$. Namely, Ax exists. Due to conditions (17) and (24), we write that

$$\sum_{k=0}^{m} |\lambda_k| \leq \sup_{n\in\mathbb{N}} \sum_k \left|\rho_{nk}^{a,b}\right| < \infty$$

for every $m \in \mathbb{N}$. As a result of this, $(\lambda_k) \in \ell_1$. So the series $\sum_k \lambda_k y_k$ obsolute convergent. Now, we substitute $t_{nk} - \lambda_k$ instead of a_{nk} in the condition (22). Then, we have

$$\sum_k (t_{nk} - \lambda_k)x_k = \sum_k \frac{1}{o} \sum_{i=k}^{\infty} \sum_{j=i}^{k} \sum_{l=0}^{k-j} \sum_{v=0}^{k-j-l} \binom{j}{i} \mu_1^{k-j-l-v} \mu_2^v \mu_3^l (-b)^{j-i}(a+b)^i a^{-j}(t_{ni} - \lambda_i) y_k \tag{25}$$

for all $n \in \mathbb{N}$. If we consider (25) and Lemma 3.3 (vi), we get

$$\lim_{n\to\infty} \sum_k (t_{nk} - \lambda_k)x_k = 0 \tag{26}$$

Finally, if we combine the circumstance (26) and the fact $(\lambda_k y_k) \in \ell_1$, we get, $Ax \in c$. It shows that $A \in \left(b_c^{a,b}(Q): c\right)$. Thus, the proof is complete.

4. Conclusion

The domain of Binomial matrix $B^{a,b} = \left(b_{nk}^{a,b}\right)$ in the sequence spaces c, c_0 and ℓ_∞ has been introduced by Bişgin (2016a, 2016b). Afterward, Meng and Song (2017) have defined the sequence spaces $b_0^{a,b}(\nabla)$, $b_c^{a,b}(\nabla)$ and $b_\infty^{a,b}(\nabla)$ and so generalized Bişgin's works. Thereafter, Sönmez (2019) has defined the sequence spaces $b_0^{a,b}(\mathrm{G})$, $b_c^{a,b}(\mathrm{G})$ and $b_\infty^{a,b}(\mathrm{G})$ and generalized Meng and Song's work. Thereafter, Bişgin (2017b) has defined the sequence spaces $b_0^{a,b}(\mathrm{D})$, $b_c^{a,b}(\mathrm{D})$ and $b_\infty^{a,b}(\mathrm{D})$ in and generalized Sönmez's work. Lastly, Bişgin (2023) has defined the quadruple band matrix $Q = \left(q_{nk}(o,t,u,v)\right)$ and has shown that it is more general than triple band matrix $\left(D(t,u,v)\right)$, double band matrix $\left(G(t,u)\right)$ and difference matrix (Δ). We define $F^{a,b} = \left(f_{nk}^{a,b}\right)$ that is the composition of the binomial matrix $B^{a,b} = \left(b_{nk}^{a,b}\right)$ and the quadruple band matrix $Q = \left(q_{nk}(o,t,u,v)\right)$. Hence, our results obtained by the matrix $F^{a,b} = B^{a,b}(Q)$ are more general than any previous work mentioned above.

Author Contributions

The author has read and agreed to the published version of the manuscript.

Conflict of Interest

All the authors declare no conflict of interest.

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