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POLAR REPRESENTATION OF COMPLEX OCTONIONS

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ABSTRACT: The complex octonions are a non-associative extension of complex quaternions, are used in areas such as quantum physics, classical electrodynamics, the representations of robotic systems, kinematics *etc.* (Kansu *et al.*, 2012, James *et al.*, 1978). In this paper, we study the complex octonions and their basic properties. We generalize in a natural way De-Moivre's and Euler's formulae for division complex octonions algebra.

Key Words: *De-Moiver's formula, Euler's formula, Complex octonion*

Kompleks Oktoniolarin Kutupsal Gösterimi

ÖZ: Kompleks oktonyonlar, kompleks kuaterniyonların birleşimli olmayan ve kuantum fiziği, klasik elektrodinamik, robotik sistemlerin gösterimleri, kinematik (Kansu et al., 2012, James et al., 1978) gibi alanlarda kullanılan bir uzantısıdır. Bu makalede, kompleks oktonyonlar ve temel özelliklerini çalıştık. De-Moivre ve Euler formüllerini Kompleks oktonyonlar cebiri için tabii bir şekilde genelleştirdik.

Anahtar Kelimeler: *De Moivre's formülü, Euler's fromülü, Kompleks oktonyonlar.*

INTRODUCTION

The octonions are the largest of the four normed division algebras. While somewhat neglected due to their non-associativity, they stand at the crossroads of many interesting fields of mathematics (Baez, 2002). A study of the classical electromagnetism's energy described by the complex octonions in sixteen dimensions is given by Kansu et al. (Kansu *et al.*, 2012). The complex exponential $e^{i\theta} = \cos\theta + i\sin\theta$ generalizes to quaternions by replacing i by any unit quaternion μ since any unit pure quaternion is a root of -1. Hence, any quaternion may be represented in the polar form $q = |q|\, e^{\mu\theta}$ where θ is a real angle. As with complex numbers and quaternions, any octonion can be written in polar form as $x = r(\cos\varphi + \vec{w}\sin\varphi)$ where $r = \sqrt{N_x}$ and $\vec{w}^2 = -1$. In this paper, we introduce the complex octonions algebra, O_C, and study some fundamental algebraic properties of them. The polar representation of complex octonions are given, and then by means of the De-Moivre's theorem, any powers of these octonions are obtained. Finally, we give some examples for more clarification.

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MATERIAL AND METHOD

A complex octonion X has an expression of the form

$$X = A_0\,e_0 + A_1\,e_1 + A_2\,e_2 + A_3\,e_3 + A_4\,e_4 + A_5\,e_5 + A_6\,e_6 + A_7\,e_7 = A_0\,e_0 + \sum_{i=1}^{7} A_i\,e_i, \tag{1}$$

where $A_0 - A_7$ are complex numbers and $e_i, (0 \le i \le 7)$ are octonionic units satisfying the equalities that are given in the table below;

Table 1. Octonionic units

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

As a consequence of this definition, a complex octonion X can be written as

$$X = x + i\,x', \tag{2}$$

where x and x', real and pure octonion components, respectively. The set of all complex octonions is denoted by O_C.

For defined octonion in equation (1), the scalar and vectorial parts can be given, respectively, as

$$S_X = A_0\,e_0, \tag{3}$$

$$\vec{V}_X = A_1\,e_1 + A_2\,e_2 + A_3\,e_3 + A_4\,e_4 + A_5\,e_5 + A_6\,e_6 + A_7\,e_7. \tag{4}$$

A complex octonion X can also be written as

$$X = (A_0\,e_0 + A_1\,e_1 + A_2\,e_2 + A_3\,e_3) + (A_4 + A_5\,e_1 + A_6\,e_2 + A_7\,e_3)e_4 = Q + Q'e, \tag{5}$$

where $e^2 = -1$ and

$$Q, Q' \in \mathrm{H}_C = \left\{ Q = A_0 + A_1 e_1 + A_2 e_2 + A_3 e_3 \;\middle|\; e_1^{\,2} = e_2^{\,2} = e_3^{\,2} = -1\,, A_i \in C \right\}, \tag{6}$$

the complex quaternion division algebra (Jafari, 2016).

For two complex octonions $X=\sum_{i=0}^{7} A_i\, e_i$ and $Y=\sum_{i=0}^{7} B_i\, e_i$, the summation and substraction processes are given as

$$X \pm Y=\sum_{i=0}^{7}(A_i \pm B_i)\, e_i. \tag{7}$$

Addition and subtraction of complex octonions is done by adding and subtracting corresponding terms and hence their coefficients, like quaternions.

The product of two complex octonions $X=S_X+\vec{V}_X$, $Y=S_Y+\vec{V}_Y$ is expressed as

$$X\,Y=S_X\, S_Y-\left\langle \vec{V}_X, \vec{V}_Y\right\rangle+S_X\, \vec{V}_Y+S_Y\, \vec{V}_X+\vec{V}_X \times \vec{V}_Y \tag{8}$$

Multiplication is distributive over addition, so the product of two octonions can be calculated by summing the product of all the terms, again like quaternions. This product can be described by a matrix-vector product as

$$X.Y=\begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 & -A_4 & -A_5 & -A_6 & -A_7 \\ A_1 & A_0 & -A_3 & A_2 & -A_5 & A_4 & A_7 & -A_6 \\ A_2 & A_3 & A_0 & -A_1 & -A_6 & -A_7 & A_4 & A_5 \\ A_3 & -A_2 & A_1 & A_0 & -A_7 & A_6 & -A_5 & A_4 \\ A_4 & A_5 & A_6 & A_7 & A_0 & -A_1 & -A_2 & -A_3 \\ A_5 & -A_4 & A_7 & -A_6 & A_1 & A_0 & A_3 & -A_2 \\ A_6 & -A_7 & -A_4 & A_5 & A_2 & -A_3 & A_0 & A_1 \\ A_7 & A_6 & -A_5 & -A_4 & A_3 & A_2 & -A_1 & A_0 \end{bmatrix}\begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \end{bmatrix}, \tag{9}$$

where $X, Y \in O_C$. Complex octonions multiplication is not associative, since

$$\begin{aligned} e_1(e_2\, e_4) &= e_1 e_6=-e_7, \\ (e_1 e_2)\, e_4 &= e_3 e_4=e_7. \end{aligned} \tag{10}$$

It is clear that subalgebra with bases $e_0, e_1, e_i, e_j\ (2 \leq i, j \leq 7)$ is isomorphic to complex quaternions algebera H_C.

SOME PROPERTIES OF COMPLEX OCTONIONS

1) The Hamilton *conjugate* of $X=\sum_{i=0}^{7} A_i\, e_i=S_X+\vec{V}_X$ is

$$\bar{X}=A_0\, e_0-\sum_{i=1}^{7} A_i\, e_i=S_X-\vec{V}_X. \tag{11}$$

The complex *conjugate* of X is

$$X^*=\bar{A}_0\, e_0+\sum_{i=1}^{7} \bar{A}_i\, e_i=(a_0-i a_0^{'}) e_0+(a_1-i a_1^{'}) e_1+\ldots+(a_7-i a_7^{'}) e_7. \tag{12}$$

The Hermitian *conjugate* of X is

$$X^{\dagger}=(\bar{X})^*=\bar{A}_0\, e_0-\sum_{i=1}^{7} \bar{A}_i\, e_i=(a_0-i a_0^{'}) e_0-(a_1-i a_1^{'}) e_1-\ldots-(a_7-i a_7^{'}) e_7. \tag{13}$$

It is clear the scalar and vector parts of X is denoted by $S_X = \frac{X+\bar{X}}{2}$ and $\vec{V}_X = \frac{X-\bar{X}}{2}$.

2) The *norm* of X is

$$N_X = X\bar{X} = \bar{X}X = \|X\|^2 = \sum_{i=0}^{7} A_i^2 \in C \tag{14}$$

If $N_X = 1,$ then X is called a unit complex octonion. We will use O_C^1 to denote the set of unit complex octonions. If $N_X = 0,$ then X is called a null complex octonion.

Lemma 1. Let $X, Y \in O_C$. The conjugate and norm of complex octonions satisfy the following properties:

$$\begin{aligned} &1)\ \bar{\bar{X}} = X,\ (X^*)^* = X,\ (X^\dagger)^\dagger = X \\ &2)\ \overline{XY} = \bar{Y}\,\bar{X},\ (XY)^* = Y^* X^*,\ (XY)^\dagger = Y^\dagger X^\dagger \\ &3)\ \overline{X+Y} = \bar{X}+\bar{Y},\ (X+Y)^* = X^*+Y^*,\ (X+Y)^\dagger = X^\dagger + Y^\dagger \\ &4)\ N_X = N_{\bar{X}},\ N_{XY} = N_X N_Y \end{aligned} \tag{15}$$

3) The *inverse* of X with $N_X \neq 0,$ is

$$X^{-1} = \frac{1}{N_X}\bar{X}. \tag{16}$$

Example 1. Consider the complex octonions

$$\begin{aligned} X_1 &= \frac{\sqrt{3}}{2} + 2e_1 + (1-i)e_2 + 2i\,e_3 + (1+i)\,e_4 + i\,e_5 + \frac{1}{2}e_6 - e_7, \\ X_2 &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}e_1 + (1-i)e_2 + 2i\,e_3 + (1+i)\,e_4 + \sqrt{2}i\,e_5 + e_6 + 2e_7, \\ X_3 &= 1 + (2-i)\,e_1 + ie_2 + e_3 + (1+i)\,e_4 + i\,e_5 + e_6 - e_7, \end{aligned} \tag{17}$$

The norms of X_1, X_2, X_3 are

$$N_{X_1} = 1,\ N_{X_2} = 0,\ N_{X_3} = 5-2i. \tag{18}$$

The conjugates of X_1, X_2, X_3 are

$$\begin{aligned} \bar{X}_1 &= \frac{\sqrt{3}}{2} - 2e_1 - (1-i)e_2 - 2i\,e_3 - (1+i)\,e_4 - i\,e_5 - \frac{1}{2}e_6 + e_7, \\ X_2^* &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}e_1 + (1+i)e_2 - 2i\,e_3 + (1-i)\,e_4 - \sqrt{2}i\,e_5 + e_6 + 2e_7, \\ X_3^\dagger &= 1 - (2+i)\,e_1 + ie_2 + e_3 - (1-i)\,e_4 + i\,e_5 + e_6 - e_7, \end{aligned} \tag{19}$$

The inverse of X_1, X_3 are

$$\begin{aligned} X_1^{-1} &= \frac{\sqrt{3}}{2} - 2e_1 - (1-i)e_2 - 2i\,e_3 - (1+i)\,e_4 - i\,e_5 - \frac{1}{2}e_6 + e_7, \\ X_3 &= \frac{1}{5-2i}\,[1 - (2-i)\,e_1 - ie_2 - e_3 - (1+i)\,e_4 - i\,e_5 - e_6 + e_7], \end{aligned} \tag{20}$$

and X_2 not invertible.

Theorem 1. The set O_C^1 of unit complex octonions is a subgroup of the group O_C^0 where $O_C^0 = O_C - [0 - \vec{0}]$.

Proof: Let $X, Y \in O_C^1$. We have $N_{XY} = 1$, *i.e.* $XY \in O_C^1$ and thus the first subgroup requirement is satisfied. Also, by the property

$$N_X = N_{\bar{X}} = N_{X^{-1}} = 1, \tag{21}$$

the second subgroup requirement $X^{-1} \in O_C^1$.

RESULT AND DISCUSSION

Trigonometric Form and De Moivre's Theorem

Every non-null complex octonion $X = \sum_{i=0}^{7} A_i e_i$ can be written in the trigonometric (polar) form

$$X = R(\cos\phi + \vec{W}\sin\phi), \tag{22}$$

with

$R = \sqrt{|N_X|} = \sqrt{\left|\sum_{i=0}^{7} A_i^2\right|}$, $\cos\phi = \dfrac{A_0}{\sqrt{|N_X|}}$ and $\sin\phi = \dfrac{\left(\sum_{i=1}^{7} A_i^2\right)^{1/2}}{\sqrt{|N_X|}}$. The unit complex vector $\vec{W} = \vec{w} + i\,\vec{w}^*$ is given by

$$\vec{W} = (w_1, w_2, ..., w_7) = \frac{1}{(\sum_{i=1}^{7} A_i^2)^{1/2}}(A_1, A_2, ..., A_7). \tag{23}$$

Example 2. The polar form of the complex octonions $X_1 = \dfrac{1}{\sqrt{2}} + (i, 1+i, 2i, 1-i, 2, 0, \sqrt{\dfrac{3}{2}})$ is

$$X_1 = \cos\frac{\pi}{4} + \vec{W}_1 \sin\frac{\pi}{4} \tag{24}$$

and $X_2 = i + (1+2i, -i+1, 2-i, -1, 2i, i+1, \sqrt{5})$ is

$$X_2 = \cos\phi + \vec{W}_2 \sin\phi,\ \ \phi = \cos^{-1} i = \frac{\pi}{2} - i\ln(-1+\sqrt{2}), \tag{25}$$

where

$$\vec{W}_1 = \sqrt{2}\,(i, 1+i, 2i, 1-i, 2, 0, \sqrt{\frac{3}{2}}) \text{ and } \vec{W}_2 = \frac{1}{\sqrt{2}}(1+2i, -i+1, 2-i, -1, 2i, i+1, \sqrt{5}). \tag{26}$$

It is clear that $N_{\vec{W}_1} = N_{\vec{W}_2} = 1$ and $\vec{W}_1\vec{W}_1 = \vec{W}_2\vec{W}_2 = -1$.

Since $\vec{W}^2 = -1$ we have a natural generalization of Euler's formula for generalized quaternions

$$\begin{aligned} e^{\vec{W}\theta} &= 1 + \vec{W}\theta - \frac{\theta^2}{2!} - \vec{W}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - ... \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ... + \vec{W}(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...) \\ &= \cos\theta + \vec{W}\sin\theta, \end{aligned} \tag{27}$$

for any dual number θ.

Lemma 2. For every unit vector $\vec{W}$, we have

$$\left(\cos\theta_1+\vec{W}\sin\theta_1\right)\left(\cos\theta_2+\vec{W}\sin\theta_2\right)=\cos\left(\theta_1+\theta_2\right)+\vec{W}\sin\left(\theta_1+\theta_2\right). \tag{28}$$

Theorem 2. (De-Moivre's formula) Let $X=\sqrt{N_x}(\cos\phi+\vec{W}\sin\phi)$ be a complex octonions. Then for any integer n;

$$X^n=(\sqrt{N_x})^n.(\cos n\phi+\vec{W}\sin n\phi) \tag{29}$$

Proof: The proof will be by induction on nonnegative integers n and let $N_X=1$.

For $n=2$ and on using the validity of theorem as lemma 1, one can show

$$(\cos\phi+\vec{W}\sin\phi)^2=\cos 2\phi+\vec{W}\sin 2\phi \tag{30}$$

Suppose that $(\cos\phi+\vec{W}\sin\phi)^n=\cos n\phi+\vec{W}\sin n\phi,$ we aim to show

$$(\cos\phi+\vec{W}\sin\phi)^{n+1}=\cos(n+1)\phi+\vec{W}\sin(n+1)\phi. \tag{31}$$

Thus

$$\begin{aligned}(\cos\phi+\vec{W}\sin\phi)^{n+1}&=(\cos\phi+\vec{W}\sin\phi)^n(\cos\phi+\vec{W}\sin\phi)\\&=(\cos n\theta+\vec{W}\sin n\phi)(\cos\phi+\vec{W}\sin\phi)\\&=\cos(n\phi+\phi)+\vec{W}\sin(n\phi+\phi)\\&=\cos(n+1)\phi+\vec{W}\sin(n+1)\phi.\end{aligned} \tag{32}$$

The formula holds for all integers n;

$$X^{-1}=\cos\phi-\vec{W}\sin\phi, \tag{33}$$

$$\begin{aligned}X^{-n}&=\cos(-n\phi)+\vec{W}\sin(-n\phi)\\&=\cos n\phi-\vec{W}\sin n\phi.\end{aligned} \tag{34}$$

■

Example 3. Let $X=-\sqrt{3}+(1+i,\ i,\ 2i,1-i,1,\ 1,\ 2)$ be a complex octonion. Every power of this octonion is found with the aid of Theorem 1. For example, *20-th* and *83-th* powers are

$$\begin{aligned}X^{20}&=2^{20}(\cos 20\frac{5\pi}{6}+\vec{W}\sin 20\frac{5\pi}{6})\\&=2^{20}(-\frac{1}{2}+\vec{W}\frac{\sqrt{3}}{2})=2^{19}[-1+\sqrt{3}(1+i,\ i,\ 2i,1-i,1,\ 1,\ 2)],\end{aligned} \tag{35}$$

and

$$\begin{aligned}x^{83}&=2^{83}(\cos 83\frac{5\pi}{6}+\vec{W}\sin 83\frac{5\pi}{6})\\&=-2^{82}(\sqrt{3}+\vec{W}).\end{aligned} \tag{36}$$

We investigate some properties of the complex octonions by separating them in two cases:

i) Complex octonions with complex angles ($\phi=\varphi+i\varphi*$); *i.e.*

$$X = \sqrt{|N_X|}\,(\cos\phi + \vec{W}\sin\phi), \tag{37}$$

ii) Complex octonions with real angles (ϕ=φ, φ*=0); *i.e.*

$$X = \sqrt{|N_X|}\,(\cos\varphi + \vec{W}\sin\varphi). \tag{38}$$

Theorem 3. De Moivre's formula implies that there are uncountably many unit complex octonion $X = \cos\varphi + \vec{W}\sin\varphi$ satisfying $X^n = 1$ for $n \geq 3$.

Proof: For every unit vector $\vec{W}$, the unit complex octonion

$$X = \cos\frac{2\pi}{n} + \vec{W}\sin\frac{2\pi}{n}, \tag{39}$$

is of order n. For $n = 1$ or $n = 2$, the complex octonion X is independent of $\vec{W}$. ■

Example 4. $X = \frac{1}{\sqrt{2}} + (i, 1+i, 2i, 1-i, 2, 0, \sqrt{\frac{3}{2}})$ is of order 8 and $X = \frac{\sqrt{3}}{2} + (1+i, i, 2i, 1-i, -1, \frac{1}{2}, 2)$ is of order 12.

Theorem 4. Let $X = \cos\varphi + \vec{W}\sin\varphi$ be a unit complex octonion. The equation $A^n = X$ has n roots, and they are

$$A_k = \cos(\frac{\varphi + 2k\pi}{n}) + \vec{W}\sin(\frac{\varphi + 2k\pi}{n}), \qquad k = 0,1,2,\ldots,n-1. \tag{40}$$

Proof: We assume that $A = \cos\vartheta + \vec{W}\sin\vartheta$ is a root of the equation $A^n = X$, since the vector parts of X and A are the same. From Theorem 2, we have

$$A^n = \cos n\vartheta + \vec{W}\sin n\vartheta, \tag{41}$$

thus, we find

$$\cos n\vartheta = \cos\varphi, \qquad \sin n\vartheta = \sin\varphi,$$

So, the n roots of X are

$$A_k = \cos(\frac{\varphi + 2k\pi}{n}) + \vec{W}\sin(\frac{\varphi + 2k\pi}{n}), \quad k = 0,1,2,\ldots,n-1. \tag{42}$$

■

Example 5. Let $X = \frac{\sqrt{3}}{2} + (1+2i, -\sqrt{3}, 2i, i-1, \frac{1}{2}, i-1, 2) = \cos\frac{\pi}{6} + \vec{W}\sin\frac{\pi}{6}$ be a unit complex octonion.

The cube roots of the octonion X are

$$X_k^{1/3} = \cos(\frac{\pi/6 + 2k\pi}{3}) + \vec{w}\sin(\frac{\pi/6 + 2k\pi}{3}), \qquad k = 0,1,2. \tag{43}$$

For $k=0$, the first root is $X_0^{1/3}=\cos\frac{\pi}{18}+\vec{W}\sin\frac{\pi}{18}=0.98+0.17\vec{W}$, and the second one for $k=1$ is $X_1^{1/3}=\cos\frac{13\pi}{18}+\vec{W}\sin\frac{13\pi}{18}=-0.64+0.76\,\vec{W}$ and third one is

$$X_2^{1/3}=\cos\frac{25\pi}{18}+\vec{W}\sin\frac{25\pi}{18}=-0.34-0.93\vec{W}. \tag{44}$$

Also, it is easy to see that $X_0^{\frac{1}{3}}+X_1^{\frac{1}{3}}+X_2^{\frac{1}{3}}=0.$

The relation between the powers of complex octonions can be found in the following Theorem.

Theorem 5. Let X be a unit complex octonion with the polar form $X=\cos\varphi+\vec{W}\sin\varphi.$

$$\text{If } m=\frac{2\pi}{\varphi}\in \mathrm{Z}^{+}-\{1\}, \text{ then } X^{n}=X^{m} \text{ if and only if } n\equiv m \ (\mathrm{mod}\ p). \tag{45}$$

Proof: Let $n\equiv m$ (mod p). Then we have $n=a.p+m,$ where $a\in \mathrm{Z}.$

$$\begin{aligned}X^{n}&=\cos n\varphi+\vec{W}\sin n\varphi\\ &=\cos(ap+m)\varphi+\vec{W}\sin(ap+m)\varphi\\ &=\cos(a\frac{2\pi}{\varphi}+m)\varphi+\vec{W}\sin(a\frac{2\pi}{\varphi}+m)\varphi\\ &=\cos(m\varphi+a2\pi)+\vec{W}\sin(m\varphi+a2\pi)\\ &=\cos m\varphi+\vec{W}\sin m\varphi\\ &=X^{m}.\end{aligned} \tag{46}$$

Now suppose

$$X^{n}=\cos n\varphi+\vec{W}\sin n\varphi \text{ and } X^{m}=\cos m\varphi+\vec{W}\sin m\varphi. \tag{47}$$

If $X^{n}=X^{m}$ then we get $\cos n\varphi=\cos m\varphi$ and $\sin n\varphi=\sin m\varphi,$ which means

$$n\varphi=m\varphi+2\pi a,\ a\in \mathrm{Z}. \tag{48}$$

Thus $n=m+\frac{2\pi}{\varphi}a$ or $n\equiv m(\mathrm{mod}\ p).$ (49)

■

Example 6. Let $X=\frac{\sqrt{2}}{2}+(1+2i,-\sqrt{3},\,2i,\,i-1,\frac{1}{\sqrt{2}},i-1,2)$ be a unit complex octonion. From Theorem 5, $m=\frac{2\pi}{\pi/4}=8,$ so we have

$$
\begin{aligned}
&X = X^{9} = X^{17} = \ldots \\
&X^{2} = X^{10} = X^{18} = \ldots \\
&X^{3} = X^{11} = X^{19} = \ldots \\
&X^{4} = X^{12} = X^{20} = \ldots = -1 \\
&\vdots \\
&X^{8} = X^{16} = X^{24} = \ldots = 1.
\end{aligned} \tag{50}
$$

CONCLUSION

In this paper, we defined and gave some of algebraic properties of complex octonions and investigated the De Moivre's formulas for these octonions. The relation between the powers of complex octonions is given in Theorem 5. We also showed that the equation $X^n = 1$ has uncountably many solutions for any unit complex octonions (Theorem 3).

FUTHER WORK

We will give a complete investigation to real matrix representations of complex octonions, and give any powers of these matrices.

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