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Four Dimensional Matrix Operators on the Double Series Spaces of Weighted Means

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Abstract

The main purpose in this study is to investigate some topological and algebraic properties of the double series space $\left|\bar{N}_{p,q}\right|_k$ defined by the absolute double weighted summability methods for $k \geq 1$. Beside this, we determine the α-dual of the double series space $\left|\bar{N}_{p,q}\right|_1$ and the $\beta(bp)$- and γ- duals of the double series space $\left|\bar{N}_{p,q}\right|_k$ for $k \geq 1$. Finally, we characterize some new four dimensional matrix transformation classes $\left(\left|\bar{N}_{p,q}\right|_k, \upsilon\right)$, $\left(\left|\bar{N}_{p,q}\right|_1, \upsilon\right)$ and $\left(\left|\bar{N}_{p,q}\right|_1, \mathcal{L}_k\right)$, where υ denotes any spaces of double sequences $\mathcal{M}_u$ and $\mathcal{C}_p$. Hence, we extend some results about weighted means to double sequences.

Keywords: Double sequences, Dual spaces, Four dimensional weighted means, Four dimensional matrix transformations, Pringsheim convergence.

1. Introduction

Recently, there has been an increased interest in studies concerned on sequence spaces (see, [1-11]). Also, an important area of study in sequence spaces is the generalization of single-sequence spaces to double sequence spaces [12-16]. The initial works on double sequences have been given by Bromwich [17]. Also, Zeltser [18] has studied both the theory of topological double sequence spaces and the theory of summability of double sequences in her PhD thesis. Later on, they were studied by Hardy [19], Móricz [20], Móricz and Rhoades [21], Mursaleen [22], Mursaleen and Başar [23], Demiriz and Duyar [24], Demiriz and Erdem [25] and many others.

A double sequence $x = (x_{rs})$ is a double infinite array of elements x_{rs} for all $r, s \in \mathbb{N}$, where $\mathbb{N} = \{0,1,2,\ldots\}$. We denote the set of all real or complex valued double sequences by Ω which forms a vector space with coordinatewise addition and scalar multiplication of double sequences. Any vector subspace of Ω is called as *a double sequence space*. We denote the space of all bounded double sequences by $\mathcal{M}_u$, i.e.,

$$\mathcal{M}_u = \left\{x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty\right\},$$

which is a Banach space with the norm $\|.\|_\infty$. Consider the double sequence $x = (x_{mn}) \in \Omega$. If for every given $\epsilon > 0$ there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ and $L \in \mathbb{C}$ such that $|x_{mn} - L| < \epsilon$ for all $m, n > n_0$, the double sequence $x = (x_{mn}) \in \Omega$ is called *convergent* to the limit point L in the *Pringsheim's sense*, where $\mathbb{C}$ denotes the complex field. Then, we write $p - \lim_{m,n\to\infty} x_{mn} = L$, and $L \in \mathbb{C}$ is called the Pringsheim limit of x. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in the Pringsheim's sense [26]. Unlike single sequences, p −convergent double sequences need not be bounded. Namely, the set $\mathcal{C}_p - \mathcal{M}_u$ is not empty. Indeed, following Boos [27], if we define the sequence $x = (x_{mn})$ by

$$x_{mn} := \begin{cases} m; n = 0, m \in \mathbb{N} \\ n; m = 0, n \in \mathbb{N} \\ 0; m, n \in \mathbb{N} - \{0\} \end{cases}$$

for all $m, n \in \mathbb{N}$, then it is trivial that $x \in \mathcal{C}_p - \mathcal{M}_u$, since $p - \lim_{m,n\to\infty} x_{mn} = 0$ but $\|x\|_\infty = \infty$. Therefore, we consider the set $\mathcal{C}_{bp}$ of double sequences which are both convergent in Pringsheim's sense and bounded, i.e, $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. Hardy [19] proved that a sequence in the space $\mathcal{C}_p$ is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and, $\mathcal{C}_r$ denotes the set of all such sequences.

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Here and after, we assume that ϑ denotes any of the symbols p, bp, or r, and k' denotes the conjugate of k, that is, $\frac{1}{k}+\frac{1}{k'}=1$ for $1<k<\infty$, and $\frac{1}{k'}=0$ for $k=1$.

Let $x=(x_{mn})$ be a double sequence and define the sequence $s=(s_{mn})$ via x by

$$s_{mn}=\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}$$

for all $m,n\in\mathbb{N}$. Then, the pair of (x,s) and the sequence $s=(s_{mn})$ are called as a double series and the sequence of partial sums of the double series, respectively. For brevity, here and in what follows we use the abbreviation $\sum_{i,j}x_{ij}$ for the summation $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}x_{ij}$. If the double sequence (s_{mn}) is convergent in the ϑ −sense, then the double series $\sum_{i,j}x_{ij}$ is said to be convergent in the ϑ −sense and it is denoted that $\vartheta-\sum_{i,j}x_{ij}=\vartheta-\lim_{m,n\to\infty}s_{mn}$.

Quite recently, Başar and Sever have introduced the Banach space $\mathcal{L}_k$ of double sequences as

$$\mathcal{L}_k=\left\{x=(x_{mn})\in\Omega:\sum_{m,n}|x_{mn}|^k<\infty\right\},$$

which corresponds to the well-known classical sequence space ℓ_k of single sequences [28]. Also, for the special case $k=1$, the space $\mathcal{L}_k$ is reduced to the space $\mathcal{L}_u$, which was introduced by Zeltser [18].

Let λ and μ be two double sequence spaces, and $A=(a_{mnij})$ be any four dimensional complex infinite matrix. Then, we say that A defines a four dimensional matrix mapping from λ into μ, if for every double sequence $x=(x_{ij})\in\lambda$, $Ax=\{(Ax)_{mn}\}_{m,n\in\mathbb{N}}$, the A-transform of x, is in μ, where

$$(Ax)_{mn}=\vartheta-\sum_{i,j}a_{mnij}x_{ij} \qquad (1.1)$$

provided that the double series exists for each $m,n\in\mathbb{N}$. By (λ,μ), we denote the set of such all four dimensional matrices transforming from the space λ into the space μ. Thus, $A=(a_{mnij})\in(\lambda,\mu)$ if and only if the double series on the right side of (1.1) converges in the sense of ϑ for each $m,n\in\mathbb{N}$ and $Ax\in\mu$ for all $x\in\lambda$.

The *α-dual* λ^{α}, *$\beta(\vartheta)$-dual* $\lambda^{\beta(\vartheta)}$ in regard to the ϑ −convergence for $\vartheta\in\{p,bp,r\}$, and *γ-dual* λ^{γ} of the double sequence space λ are defined by, respectively,

$$\lambda^{\alpha}:=\left\{a=(a_{kl})\in\Omega\ :\sum_{k,l}|a_{kl}x_{kl}|<\infty,\text{for all }(x_{kl})\in\lambda\right\},$$

$$\lambda^{\beta(\vartheta)}:=\left\{a=(a_{kl})\in\Omega\ :\vartheta-\sum_{k,l}a_{kl}x_{kl}\text{ exists, for all }(x_{kl})\in\lambda\right\},$$

and

$$\lambda^{\gamma}:=\left\{a=(a_{kl})\in\Omega\ :\sup_{m,n\in\mathbb{N}}\left|\sum_{k,l=0}^{m,n}a_{kl}x_{kl}\right|<\infty,\text{for all }(x_{kl})\in\lambda\right\}.$$

We define the ϑ −summability domain $\lambda_A^{(\vartheta)}$ of $A=(a_{mnij})$ in a space λ of double sequences by

$$\lambda_A^{(\vartheta)}=\left\{x=(x_{ij})\in\Omega : Ax=\left(\vartheta-\sum_{i,j}a_{mnij}x_{ij}\right)_{m,n\in\mathbb{N}}\text{ exists and is in }\lambda\right\}.$$

We write throughout for simplicity in notation for all $m,n,k,l\in\mathbb{N}$ that

$$\Delta_{10}x_{mn}=x_{mn}-x_{m+1,n}\ ,$$
$$\Delta_{01}x_{mn}=x_{mn}-x_{m,n+1}\ ,$$
$$\Delta_{11}x_{mn}=\Delta_{01}(\Delta_{10}x_{mn})=\Delta_{10}(\Delta_{01}x_{mn})$$

and

$$\Delta_{10}^{kl}a_{mnkl}=a_{mnkl}-a_{mn,k+1,l}\ ,$$
$$\Delta_{01}^{kl}a_{mnkl}=a_{mnkl}-a_{mnk,l+1}\ ,$$
$$\Delta_{11}^{kl}a_{mnkl}=\Delta_{01}^{kl}(\Delta_{10}^{kl}a_{mnkl})=\Delta_{10}^{kl}(\Delta_{01}^{kl}a_{mnkl}).$$

Now, we give some definitions about fundamental four dimensional matrix methods. The four dimensional Cesàro matrix $C=(c_{mnij})$ of order one is defined by

$$c_{mnij}=\begin{cases}\dfrac{1}{mn}, & 1\le i\le m,\quad 1\le j\le n\\ 0, & \text{otherwise}\end{cases}$$

for all $m,n,i,j\in\mathbb{N}-\{0\}$ [23]. The four dimensional Cesàro matrix $C=(c_{mnij})$ is extended by means of the four dimensional Riesz matrix.

Let $p=(p_k)$, $q=(q_k)$ be two sequences of positive numbers, and $P_n=\sum_{k=0}^{n}p_k$, and $Q_n=\sum_{k=0}^{n}q_k$. Then, the four dimensional Riesz matrix $R^{pq}=(r_{mnij}^{pq})$ is defined by

$$r_{mnij}^{pq}=\begin{cases}\dfrac{p_iq_j}{P_mQ_n}, & 0\le i\le m,\quad 0\le j\le n\\ 0, & \text{otherwise}\end{cases}$$

for all $m,n,i,j\in\mathbb{N}$ [29]. Note that in the case $p_k=q_k=1$ for all $k\in\mathbb{N}$, the Riesz matrix R^{pq} is reduced to the four dimensional Cesàro matrix of order one.

Let $\sum_{i,j} x_{ij}$ be an infinite double series with partial sums (s_{mn}). The double weighted mean transformation R_{mn}^{pq} of a double sequence $s = (s_{mn})$ by means of the four dimensional Riesz matrix is defined as

$$R_{mn}^{pq}(s) = \frac{1}{P_m Q_n} \sum_{i=0}^{m} \sum_{j=0}^{n} p_i q_j s_{ij}\,, \qquad (m,n \in \mathbb{N}).$$

We say that $s = (s_{mn})$ is $(\bar{N}, p_n, q_n)$ summable or Riesz summable to some number ℓ if, (see, [30])

$$p - \lim_{m,n\to\infty} R_{mn}^{pq}(s) = \ell.$$

A double series $\sum_{i,j} x_{ij}$ is called summable $|\bar{N}, p_n, q_n|_k$ or absolute double weighted summability ([31]), if

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{P_m Q_n}{p_m q_n}\right)^{k-1} \left|\Delta_{11} R_{m-1,n-1}^{pq}(s)\right|^k < \infty,$$

where, $k \geq 1$ and for $m, n \geq 1$

$$\Delta_{11}\left(R_{m-1,-1}^{pq}\right) = R_{m0}^{pq} - R_{m-1,0}^{pq},$$
$$\Delta_{11}\left(R_{-1,n-1}^{pq}\right) = R_{0n}^{pq} - R_{0,n-1}^{pq},$$

and

$$\Delta_{11}\left(R_{m-1,n-1}^{pq}\right) = R_{mn}^{pq} - R_{m-1,n}^{pq} - R_{m,n-1}^{pq} + R_{m-1,n-1}^{pq}.$$

Further, it is easily seen that

$$\begin{aligned} R_{mn}^{pq}(s) &= \frac{1}{P_m Q_n} \sum_{i=0}^{m} \sum_{j=0}^{n} p_i q_j s_{ij} \\ &= \frac{1}{P_m Q_n} \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} (P_m - P_{i-1})(Q_n - Q_{j-1}) \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} \left(1 - \frac{P_{i-1}}{P_m}\right)\left(1 - \frac{Q_{j-1}}{Q_n}\right). \qquad (1.2) \end{aligned}$$

Also, throughout the paper for brevity, we show $R_{mn}^{pq}(s)$ defined as in (1.2) by R_{mn}^{pq}.

So, we can calculate for $m, n = 0$,

$$\Delta_{11} R_{m-1,n-1}^{pq} = x_{00}\,, \qquad (1.3)$$

and, for $m, n \geq 1$,

$$\Delta_{11} R_{m-1,-1}^{pq} = \frac{p_m}{P_m P_{m-1}} \sum_{i=1}^{m} P_{i-1} x_{i0}\,, \qquad (1.4)$$

$$\Delta_{11} R_{-1,n-1}^{pq} = \frac{q_n}{Q_n Q_{n-1}} \sum_{j=1}^{n} Q_{j-1} x_{0j}\,, \qquad (1.5)$$

and

$$\Delta_{11} R_{m-1,n-1}^{pq} = \frac{p_m q_n}{P_m P_{m-1} Q_n Q_{n-1}} \sum_{i=1}^{m} \sum_{j=1}^{n} P_{i-1} Q_{j-1} x_{ij}. \qquad (1.6)$$

Now, considering Sarıgöl [31], we show the double series space $\left|\bar{N}_{p,q}\right|_k$ by the set of all double series summable by absolute double weighted summability method $|\bar{N}, p_m, q_n|_k$, that is,

$$\left|\bar{N}_{p,q}\right|_k = \left\{x = (x_{ij}) \in \Omega : \sum\sum x_{ij} \text{ is summable } |\bar{N}, p_m, q_n|_k\right\},$$

which is a Banach space [15].

Many single sequence spaces have been defined by using the matrix domain of Riesz means [32,33,34]. Bodur and Güleç [13] have essentially studied some topological properties of double series space $\left|C_{1,1}\right|_k$, determined certain dual spaces and characterized the classes of four dimensional matrix transformations. Also, Yeşilkayagil and Başar [29,35] have introduced the spaces $(\mathcal{M}_u)_{R^{qt}}, (\mathcal{C}_p)_{R^{qt}}, (\mathcal{C}_{bp})_{R^{qt}}, (\mathcal{C}_r)_{R^{qt}}$ and $(\mathcal{L}_s)_{R^{qt}}$ as the domain of four dimensional Riesz mean R^{qt} in the spaces $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_s$, respectively. In this paper, we investigate some topological and algebraic properties of the absolutely double series space $\left|\bar{N}_{p,q}\right|_k$ taking account of absolute double weighted summability method for $k \geq 1$. Beside this, we determine the α-dual of the double series space $\left|\bar{N}_{p,q}\right|_1$ and the $\beta(bp)$- and γ- duals of the double series spaces $\left|\bar{N}_{p,q}\right|_k$ for $k \geq 1$. Finally, we characterize some new classes of the four dimensional matrix transformations.

2. The Absolutely Double Series Space of Double Weighted Means

In this section, we give some properties of the absolutely double weighted series spaces $\left|\bar{N}_{p,q}\right|_k$ for $k \geq 1$. Also, we determine the α- dual of the double series space $\left|\bar{N}_{p,q}\right|_1$, $\beta(bp)$- and γ- duals of the double series spaces $\left|\bar{N}_{p,q}\right|_k$ for $1 \leq k < \infty$.

Theorem 2.1. The set $\left|\bar{N}_{p,q}\right|_k$ becomes a linear space with the coordinatewise addition and scalar multiplication, and $\left|\bar{N}_{p,q}\right|_k$ is a Banach space with the norm

$$\|x\|_{\left|\bar{N}_{p,q}\right|_k} = \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{P_m Q_n}{p_m q_n}\right)^{k-1} \left|\Delta_{11} R_{m-1,n-1}^{pq}\right|^k\right)^{1/k} < \infty, \qquad (2.1)$$

and it is linearly norm isomorphic to the space $\mathcal{L}_k$ for $1 \leq k < \infty$, where R_{mn}^{pq} is defined as in $(1.3 - 1.6)$.

Proof. Since the initial assertion is routine verification and so we omit it.
To prove the fact that $\left|\bar{N}_{p,q}\right|_k$ is norm isomorphic to the space $\mathcal{L}_k$, we should show the existence of a linear and norm preserving bijection between the spaces $\left|\bar{N}_{p,q}\right|_k$ and $\mathcal{L}_k$ for $1 \leq k < \infty$. Consider the transformation U defined by

$$U : \left|\bar{N}_{p,q}\right|_k \to \mathcal{L}_k$$
$$x \to y = U(x),$$

where $U(x) = \left(U_{mn}(x)\right) = (y_{mn})$ is stated by

$$U_{mn}(x) = y_{mn} = \left(\frac{P_m Q_n}{p_m q_n}\right)^{1-1/k} \Delta_{11} R_{m-1,n-1}^{pq} \qquad (2.2)$$

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for $m,n \geq 0$ and $\Delta_{11} R^{pq}_{m-1,n-1}$ is given as in $(1.3-1.6)$. The linearity of U is clear. Also, $x=\theta$ whenever $U(x)=\theta$, where θ denotes the zero vector. This says us that U is injective.

Let $y=(y_{mn}) \in \mathcal{L}_k$ and define the sequence $x=(x_{mn})$ via $y=(y_{mn})$ by

$$x_{mn} = \frac{1}{P_{m-1}Q_{n-1}} \Delta_{11}\left(y_{m-1,n-1}\left(\frac{P_{m-1}Q_{n-1}}{p_{m-1}q_{n-1}}\right)^{1/k} P_{m-2}Q_{n-2}\right), \quad (2.3)$$

$$x_{m0} = \frac{1}{P_{m-1}} \overline{\Delta}_{10}\left(y_{m0}\left(\frac{P_m}{p_m}\right)^{1/k} P_{m-1}\right), \quad (2.4)$$

$$x_{0n} = \frac{1}{Q_{n-1}} \overline{\Delta}_{01}\left(y_{0n}\left(\frac{Q_n}{q_n}\right)^{1/k} Q_{n-1}\right), \quad (2.5)$$

for $m,n \geq 1$, and

$$x_{00} = y_{00}, \quad (2.6)$$

where $\overline{\Delta}_{10}$ and $\overline{\Delta}_{01}$ refer to the back difference notations, that is, $\overline{\Delta}_{10}(x_{mn}) = x_{m,n} - x_{m-1,n}$, $\overline{\Delta}_{01}(x_{mn}) = x_{m,n} - x_{m,n-1}$ for all $m,n \in \mathbb{N}$.

In that case, it is seen that

$$\|x\|_{|\overline{N}_{p,q}|_k} = \|U(x)\|_{\mathcal{L}_k} = \left(\sum_{m,n} |U_{mn}(x)|^k\right)^{1/k} = \|y\|_{\mathcal{L}_k} < \infty$$

for $1 \leq k < \infty$. So, this yields that U is surjective and norm preserving. Thus, U is a linear and norm preserving bijection which says the spaces $|\overline{N}_{p,q}|_k$ and $\mathcal{L}_k$ are norm isomorphic for $1 \leq k < \infty$, as desired.

Now, we may show that $|\overline{N}_{p,q}|_k$ is a Banach space with the norm defined by (2.1). To prove this, we can consider "Let (E,ρ) and (F,σ) be semi-normed spaces and $\Psi : (E,\rho) \to (F,\sigma)$ be an isometric isomorphism. Then (E,ρ) is complete if and only if (F,σ) is complete. In particular, (E,ρ) is a Banach space if and only if (F,σ) is a Banach space." which can be found section (b) of Corollary 6.3.41 in [27]. Since the transformation U defined from $|\overline{N}_{p,q}|_k$ into $\mathcal{L}_k$ by (2.2) is an isometric isomorphism and the double sequence space $\mathcal{L}_k$ is a Banach space from Theorem 2.1 in [28], we deduce that the space $|\overline{N}_{p,q}|_k$ is a Banach space. This is the result that we desired.

Now, we calculate the α-, $\beta(bp)$- and γ- duals of the double series spaces $|\overline{N}_{p,q}|_k$ for $k \geq 1$. Before we give some results based on their duals, we need to state the following significant lemma which is essential for proving next theorems.

Lemma 2.2. [35] Let $A=(a_{mnij})$ be any four dimensional infinite matrix. At that case, the following statements are satisfied:

(a) Let $0 < k \leq 1$. Then, $A=(a_{mnij}) \in (\mathcal{L}_k, \mathcal{M}_u)$ iff

$$\Lambda_1 = \sup_{m,n,i,j \in \mathbb{N}} |a_{mnij}| < \infty. \quad (2.7)$$

(b) Let $1 < k < \infty$. Then, $A=(a_{mnij}) \in (\mathcal{L}_k, \mathcal{M}_u)$ iff

$$\Lambda_2 = \sup_{m,n \in \mathbb{N}} \sum_{i,j} |a_{mnij}|^{k'} < \infty. \quad (2.8)$$

(c) Let $0 < k \leq 1$ and $1 \leq k_1 < \infty$. Then, $A=(a_{mnij}) \in (\mathcal{L}_k, \mathcal{L}_{k_1})$ iff

$$\sup_{i,j \in \mathbb{N}} \sum_{m,n} |a_{mnij}|^{k_1} < \infty.$$

(d) Let $0 < k \leq 1$.Then, $A=(a_{mnij}) \in (\mathcal{L}_k, \mathcal{C}_{bp})$ iff the condition (2.7) holds and there exists a $(\lambda_{ij}) \in \Omega$ such that

$$bp - \lim_{m,n \to \infty} a_{mnij} = \lambda_{ij}. \quad (2.9)$$

(e) Let $1 < k < \infty$. Then, $A=(a_{mnij}) \in (\mathcal{L}_k, \mathcal{C}_{bp})$ iff (2.8) and (2.9) are satisfied.

To shorten the following theorems and their proofs, let us denote the sets E_k with $k \in \{1,2,3,4\}$ as follows for $\xi = (\xi_{mn}) \in \Omega$:

$$E_1 = \left\{\xi \in \Omega : \sup_{i,j \in \mathbb{N}} \sum_{m,n} |f_{mnij}| < \infty\right\}, \quad (2.10)$$

$$E_2 = \left\{\xi \in \Omega : bp - \lim_{m,n \to \infty} d^{(k)}_{mnij} \text{ exists}\right\}, \quad (2.11)$$

$$E_3 = \left\{\xi \in \Omega : \sup_{m,n,i,j \in \mathbb{N}} |d^{(1)}_{mnij}| < \infty\right\}, \quad (2.12)$$

$$E_4 = \left\{\xi \in \Omega : \sup_{m,n \in \mathbb{N}} \sum_{i,j} |d^{(k)}_{mnij}|^{k'} < \infty\right\}, \quad (2.13)$$

where the four dimensional matrices $D^{(k)} = (d^{(k)}_{mnij})$ and $F=(f_{mnij})$ are defined by

$$d_{mnij}^{(k)}=\begin{cases}\xi_{00}, & m=n=0,\\ \xi_{m0}\left(\dfrac{P_m}{p_m}\right)^{1/k}, & n=0 \text{ and } i=m,\\ \xi_{0n}\left(\dfrac{Q_n}{q_n}\right)^{1/k}, & m=0 \text{ and } j=n,\\ P_{i-1}\left(\dfrac{P_i}{p_i}\right)^{1/k}\Delta_{10}\left(\dfrac{\xi_{i0}}{P_{i-1}}\right), & n=0 \text{ and } 1\le i\le m-1,\\ Q_{j-1}\left(\dfrac{Q_j}{q_j}\right)^{1/k}\Delta_{01}\left(\dfrac{\xi_{0j}}{Q_{j-1}}\right), & m=0 \text{ and } 1\le j\le n-1,\\ \Delta_{11}\left(\dfrac{\xi_{ij}}{P_{i-1}Q_{j-1}}\right)\left(\dfrac{P_iQ_j}{p_iq_j}\right)^{1/k}P_{i-1}Q_{j-1}, & 1\le i\le m-1 \text{ and } 1\le j\le n-1,\\ \Delta_{10}\left(\dfrac{\xi_{in}}{P_{i-1}}\right)\left(\dfrac{P_iQ_n}{p_iq_n}\right)^{1/k}P_{i-1}, & 1\le i\le m-1 \text{ and } j=n,\\ \Delta_{01}\left(\dfrac{\xi_{mj}}{Q_{j-1}}\right)\left(\dfrac{P_mQ_j}{p_mq_j}\right)^{1/k}Q_{j-1}, & i=m \text{ and } 1\le j\le n-1,\\ \xi_{mn}\left(\dfrac{P_mQ_n}{p_mq_n}\right)^{1/k}, & i=m \text{ and } j=n,\\ 0, & \text{otherwise},\end{cases} \tag{2.14}$$

and

$$f_{mnij}=\begin{cases}\xi_{00}, & m=n=0,\\ \xi_{m0}\dfrac{P_m}{p_m}, & n=0 \text{ and } i=m,\\ -\dfrac{\xi_{m0}P_{m-2}}{p_{m-1}}, & n=0 \text{ and } i=m-1,\\ \xi_{0n}\dfrac{Q_n}{q_n}, & m=0 \text{ and } j=n,\\ -\dfrac{\xi_{0n}Q_{n-2}}{q_{n-1}}, & m=0 \text{ and } j=n-1,\\ \dfrac{\xi_{mn}P_{m-2}Q_{n-2}}{p_{m-1}q_{n-1}}, & i=m-1 \text{ and } j=n-1,\\ -\dfrac{\xi_{mn}P_{m-2}Q_n}{p_{m-1}q_n}, & i=m-1 \text{ and } j=n,\\ -\dfrac{\xi_{mn}Q_{n-2}P_m}{p_mq_{n-1}}, & i=m \text{ and } j=n-1,\\ \xi_{mn}\dfrac{P_mQ_n}{p_mq_n}, & i=m \text{ and } j=n,\\ 0, & \text{otherwise}\end{cases} \tag{2.15}$$

respectively.

Now, we give the theorems determining the α- dual of the double series space $\left|\bar{N}_{p,q}\right|_1$ and $\beta(bp)$- and γ- duals of the double series spaces $\left|\bar{N}_{p,q}\right|_k$ for $k\ge 1$.

Theorem 2.3. Let the set E_1 and the four dimensional matrix $F=\left(f_{mnij}\right)$ be defined as in (2.10) and (2.15), respectively. Then, $\left(\left|\bar{N}_{p,q}\right|_1\right)^{\alpha}=E_1$.

Proof. Let $x=(x_{mn})\in\left|\bar{N}_{p,q}\right|_1$, $\xi=(\xi_{mn})\in\Omega$. Taking account of relations in $(2.3-2.6)$ for $m,n\ge 0$, we can calculate the following equalities:
for $m,n\ge 1$,

$$\xi_{mn}x_{mn}=\frac{\xi_{mn}}{P_{m-1}Q_{n-1}}(\left(\frac{P_mQ_n}{p_mq_n}\right)P_{m-1}Q_{n-1}y_{mn}-\left(\frac{P_mQ_{n-1}}{p_mq_{n-1}}\right)P_{m-1}Q_{n-2}y_{m,n-1}-\left(\frac{P_{m-1}Q_n}{p_{m-1}q_n}\right)P_{m-2}Q_{n-1}y_{m-1,n}+\left(\frac{P_{m-1}Q_{n-1}}{p_{m-1}q_{n-1}}\right)P_{m-2}Q_{n-2}y_{m-1,n-1})$$

$$=\frac{\xi_{mn}}{P_{m-1}Q_{n-1}}\sum_{i=m-1}^{m}\sum_{j=n-1}^{n}(-1)^{m+n-(i+j)}\left(\frac{P_iQ_j}{p_iq_j}\right)y_{ij}P_{i-1}Q_{j-1}=(Fy)_{mn},$$

for $n=0$ and $m\geq 1$,

$$\xi_{m0}x_{m0}=\xi_{m0}\frac{1}{P_{m-1}}\left(\left(\frac{P_m}{p_m}\right)P_{m-1}y_{m0}-P_{m-2}\left(\frac{P_{m-1}}{p_{m-1}}\right)y_{m-1,0}\right)$$

$$=\xi_{m0}\frac{1}{P_{m-1}}\sum_{i=m-1}^{m}(-1)^{m-i}\left(\frac{P_i}{p_i}\right)P_{i-1}y_{i0}=(Fy)_{m0},$$

for $m=0$ and $n\geq 1$,

$$\xi_{0n}x_{0n}=\xi_{0n}\frac{1}{Q_{n-1}}\left(\left(\frac{Q_n}{q_n}\right)Q_{n-1}y_{0n}-Q_{n-2}\left(\frac{Q_{n-1}}{q_{n-1}}\right)y_{0,n-1}\right)$$

$$=\xi_{0n}\frac{1}{Q_{n-1}}\sum_{j=n-1}^{n}(-1)^{n-j}\left(\frac{Q_j}{q_j}\right)Q_{j-1}y_{0j}=(Fy)_{0n},$$

and for $n=m=0$,

$$\xi_{00}x_{00}=\xi_{00}y_{00}=(Fy)_{00},$$

where the four dimensional matrix $F=(f_{mnij})$ is defined by (2.15). In this fact, we obtain that $\xi x=(\xi_{mn}x_{mn})\in\mathcal{L}_u$ whenever $x\in\left|\bar{N}_{p,q}\right|_1$ if and only if $Fy\in\mathcal{L}_u$ whenever $y\in\mathcal{L}_u$. This implies that $\xi=(\xi_{mn})\in\left(\left|\bar{N}_{p,q}\right|_1\right)^{\alpha}$ iff $F\in(\mathcal{L}_u,\mathcal{L}_u)$. Then, we deduce by using (c) of Lemma 2.2 with $k_1=k=1$ that

$$\sup_{i,j\in\mathbb{N}}\sum_{m,n}|f_{mnij}|<\infty.$$

Hence, we get $\left(\left|\bar{N}_{p,q}\right|_1\right)^{\alpha}=E_1$, as desired.

This step concludes the proof.

Theorem 2.4. Let the sets E_2, E_3, E_4 and the four dimensional matrix $D^{(k)}=(d_{mnij}^{(k)})$ be defined as in $(2.11-2.13)$ and (2.14), respectively. Then, we have $\left(\left|\bar{N}_{p,q}\right|_1\right)^{\beta(bp)}=E_2\cap E_3$ and $\left(\left|\bar{N}_{p,q}\right|_k\right)^{\beta(bp)}=E_2\cap E_4$ for $1<k<\infty$.

Proof. Let $\xi=(\xi_{mn})\in\Omega$ and $x=(x_{mn})\in\left|\bar{N}_{p,q}\right|_k$ be given. Then, we deduce from Theorem 2.1 that there exists a double sequence $y=(y_{ij})\in\mathcal{L}_k$. Therefore, by using the equations $(2.3-2.6)$ we can calculate that

$$z_{mn}=\sum_{i=0}^{m}\sum_{j=0}^{n}\xi_{ij}x_{ij} \tag{2.16}$$

$$=\xi_{00}y_{00}+\sum_{i=1}^{m}\xi_{i0}y_{i0}\left(\frac{P_i}{p_i}\right)^{1/k}-\sum_{i=2}^{m}\xi_{i0}y_{i-1,0}\left(\frac{P_{i-1}}{p_{i-1}}\right)^{1/k}\frac{P_{i-2}}{P_{i-1}}+\sum_{j=1}^{n}\xi_{0j}y_{0j}\left(\frac{Q_j}{q_j}\right)^{1/k}-\sum_{j=2}^{n}\xi_{0j}y_{0,j-1}\left(\frac{Q_{j-1}}{q_{j-1}}\right)^{\frac{1}{k}}\frac{Q_{j-2}}{Q_{j-1}}+\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\xi_{ij}}{P_{i-1}Q_{j-1}}r_{ij},$$

where

$$r_{ij}=\Delta_{11}\left(y_{i-1,j-1}\left(\frac{P_{i-1}Q_{j-1}}{p_{i-1}q_{j-1}}\right)^{1/k}P_{i-2}Q_{j-2}\right)$$

for every $i,j\in\mathbb{N}$. Also, by the generalized Abel transformation for double sequences we obtain that

$$u_{mn}=\sum_{k,l=0}^{m,n}a_{kl}x_{kl}=\sum_{k,l=0}^{m-1,n-1}s_{kl}\Delta_{11}a_{kl}+\sum_{k=0}^{m-1}s_{kn}\Delta_{10}a_{kn}+\sum_{l=0}^{n-1}s_{ml}\Delta_{01}a_{ml}+s_{mn}a_{mn},$$

where

$$s_{mn}=\sum_{k,l=0}^{m,n}x_{kl}$$

for every $m,n\in\mathbb{N}$ [29]. With the generalized Abel transformation for double sequences, we can calculate relation (2.16) as follows:

$$z_{mn}=\xi_{00}y_{00}+\xi_{m0}y_{m0}\left(\frac{P_m}{p_m}\right)^{1/k}+\sum_{i=1}^{m-1}y_{i0}P_{i-1}\left(\frac{P_i}{p_i}\right)^{1/k}\Delta_{10}\left(\frac{\xi_{i0}}{P_{i-1}}\right)+\xi_{0n}y_{0n}\left(\frac{Q_n}{q_n}\right)^{1/k}+\sum_{j=1}^{n-1}y_{0j}Q_{j-1}\left(\frac{Q_j}{q_j}\right)^{1/k}\Delta_{01}\left(\frac{\xi_{0j}}{Q_{j-1}}\right)$$

$$+\sum_{i,j=1}^{m-1,n-1}\Delta_{11}\left(\frac{\xi_{ij}}{P_{i-1}Q_{j-1}}\right)y_{ij}\left(\frac{P_iQ_j}{p_iq_j}\right)^{1/k}P_{i-1}Q_{j-1}$$
$$+\sum_{i=1}^{m-1}\Delta_{10}\left(\frac{\xi_{in}}{P_{i-1}Q_{n-1}}\right)y_{in}\left(\frac{P_iQ_n}{p_iq_n}\right)^{1/k}P_{i-1}Q_{n-1}$$
$$+\sum_{j=1}^{n-1}\Delta_{01}\left(\frac{\xi_{mj}}{P_{m-1}Q_{j-1}}\right)y_{mj}\left(\frac{P_mQ_j}{p_mq_j}\right)^{1/k}P_{m-1}Q_{j-1}$$
$$+\xi_{mn}y_{mn}\left(\frac{P_mQ_n}{p_mq_n}\right)^{1/k}$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n}d_{mnij}^{(k)}y_{ij}$$
$$=\left(D^{(k)}(y)\right)_{mn}.$$

Thus, we see that $\xi x=(\xi_{mn}x_{mn})\in\mathcal{CS}_{bp}$ whenever $x=(x_{mn})\in|\bar{N}_{p,q}|_k$ if and only if $z=(z_{mn})\in\mathcal{C}_{bp}$ whenever $y=(y_{ij})\in\mathcal{L}_k$. This leads to the fact that $\xi=(\xi_{mn})\in\left(|\bar{N}_{p,q}|_k\right)^{\beta(bp)}$ if and only if $D^{(k)}\in(\mathcal{L}_k,\mathcal{C}_{bp})$, where the four dimensional matrix $D^{(k)}=\left(d_{mnij}^{(k)}\right)$ is defined in (2.14) for every $m,n,i,j\in\mathbb{N}$. Hence, we deduce $\left(|\bar{N}_{p,q}|_1\right)^{\beta(bp)}=E_2\cap E_3$ and $\left(|\bar{N}_{p,q}|_k\right)^{\beta(bp)}=E_2\cap E_4$ for $1<k<\infty$ from parts (d) and (e) of Lemma 2.2, respectively.

Theorem 2.5. Let the sets E_3, E_4 and the four dimensional matrix $D^{(k)}=\left(d_{mnij}^{(k)}\right)$ be defined as in (2.12), (2.13) and (2.14), respectively. Then, $\left(|\bar{N}_{p,q}|_1\right)^{\gamma}=E_3$ and $\left(|\bar{N}_{p,q}|_k\right)^{\gamma}=E_4$ for $1<k<\infty$.

Proof. The proof of this theorem is similar to the proof Theorem 2.4 using Parts (a) and (b) of Lemma 2.2 in place of parts (d) and (e) of Lemma 2.2, respectively. To avoid the repetition of similar statements, we omit the details.

3. Characterizations of Some Classes of Four Dimensional Matrices

In the present section, we characterize some four dimensional matrix mappings from the double series spaces $|\bar{N}_{p,q}|_1$ and $|\bar{N}_{p,q}|_k$ to the double sequence spaces $\mathcal{M}_u$, $\mathcal{C}_{bp}$ and $\mathcal{L}_k$ for $1\le k<\infty$.

Theorem 3.1. Suppose that $A=(a_{mnij})$ be an arbitrary four dimensional infinite matrix. Then, the following statements hold:

(a) $A=(a_{mnij})\in\left(|\bar{N}_{p,q}|_1,\mathcal{M}_u\right)$ if and only if

$$A_{mn}\in\left(|\bar{N}_{p,q}|_1\right)^{\beta(bp)} \tag{3.1}$$

and

$$\sup_{m,n,i,j\in\mathbb{N}}\left|\Delta_{11}^{(i,j)}\left(\frac{a_{mnij}}{P_{i-1}Q_{j-1}}\right)\left(\frac{P_iQ_j}{p_iq_j}\right)P_{i-1}Q_{j-1}\right|<\infty. \tag{3.2}$$

(b) Let $1<k<\infty$. Then, $A=(a_{mnij})\in\left(|\bar{N}_{p,q}|_k,\mathcal{M}_u\right)$ if and only if

$$A_{mn}\in\left(|\bar{N}_{p,q}|_k\right)^{\beta(bp)} \tag{3.3}$$

and

$$\sup_{m,n\in\mathbb{N}}\sum_{i,j}\left|\Delta_{11}^{(i,j)}\left(\frac{a_{mnij}}{P_{i-1}Q_{j-1}}\right)\left(\frac{P_iQ_j}{p_iq_j}\right)^{1/k}P_{i-1}Q_{j-1}\right|^{k'}<\infty. \tag{3.4}$$

Proof. Part (a) can be proved using Lemma 2.2 (a) in a manner similar to that used in the proof of part (b) of Theorem 3.1. To avoid repeating similar statements, we prove for $1<k<\infty$.

(b) Let $1<k<\infty$ and $x=(x_{ij})\in|\bar{N}_{p,q}|_k$. Then, from Theorem 2.1 there exists a double sequence $y=(y_{mn})\in\mathcal{L}_k$ via x by (2.2). Then, using the equalities $(2.3-2.6)$, for (s,t)th rectangular partial sum of the series $\sum_{i,j}a_{mnij}x_{ij}$, we obtain that

$$(Ax)_{mn}^{[s,t]}=\sum_{i,j}^{s,t}a_{mnij}x_{ij} \tag{3.5}$$
$$=a_{mn00}y_{00}+a_{mns0}y_{s0}\left(\frac{P_s}{p_s}\right)^{1/k}+a_{mn0t}y_{0t}\left(\frac{Q_t}{q_t}\right)^{1/k}$$
$$+\sum_{i=1}^{s-1}y_{i0}P_{i-1}\left(\frac{P_i}{p_i}\right)^{1/k}\Delta_{10}^{(ij)}\left(\frac{a_{mni0}}{P_{i-1}}\right)+\sum_{j=1}^{t-1}y_{0j}Q_{j-1}\left(\frac{Q_j}{q_j}\right)^{1/k}\Delta_{01}^{(ij)}\left(\frac{a_{mn0j}}{Q_{j-1}}\right)$$
$$+\sum_{i,j=1}^{s-1,t-1}\Delta_{11}^{(ij)}\left(\frac{a_{mnij}}{P_{i-1}Q_{j-1}}\right)y_{ij}\left(\frac{P_iQ_j}{p_iq_j}\right)^{1/k}P_{i-1}Q_{j-1}$$
$$+\sum_{i=1}^{s-1}\Delta_{10}^{(ij)}\left(\frac{a_{mnit}}{P_{i-1}Q_{t-1}}\right)y_{it}\left(\frac{P_iQ_t}{p_iq_t}\right)^{1/k}P_{i-1}Q_{t-1}$$
$$+\sum_{j=1}^{t-1}\Delta_{01}^{(ij)}\left(\frac{a_{mnsj}}{P_{s-1}Q_{j-1}}\right)y_{sj}\left(\frac{P_sQ_j}{p_sq_j}\right)^{1/k}P_{s-1}Q_{j-1}$$
$$+a_{mnst}y_{st}\left(\frac{P_sQ_t}{p_sq_t}\right)^{1/k}$$
$$=\sum_{i,j}^{s,t}g_{stij}^{mn}y_{ij}=(G_{mn}y)_{[s,t]}$$

for every $t,s,m,n\in\mathbb{N}$, where the four dimensional matrix $G_{mn}=\left(g_{stij}^{mn}\right)$ is defined by

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$$g_{stij}^{mn}=\begin{cases} a_{mn00}, & s=t=0,\\ a_{mns0}\left(\frac{P_s}{p_s}\right)^{1/k}, & t=0 \text{ and } i=s,\\ a_{mn0t}\left(\frac{Q_t}{q_t}\right)^{1/k}, & s=0 \text{ and } j=t,\\ P_{i-1}\left(\frac{P_i}{p_i}\right)^{1/k}\Delta_{10}^{(ij)}\left(\frac{a_{mni0}}{P_{i-1}}\right), & t=0 \text{ and } 1\le i\le s-1,\\ Q_{j-1}\left(\frac{Q_j}{q_j}\right)^{1/k}\Delta_{01}^{(ij)}\left(\frac{a_{mn0j}}{Q_{j-1}}\right), & s=0 \text{ and } 1\le j\le t-1,\\ \Delta_{11}^{(ij)}\left(\frac{a_{mnij}}{P_{i-1}Q_{j-1}}\right)\left(\frac{P_iQ_j}{p_iq_j}\right)^{1/k}P_{i-1}Q_{j-1}, & 1\le i\le s-1 \text{ and } 1\le j\le t-1,\\ \Delta_{10}^{(ij)}\left(\frac{a_{mnit}}{P_{i-1}}\right)\left(\frac{P_iQ_t}{p_iq_t}\right)^{1/k}P_{i-1}, & 1\le i\le s-1 \text{ and } j=t,\\ \Delta_{01}^{(ij)}\left(\frac{a_{mnsj}}{Q_{j-1}}\right)\left(\frac{P_sQ_j}{p_sq_j}\right)^{1/k}Q_{j-1}, & i=s \text{ and } 1\le j\le t-1,\\ a_{mnst}\left(\frac{P_sQ_t}{p_sq_t}\right)^{1/k}, & i=s \text{ and } j=t\\ 0, & \text{otherwise} \end{cases}$$

for every $s,t,i,j\in\mathbb{N}$. Then, from (3.5), we have

$$(Ax)_{mn}^{[s,t]}=(G_{mn}y)_{[s,t]}. \quad (3.6)$$

Therefore, it follows from (3.6) that the bp-convergence of $(Ax)_{mn}^{[s,t]}$ and the statement $G_{mn}\in(\mathcal{L}_k,\mathcal{C}_{bp})$ are equivalent for all $x\in|\bar{N}_{p,q}|_k$ and $m,n\in\mathbb{N}$. Hence, the condition (3.3) is satisfied for each fixed $m,n\in\mathbb{N}$, that is, $A_{mn}\in\left(|\bar{N}_{p,q}|_k\right)^{\beta(bp)}$ for each fixed $m,n\in\mathbb{N}$ and $1<k<\infty$.

If we take bp-limit in the terms of the matrix $G_{mn}=(g_{stij}^{mn})$ while $s,t\to\infty$, we obtain that

$$bp-\lim_{s,t\to\infty} g_{stij}^{mn}$$
$$=\left(\frac{P_iQ_j}{p_iq_j}\right)^{\frac{1}{k}}P_{i-1}Q_{j-1}\Delta_{11}^{(i,j)}\left(\frac{a_{mnij}}{P_{i-1}Q_{j-1}}\right). \quad (3.7)$$

With the relation (3.7), we can define the four dimensional matrix $G=(g_{mnij})$ as

$$g_{mnij}=\left(\frac{P_iQ_j}{p_iq_j}\right)^{1/k}P_{i-1}Q_{j-1}\Delta_{11}^{(i,j)}\left(\frac{a_{mnij}}{P_{i-1}Q_{j-1}}\right).$$

In this situation, we deduce from the equations (3.6) and (3.7) that

$$bp-\lim_{s,t\to\infty}(Ax)_{mn}^{[s,t]}=bp-\lim(Gy)_{mn}. \quad (3.8)$$

Thus, one can write that $A=(a_{mnij})\in\left(|\bar{N}_{p,q}|_k,\mathcal{M}_u\right)$ if and only if $G\in(\mathcal{L}_k,\mathcal{M}_u)$, by considering the relation (3.8).

Therefore, using Lemma 2.2 (b), we calculate that

$$\sup_{m,n\in\mathbb{N}}\sum_{i,j}\left|\left(\frac{P_iQ_j}{p_iq_j}\right)^{1/k}P_{i-1}Q_{j-1}\Delta_{11}^{(i,j)}\left(\frac{a_{mnij}}{P_{i-1}Q_{j-1}}\right)\right|^{k'}<\infty,$$

which gives the condition (3.4).

So, we conclude that $A=(a_{mnij})\in\left(|\bar{N}_{p,q}|_k,\mathcal{M}_u\right)$ if and only if the conditions (3.3) and (3.4) are satisfied.

This step completes the proof.

Theorem 3.2. Suppose that $A=(a_{mnij})$ be an arbitrary four dimensional infinite matrix. In that case, the following statements hold:

(a) $A=(a_{mnij})\in\left(|\bar{N}_{p,q}|_1,\mathcal{C}_{bp}\right)$ if and only if (3.1), (3.2) are satisfied, and there exists $(\alpha_{ij}^{(1)})\in\Omega$ such that

$$bp-\lim_{m,n\to\infty}\left(\frac{P_iQ_j}{p_iq_j}\right)P_{i-1}Q_{j-1}\Delta_{11}^{(i,j)}\left(\frac{a_{mnij}}{P_{i-1}Q_{j-1}}\right)=\alpha_{ij}^{(1)}.$$

(b) Let $1<k<\infty$. Then, $A=(a_{mnij})\in\left(|\bar{N}_{p,q}|_k,\mathcal{C}_{bp}\right)$ if and only if (3.3), (3.4) are satisfied, and there exists $(\alpha_{ij}^{(k)})\in\Omega$ such that

$$bp-\lim_{m,n\to\infty}\left(\frac{P_iQ_j}{p_iq_j}\right)^{1/k}P_{i-1}Q_{j-1}\Delta_{11}^{(i,j)}\left(\frac{a_{mnij}}{P_{i-1}Q_{j-1}}\right)=\alpha_{ij}^{(k)}.$$

Proof. This theorem is easily proved by proceeding as in the proof of Theorem 3.1 by using parts (d) and (e) of Lemma 2.2.

Theorem 3.3. Suppose that $A = (a_{mnij})$ be an arbitrary four dimensional infinite matrix. Then, $A \in \left(\left|\bar{N}_{p,q}\right|_1, \mathcal{L}_k\right)$ if and only if (3.1) and

$$\sup_{i,j\in\mathbb{N}} \sum_{m,n} \left| \frac{P_i Q_j}{p_i q_j} P_{i-1} Q_{j-1} \Delta_{11}^{(i,j)} \left(\frac{a_{mnij}}{P_{i-1} Q_{j-1}} \right) \right|^k < \infty$$

holds for $1 \leq k < \infty$.

Proof. This theorem is easily proved by proceeding as in the proof of Theorem 3.1 by using part (c) of Lemma 2.2.

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Author's Contributions

Feride Çalışır: Drafted and wrote the manuscript, performed the experiment and result analysis.

Canan Hazar Güleç: Assisted in analytical analysis on the structure, supervised the experiment's progress, result interpretation and helped in manuscript preparation.

Ethics

There are no ethical issues after the publication of this manuscript.

References

[1]. Demiriz, S., Erdem, S., 2024. Mersenne matrix operator and its application in p−summable sequence space. *Communications in Advanced Mathematical Sciences*; 7(1): 42-55.

[2]. Devletli U., Kara, M.I., 2023. New Banach sequence spaces defined by Jordan totient function. *Communications in Advanced Mathematical Sciences*; 6(4): 211-225.

[3]. Ellidokuzoğlu, H.B., Demiriz, S., Köseoğlu, A., 2018. On the paranormed binomial sequence spaces. *Universal Journal of Mathematics and Applications;* 1(3): 137-147.

[4]. Hazar Güleç, G.C., 2019. Characterization of some classes of compact and matrix operators on the sequence spaces of Cesàro means. *Operators and Matrices*; 13(3): 809-822.

[5]. Hazar Güleç, G.C., 2020. Compact matrix operators on absolute Cesàro spaces. *Numerical Functional Analysis and Optimization*; 41(1): 1-15.

[6]. Hazar Güleç, G.C., Sarıgöl, M.A., 2020. Matrix mappings and norms on the absolute Cesàro and weighted spaces. *Quaestiones Mathematicae*; 43(1): 117-130.

[7]. İlkhan M., Bayrakdar, M.A., 2021. A study on matrix domain of Riesz-Euler totient matrix in the space of p-absolutely summable sequences. *Communications in Advanced Mathematical Sciences*; 4(1): 14-25.

[8]. Polat, H., 2018. Some new Cauchy sequence spaces, *Universal Journal of Mathematics and Applications;* 1(4): 267-272.

[9]. Sezer, S.A., Çanak, İ., 2023. On the convergence of weighted mean summable improper integrals over $R \geq 0$. *The Journal of Analysis;* 31(2): 1029-1039.

[10]. Sezer, S.A., Çanak, İ., 2023. Tauberian theorems concerning weighted mean summable integrals. *Periodica Mathematica Hungarica*; 87: 315-323.

[11]. Önder, Z., Savaş, E., Çanak, İ., 2023. On the weighted generator of double sequences and its Tauberian conditions. *Advances in Operator Theory*; 8(3): 38.

[12]. Başar, F., Savaşcı, M.Y., Double Sequence Spaces and Four-Dimensional Matrices (1st ed.). Chapman and Hall/CRC, 2022.

[13]. Bodur, O., Güleç, C.H., 2024. Characterizations of some new classes of four-dimensional matrices on the double series spaces of first order Cesàro means. *Mathematical Sciences and Applications E-Notes*; 12(4): 196-206.

[14]. Ceylan, A.I., Güleç, C.H., 2024. A new double series space derived by factorable matrix and four-dimensional matrix transformations. *AIMS Mathematics*; 9(11): 30922-30938.

[15]. Sarıgöl, M.A.,2023. Four dimensional matrix mappings on double summable spaces. *Filomat*; 37: 1277-1290.

[16]. Sarıgöl, M. A., 2023. Four dimensional matrix mappings and applications. *Kuwait Journal of Science*; 50(2A): 1-12.

[17]. Bromwich, T. J., An Introduction to the Theory of Infinite Series; Macmillan, New York, 1965.

[18]. Zeltser, M, 2022. On conservative matrix methods for double sequence spaces. *Acta Mathematica Hungarica*; 95(3): 225-242.

[19]. Hardy, G. H., 1917. On the convergence of certain multiple series. *Proceedings of the Cambridge Philosophical Society*; 19: 86-95.

[20]. Móricz, F., 1991. Extensions of the spaces c and c_0 from single to double sequences. *Acta Mathematica Hungarica*; 57, (1-2): 129-136.

[21]. Móricz, F., Rhoades, B.E., 1988. Almost convergence of double sequences and strong regularity of summability matrices. *Mathematical Proceedings of the Cambridge Philosophical Society*; 104 (2): 283-294.

[22]. Mursaleen, M., 2004. Almost strongly regular matrices and a core theorem for double sequences. *Journal of Mathematical Analysis and Applications*; 293(2): 523–531.

[23]. Mursaleen, M., Başar, F., 2014. Domain of Cesàro mean of order one in some spaces of double sequences. *Studia Scientiarum Mathematicarum Hungarica*; 51: 335-356.

[24]. Demiriz, S., Duyar, O., 2015. Domain of difference matrix of order one in some spaces of double sequences. *Gulf Journal of Mathematics*; 3(3): 85-100.

[25]. Demiriz, S., Erdem, S., 2020. Domain of Euler-totient matrix operator in the space $\mathcal{L}_p$. *Korean Journal of Mathematics*; 28(2): 361-378.

[26]. Pringsheim, A., 1900. Zur theorie der zweifach unendlichen Zahlenfolgen. *Mathematische Annalen*; 53: 289-321.

[27]. Boos, J., Classical and Modern Methods in Summability; Oxford University Press, New York, 2000.

[28]. Başar, F., Sever, Y., 2009. The space $\mathcal{L}_k$ of double sequences. *Mathematical Journal of Okayama University*; 51: 149–157.

[29]. Yeşilkayagil, M., Başar, F., 2018. Domain of Riesz mean in some spaces of double sequences. *Indagationes Mathematicae*; 29(3): 1009-1029.

[30]. Alotaibi, A.M., Çakan, C., 2012. The Riesz convergence and Riesz core of double sequences. *Journal of Inequalities and Applications*; 2012, 56.

[31]. Sarıgöl, M.A., 2021. On equivalence of absolute double weighted mean methods. *Quaestiones Mathematicae*; 44(6): 755-764.

[32]. Altay, B., Başar, F., 2006. Some paranormed Riesz sequence spaces of non-absolute type. *Southeast Asian Bulletin of Mathematics*; 30(4): 591-608.

[33]. Başarır, M., Kara, E.E., 2011. On compact operators on the Riesz Bm-difference sequence space. *Iranian Journal of Science and Technology (Sciences);* 35(4): 279-285.

[34]. Sarıgöl, M.A., 2016. Norms and compactness of operators on absolute weighted mean summable series. *Kuwait Journal of Science*; 43(4): 68-74.

[35]. Yeşilkayagil, M., Başar, F., 2017. On the domain of Riesz mean in the space $\mathcal{L}_k^*$. *Filomat*; 31(4): 925-940.