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A New Goodness of Fit Test for Complete or Type II Right Censored Samples

Anıl KOYUNCU[1], Mehmet KARAHASAN*[2]

[1] Milli Eğitim Bakanlığı, Muğla Gazi Anadolu Lisesi, 48000, Muğla, Türkiye
[2*]Muğla Sıtkı Koçman Üniversitesi, Fen Fakültesi, İstatistik Bölümü, 48170, Muğla, Türkiye

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Abstract: This study proposes a new goodness-of-fit test based on the empirical distribution function for complete or type II right-censored random samples, which are drawn from either the exponential or log-normal distributions. Some simulation studies were conducted to compare the newly proposed test with some of the well-known goodness-of-fit tests, such as Kolmogorov-Smirnov, Cramer-von Mises, and Anderson-Darling, in terms of power over various sample sizes and censoring rates. The simulation results show that the newly proposed goodness of fit test generally seems to perform well compared to the other goodness of fit tests considered. In addition, the newly proposed test and the other goodness of fit tests are illustrated by applying them to some real data sets obtained from the relevant literature.

Tam ya da II. Tür Sağdan Durdurulmuş Örneklemler için Yeni Bir Uyum İyiliği Testi

Anahtar Kelimeler
Uyum iyiliği testleri,
Deneysel dağılım fonksiyonu,
II. tür sağdan durdurma
Ölçek ailesi,
Konum-ölçek ailesi

Özet: Bu çalışmada, üstel veya log-normal dağılımlardan alınan tam veya II. tür sağdan durdurulmuş rastgele örneklemler için deneysel dağılım fonksiyonuna dayalı yeni bir uyum iyiliği testi önerilmektedir. Yeni önerilen uyum iyiliği testi Kolmogorov-Smirnov, Cramer-von Mises ve Anderson-Darling gibi iyi bilinen bazı uyum iyiliği testleri ile çeşitli örneklem büyüklükleri ve durdurma oranları üzerinde güç açısından karşılaştırmak için bazı simülasyon çalışmaları yapılmıştır. Simülasyon sonuçları, yeni önerilen uyum iyiliği testinin, dikkate alınan diğer uyum iyiliği testlerine kıyasla genel olarak iyi performans gösterdiğini ortaya koymaktadır. Ayrıca, yeni önerilen uyum iyiliği testi ve diğer uyum iyiliği testleri, ilgili literatürden elde edilen bazı gerçek veri setlerine uygulanarak gösterilmiştir.

1. Introduction

For several reasons, such as the physical constraints and cost, data in reliability and life testing studies are generally censored. Consequently, statistical analyses are usually performed on censored data. In particular, it is important to the fit of a family of probability distributions to data in parametric analyses when the data are censored. The problem can be expressed as follows. Let X denote the random variable with the distribution function F and $X_1, X_2, \dots, X_n$ a random sample from the distribution and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n}$ the corresponding type II right censored random sample with censoring rate $p = r/n$. Then the hypothesis of interest is as follows:

$$H_0\colon F \in \{F_\theta\} \text{ versus } H_A\colon F \notin \{F_\theta\}, \qquad (1)$$

where $\{F_\theta\}$ is a parametric family of distributions indexed by $\theta \in \Theta \subseteq \mathbb{R}^k$. To deal with the problem, or its simpler version claiming that the data come from a fully specified distribution, several goodness-of-fit tests have been proposed. One of the most popular approaches is based on empirical distribution functions, i.e. EDF-based goodness-of-fit tests. Kolmogorov-Smirnov by [1] and [2], Cramer-von Mises by [3] and [4], and Anderson and Darling [5] are some of the well-known examples of this type of goodness-fit tests. Kuiper and Watson tests by [6] and [7] are other examples of EDF based goodness-of-fit tests. These tests essentially measure some kind of discrepancy between the empirical distribution function F_n and the theoretical distribution function F_θ [8]. While some tests such as Kolmogorov-Smirnov consider absolute distance between the two functions, some tests such as Cramer-von Mises consider the square distance.

Goodness of fit tests based on EDF statistics have been discussed for complete and censored samples in various works in the literature. Some of the works include: [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], and [18].

* Corresponding Author: mkarahas@mu.edu.tr

Since goodness-of-fit tests is a commonly used statistical tool in scientific research in many disciplines, the effort to improve existing goodness-of-fit tests or develop new ones continues. Some of the works in this direction are as follows: [19], [20], [21], [22], [23], [24], [25], and [26].

In this study, a new goodness-of-fit test, which we will call W_{AKM}, is introduced in the cases of uncensored and Type II right censored samples coming from either the exponential or log-normal distribution. Then, the power performance of the newly proposed test compared to some of the EDF type goodness-of-fit tests is investigated through simulation studies for some scale or location-scale families. The goodness-of-fit tests considered are the following: Kolmogorov-Smirnov, Kuiper, Cramer-von Mises, Watson and Anderson-Darling tests.

The rest of the paper is organized as follows. In Section 2, the new goodness-of-fit test is first introduced. Then, in the same section, the settings of the simulation studies to be performed are explained before describing some real data sets used to illustrate the new goodness-of-fit test as well as the other tests considered. Section 3 presents the results of the comparisons from the simulation studies as well as the results obtained from applying the methods to the data sets. Finally, some discussion and concluding remarks are given in Section 4.

2. Material and Method

In this section, a new goodness-of-fit test is proposed for the cases of both complete and Type II right censored samples from the exponential distribution, a scale family, or the log-normal distribution, a log location-scale family. Let $X_1, X_2, \dots, X_n$ be a random sample from such a distribution F and $X_{1:n} \le X_{2:n} \le \dots \le X_{r:n}$ the corresponding type II right censored sample.

The parameters of the distribution F are to be estimated from the sample. Since the parameters are unknown, the sample $U_1, U_2, \dots, U_n$ from the uniform (0,1) distribution or the corresponding type II censored sample $U_{1:n} \le U_{2:n} \le \dots \le U_{r:n}$ cannot be obtained by using the probability integral transformation $U_i = F(X_i)$. However, by replacing the parameters with their estimates in F, i.e. $U_i^* = F_{\hat{\theta}}(X_i)$, the sample $U_1^*, U_2^*, \dots, U_n^*$ and the corresponding type II censored sample $U_{1:n}^* \le U_{2:n}^* \le \dots \le U_{r:n}^*$ are formed. The sample $U_1^*, U_2^*, \dots, U_n^*$ will not be a sample from the uniform distribution [8]. In such a case, the critical values of EDF test statistics will generally depend on the family of distributions, the parameters, the method of estimation and the sample size [8]. However, for scale or location-scale families of distributions, the critical values do not depend on the true values of the parameters but the sample size n and the specific family of distributions.

The well-known Cramer-von Mises test is based on the quadratic distance between the empirical distribution function and the theoretical distribution, giving equal weight to all parts of the distribution as in (2).

$$W^2 = n\int_{-\infty}^{\infty}\{F_n(x) - F(x)\}^2 dF(x) \qquad (2)$$

On the other hand, the Anderson-Darling test, another well-known test based on the square distance, emphasizes the tails of the distribution by using the weights as in (3).

$$A^2 = n\int_{-\infty}^{\infty}\{F_n(x) - F(x)\}^2 \left\{\frac{1}{F(x)[1-F(x)]}\right\} dF(x) \qquad (3)$$

Unlike the two tests, the Watson test uses the following type of difference between $F_n(x)$ and $F(x)$ as in (4),

$$U^2 = n\int_{-\infty}^{\infty}\Big\{F_n(x) - F(x) - \int_{-\infty}^{\infty}[F_n(x)F(x)]dF(x)\Big\}^2 dF(x) \qquad (4)$$

The idea behind the new goodness-of-fit test can be explained as follows. Unlike the three tests, this paper proposes a goodness-of-fit test that takes into account both the quadratic and absolute differences between the empirical distribution and the theoretical distribution functions. The explicit formula for the new test W_{AKM} is expressed in (5):

$$W_{AKM} = \Big|n\Big\{\int_{-\infty}^{\infty}[F_n(x) - F(x)]^2 dF(x) - \int_{-\infty}^{\infty}[F_n(x) - F(x)]dF(x)\Big\}\Big| \qquad (5)$$

In addition, the proposed test is similar to the Watson test; thus, it can be considered as another modification of the Cramer von-Mises test.

Table A1. and Table A2. in Appendix show the simulated critical values of some of the EDF tests for the exponential and log-normal distribution families, respectively, over various sample sizes and Type II censoring rates for $\alpha = 0.05$ over 100000 replicates.

In the following subsections, when introducing the new goodness of fit test, the notation $U_1, U_2, \dots, U_n$ and $U_{1:n} \le U_{2:n} \le \dots \le U_{r:n}$ are used for complete and type II censored samples, respectively, instead of $U_1^*, U_2^*, \dots, U_n^*$ and $U_{1:n}^* \le U_{2:n}^* \le \dots \le U_{r:n}^*$ for simplicity.

2.1. Type II right censored case

Since the expression of the new test statistic is very similar to that of the Watson test statistic, the computational formula for the new test is easily derived from that of the Watson test. Recall the formula for the Watson test statistic for the Type II right-censored data,

$$U_{r,n}^2 = W_{r,n}^2 - nU_{r:n}\left[\frac{r}{n} - \frac{U_{r:n}}{2} - \frac{r\bar{U}}{nU_{r:n}}\right]^2 \qquad (6)$$

where $\bar{U} = \sum_{i=1}^{r}\frac{U_{i:n}}{r}$ and $W_{r,n}^2$ denotes Cramer-von Mises goodness-of-fit test statistic for the Type II right-censored data. By making the necessary changes in (6), the modified statistic W_{AKM} is obtained as in (7).

$$W_{AKM} = \left| W_{r,n}^2 - n\sqrt{U_{r:n}}\left(\frac{r}{n} - \frac{U_{r:n}}{2} - \frac{r\bar{U}}{nU_{r:n}}\right)\right| \qquad (7)$$

where $\bar{U}$ is defined as in (6).

2.2. Complete sample case

The new test for the full sample case is again obtained by modifying the original Watson test for the case of Type II censored case. In other words, a complete sample is considered as if it were a Type II censored sample and then the statistic in (8) is proposed as a new modification of the Watson statistic:

$$W_{AKM} = \left| W_{n,n}^2 - n\sqrt{U_{n:n}}\left(1 - \frac{U_{n:n}}{2} - \frac{\bar{U}}{U_{n:n}}\right)\right| \qquad (8)$$

where $\bar{U} = \sum_{i=1}^{n}\frac{U_{i:n}}{n}$ and $W_{n,n}^2$ is the statistic of Cramer-von Mises test for the Type II right-censored data with r replaced by n:

$$W_{n,n}^2 = \sum_{i=1}^{n}\left(U_{i:n} - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n} + \frac{n}{3}(U_{n:n} - 1)^3 \qquad (9)$$

2.3. Settings of simulation study

The power performance of some of the well-known goodness-of-fit tests based on EDF are compared with that of the goodness-of-fit test proposed in this study through simulations.

The setting of the simulations can be explained as follows. First, the hypothesis testing problem to be considered is that the random sample is drawn from a specified family of location-scale distributions, as indicated below.

H_0: The relevant random sample $X_1, X_2, \dots, X_n$ or the Type II right censored sample $X_{1:n}, X_{2:n}, \dots, X_{r:n}$ comes from the specified family of scale or location-scale distributions.

H_A: The relevant random sample $X_1, X_2, \dots, X_n$ or the Type II right censored sample $X_{1:n}, X_{2:n}, \cdots, X_{r:n}$ comes from another specified family of distributions.

The samples of size $n = 10, 20, 30, 40, 50$ *and* 100 were generated as either complete or Type II right censored with censoring rates $p = \frac{r}{n} = 0.20, 0.40, 0.60, 0.80,$ *and* 1.00 from the hypothesized distributions in H_A. Then, the pseudo-random sample is considered as if it came from the distribution specified in H_0 and the parameters, in turn, the $U_i = F(X_i)$ is estimated to obtain the "uniform" sample $U_1, U_2, \dots, U_n$ or the corresponding Type II right-censored data $U_{1:n}, U_{2:n}, \dots, U_{r:n}$.

Finally, the relevant statistics for the goodness-of-fit tests are computed and the decision is made to reject or not reject the null hypothesis H_0. The simulated power of the tests are obtained for $\alpha = 0.05$ over 25000 replicates for each case. For each test, the ratio of the number of rejections to the number of repetitions, namely 25000, gives the simulated power for that particular case.

The simulations treat the parameters as if they were unknown. Therefore, they are estimated by the method of maximum likelihood. The corresponding estimates are obtained using the R computing environment via the package fitdistriplus, using the functions fitdistr and fitdistcens for complete and Type II right censored samples are used, respectively. In some cases, the estimates cannot be obtained because of the convergence problems associated with the algorithms for finding maximum likelihood estimates, especially when testing the log-normal distribution against the alternative distributions when the sample size n is 10 and the censoring rate is 0.20. Such cases are omitted from the simulations.

Regarding the precision of the rejection rates, with 25000 replicates, the largest possible standard error for the rejection rate is

$$\sigma = \sqrt{\frac{0.5(1-0.5)}{25000}} = 0.0032 \qquad (10)$$

Therefore, an approximate 99% confidence interval for the proportion of rejections is formed as the simulated power value$\pm 3(0.0032)$, i.e., the simulated power value ± 0.00964. This means that if any two simulated power values associated with any two tests for a situation differ by more than 0.01 for a situation, the power of the two tests for that special case are statistically different at the $\alpha = 0.01$ significance level. Note that all computations in this study are performed using computer programs of R code developed specifically for this study.

The formulae for the goodness of fit tests considered in the case of the complete sample are as follows.

The Kolmogorov-Smirnov test statistic is defined as $D = \max(D^+, D^-)$ where,

$$D^+ = \max_i\left(\frac{i}{n} - U_{i:n}\right) \; D^- = \max_i\left(U_{i:n} - \frac{i-1}{n}\right) \qquad (11)$$

By using (11) Kuiper test statistic is expressed as $V = D^+ + D^-$. As for the Cramer-von Mises test statistic, it is as follows:

$$W^2 = \sum_{i=1}^{n}\left\{U_{i:n} - \frac{2i-1}{2n}\right\}^2 + \frac{1}{12n} \quad (12)$$

Next, the test that is closely related to the Cramer-von Mises test, namely the Watson test has the following test statistic

$$U^2 = W^2 - n\left(\bar{U} - \frac{1}{2}\right)^2 \quad (13)$$

where $\bar{U} = \sum_{i=1}^{n}\frac{U_{i:n}}{n}$. Finally, the Anderson-Darling test has the test statistic of the form in (14).

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\ln(U_{i:n}) + \ln(1 - U_{n+1-i:n})\} \quad (14)$$

In all of the above formulas $U_{i:n} = F(X_{i:n})$ denotes the ith order statistic for a random sample size of n from the uniform distribution over $(0, 1)$. In the case of the type II censored samples, the statistics for these EDF tests are expressed as follows.

First, the test statistic for the Kolmogorov-Smirnov test is

$$D_{r,n} = \max_{1\le i\le r}\left\{\frac{i}{n} - U_{i:n}\,,\ U_{i:n} - \frac{i-1}{n}\right\} \quad (15)$$

Then, that of the Cramer-von Mises test is as follows:

$$W_{r,n}^2 = \sum_{i=1}^{r}\left(U_{i:n} - \frac{2i-1}{2n}\right)^2 + \frac{r}{12n^2} + \frac{n}{3}\left(U_{r:n} - \frac{r}{n}\right)^3 \quad (16)$$

Next, the Watson test has the statistic of the following form in (17):

$$U_{r,n}^2 = W_{r,n}^2 - nU_{r:n}\left[\frac{r}{n} - \frac{U_{r:n}}{2} - \frac{r\bar{U}}{nU_{r:n}}\right]^2 \quad (17)$$

where $\bar{U} = \sum_{i=1}^{r}\frac{U_{i:n}}{r}$. Finally, the Anderson-Darling test uses the statistic in (18).

$$A_{r,n}^2 = -\frac{1}{n}\sum_{i=1}^{r}(2i-1)\left[\ln U_{i:n} - \ln\{1 - U_{i:n}\}\right] - 2\sum_{i=1}^{r}\ln\{1 - U_{i:n}\} - \frac{1}{n}\left[(r-n)^2\ln\{1 - U_{r:n}\} - r^2\ln U_{r:n} + n^2U_{r:n}\right] \quad (18)$$

In the simulations, the exponential and log-normal distributions are considered as the null distributions. Since the log-normal distribution is not directly a location-scale distribution, the simulated observations from these distributions are transformed using the natural logarithm to obtain a location-scale distribution, namely the normal distribution. Therefore, all computations including the critical values of the goodness-of-fit tests related to the log-normal distribution were performed on the normal distribution.

The alternative distributions considered are exponential, Weibull, gamma and log-normal. With this choice of the distributions, constant, increasing, decreasing and non-monotonic hazard functions are employed as alternatives. The specific distributions for the null and alternative hypotheses are shown in Table 1. The values of the parameters are chosen so that the mean lifetimes for the null and alternative distributions are approximately equal. The critical values do not depend on the specific chosen parameter values, because they are computed after the transformation to location-scale distribution in the case of the log-normal distribution. The same is true for the exponential distribution because it is a scale family of distributions.

2.4. Some real data sets

In order to illustrate the newly proposed goodness of fit test W_{AKM} in comparison to some well-known goodness of fit tests, some data sets from the literature have been used.

The first data set we consider consists of the times of successive failures of the air conditioning system of each member of a fleet of Boeing 720 jets, given by [27]. The flight hours between 30 failures on aircraft 7912 are as follows.

23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95

In [27] it is stated that the distribution from which the data come from is the exponential distribution.

The second data set we consider is from [29] and contains the following 20 failure times in hours of electronic parts of equipment [28].

154, 419, 590, 603, 770, 845, 848, 891, 899, 953, 954, 982, 1044, 1059, 1126, 1127, 1294, 1678, 1831, 1847

Furthermore, the Weibull distribution is assumed to be an appropriate model to describe the data in [29].

The third data set we consider is given by [31] and it consists of 23 observations representing the number of millions of revolutions before failure for 23 ball bearings in a life test [30]. The data are as follows.
17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40

Furthermore, this data set is found to be a good fit to the log-logistic distribution [30]. In this study, the first 20 observations of the data are used in the goodness-of-fit tests.
Finally, the fourth data set we consider is provided by [32] and gives the average duration of hypopnea in

seconds for 25 subjects with obstructive sleep apnea, which is also analyzed in [33].

14.7, 17.8, 16.5, 17.7, 28.5, 18.1, 32.2, 27.6, 22.3, 31.8, 22.0, 23.1, 31.6, 18.4, 28.3, 16.5, 21.8, 23.7, 27.6, 17.2, 20.0, 20.6, 19.0, 18.7, 19.2

Furthermore, the log-normality of the data cannot be rejected [33]. In this study, the first 20 observations of the data are considered in the goodness-of-fit tests performed.

3. Results

The simulation results of performance comparisons of some common EDF goodness-of-fit tests are presented in Tables 2. through Table 5. In these tables, KS stands for Kolmogorov-Smirnov test; K, Kuiper test; CVM, Cramer-von Mises test; W, Watson test; AD, Anderson-Darling test; W_{AKM}, the new test proposed in this paper.

In addition, in the cases where the new test W_{AKM} provides an improvement in performance over the other tests, the "increase%" column is added to the tables to quantify the improvement. This column shows the percentage increase in power between the test W_{AKM} and the test that is closest in power to the test W_{AKM}.

$$increase\% = \frac{the\ power\ of\ W_{AKM} - the\ closest\ power\ to\ that\ of W_{AKM}}{the\ closest\ power\ to\ that\ of W_{AKM}} \times 100$$

In this section, the families of exponential distribution and log-normal distributions are hypothesized under the null hypothesis H_0 versus Weibull, gamma, exponential and log-normal distributions under the alternative hypothesis H_A.

Table 2. shows the simulation results for testing the exponential distribution against the Weibull distribution with increasing hazard function. According to Table 2., the following comments can be made:

- First, when the sample size n is 10, the power of all goodness-of-fit tests for all censoring rates is low. Among these low power values, the power of the W_{AKM} test is higher than the power of the other tests when the censoring rate p is 0.60 and 0.80 and for the full sample case. Furthermore, in terms of power, the W_{AKM} test is followed by the CVM and KS tests in these cases. Similarly, KS, CVM and W_{AKM} tests are observed as the most powerful tests when the censoring ratio p is 0.20 and 0.40.

- For sample sizes n=20 and n=30 and censoring ratio $p \geq 0.40$ and as well as for the full sample case, the power performance of the W_{AKM} test is better than the other tests. Moreover, the CVM, KS and AD tests are remarkable in terms of power in these cases after the W_{AKM} test. In addition, the percentage power increase provided by the W_{AKM} test tends to be higher when the sample size n is 20 than when the sample size n is 30. Finally, when the censoring rate p is 0.20, the W_{AKM}, CVM and KS tests are more powerful than other tests.

- The W_{AKM} test outperforms the other tests for sample sizes n of 40 and 50, for all censoring rates, and for the full sample case. Other tests for the magnitude of power that stand out under the same cases are the CVM and AD tests. The percentage power improvement provided by the W_{AKM} test in these cases tends to decrease with increasing sample size.

- The W_{AKM} test has slightly the highest power when the sample size n is 100 and the censoring ratios p's are 0.20, 0.40 and 0.60. Moreover, the closest power performance to W_{AKM} comes from CVM and AD tests. When the censoring ratio p is 0.80, the tests with the highest power are the W_{AKM}, AD and CVM tests. For the complete sample case, all tests are very close to a power of almost 1

In this paper, simulation results are not given for the following two cases due to the low power or no significant improvement in power provided by the proposed W_{AKM} test: (i) the case of exponential distribution versus gamma distribution with decreasing hazard function and (ii) the case of exponential distribution versus log-normal distribution.

The simulation results for testing the log-normal distribution against the Weibull distribution with increasing hazard function are presented in Table 3. From Table 3, the followings can be said:

- As expected when the sample size n is 10, the power of all goodness-of-fit tests is very low for all censoring rates and for the full sample case. The best performance comes from the W_{AKM} test when the censoring ratio p is 0.60,

Table 1. The distributions for the null and alternative hypotheses in the simulations

Distributions for H_A	Hazard Function	Distributions for H_0 Exponential (1)	Lognormal (0, 0.7)
Weibull (1.5, 1.1)	increasing	√	√
Gamma (0.5, 0.7)	decreasing	√	√

Exponential (1)	constant		√
Log-normal (0, 0.7)	non-monotonic	√	

Table 2. Simulated power values for exponential distribution against the Weibull distribution with increasing hazard function (25,000 repetitions)

n	p	KS	K	CVM	W	AD	W_{AKM}	Increase%
10	0.20	0.0766	0.0474	0.0764	0.0089	0.0390	0.0755	None
	0.40	0.1118	0.0568	0.1192	0.0144	0.0700	0.1289	None
	0.60	0.1364	0.0788	0.1472	0.0222	0.0911	0.1896	28.8
	0.80	0.1684	0.1245	0.1776	0.0444	0.1216	0.2485	39.9
	1.00	0.2128	0.2050	0.2481	0.2206	0.1854	0.3540	42.7
20	0.20	0.1126	0.0516	0.1248	0.0282	0.0757	0.1282	None
	0.40	0.1617	0.0804	0.1870	0.0372	0.1349	0.2239	19.7
	0.60	0.2304	0.1430	0.2564	0.0665	0.2110	0.3387	32.1
	0.80	0.2916	0.2340	0.3326	0.1276	0.2860	0.4612	38.7
	1.00	0.3979	0.3703	0.4857	0.4076	0.4440	0.6154	26.7
30	0.20	0.1450	0.0608	0.1627	0.0383	0.1088	0.1753	None
	0.40	0.2252	0.1214	0.2590	0.0636	0.2099	0.3125	20.6
	0.60	0.3098	0.2126	0.3593	0.1183	0.3295	0.4592	27.8
	0.80	0.4247	0.3540	0.4888	0.2208	0.4599	0.6112	25.0
	1.00	0.5666	0.5278	0.6722	0.5717	0.6518	0.7941	18.1
40	0.20	0.1718	0.0758	0.1974	0.0508	0.1425	0.2236	13.3
	0.40	0.2862	0.1599	0.3258	0.0984	0.2927	0.3920	20.3
	0.60	0.3994	0.2911	0.4714	0.1780	0.4514	0.5710	21.1
	0.80	0.5385	0.4756	0.6230	0.3330	0.6118	0.7384	18.5
	1.00	0.7046	0.6712	0.8125	0.7120	0.8068	0.9011	10.9
50	0.20	0.1977	0.0852	0.2308	0.0604	0.1794	0.2637	14.2
	0.40	0.3426	0.2076	0.4055	0.1330	0.3760	0.4679	15.4
	0.60	0.4933	0.3740	0.5665	0.2426	0.5596	0.6562	15.8
	0.80	0.6488	0.5861	0.7402	0.4373	0.7387	0.8358	12.9
	1.00	0.8107	0.7776	0.8998	0.8158	0.9016	0.9559	6.0
100	0.20	0.3343	0.1638	0.3960	0.1250	0.3584	0.4380	10.6
	0.40	0.5862	0.4229	0.6712	0.3123	0.6844	0.7323	7.0
	0.60	0.7999	0.7148	0.8699	0.5704	0.8882	0.9111	2.6
	0.80	0.9284	0.9087	0.9679	0.8287	0.9747	0.9853	None
	1.00	0.9878	0.9828	0.9974	0.9888	0.9984	0.9994	None

0.80, followed by the KS, AD and CVM tests. For a censoring rate of p=0.40, the power ranking is W_{AKM}, CVM, KS and AD tests. Also, the most powerful tests after W_{AKM} are the AD and CVM tests for the full sample case.

- The W_{AKM} test is superior to the other tests in terms of power when the sample size n is 20 and 30, and for all censoring rates except p = 0.20. In the same cases, the performance of the AD and CVM tests follows the performance of the W_{AKM} test. In addition, for a censoring rate of p=0.20, the most powerful tests for sample sizes of 20 and 30 are the W_{AKM}, CVM, and AD tests.

- The W_{AKM} test has the best power performance among the other tests when the sample size n is 40, 50, and 100 and for all censoring rates. The next good performance comes from the AD and CVM tests.

- The increase in percent power delivered by the W_{AKM} test in Table 3 is typically higher than that in Table 2.

Table 4. gives the simulation results for testing the log-normal distribution versus the gamma distribution with decreasing hazard function. The following points can be made from Table 4:

- Again low powers are obtained for all tests and all censoring ratios when the sample size n is 10. The W_{AKM} test exceeds the other tests with respect to power for censoring ratios of 0.60 and 0.80. The same is true for the complete sample case.

Table 3. Simulated power values for the log-normal distribution against the Weibull distribution (25,000 repetitions)

n	p	KS	K	CVM	W	AD	W_{AKM}	Increase%
	0.40	0.0672	0.0554	0.0687	0.0491	0.0664	0.0774	None

10	0.60	0.0774	0.0505	0.0743	0.0440	0.0771	0.1116	44.2
	0.80	0.0955	0.0709	0.0860	0.0486	0.0927	0.1591	66.6
	1.00	0.1169	0.1086	0.1317	0.1209	0.1392	0.2351	68.9
20	0.20	0.0671	0.0582	0.0688	0.0558	0.0650	0.0704	None
	0.40	0.0844	0.0587	0.0854	0.0529	0.0908	0.1224	34.8
	0.60	0.1052	0.0713	0.1052	0.0555	0.1180	0.1886	59.8
	0.80	0.1391	0.1056	0.1398	0.0730	0.1547	0.2892	86.9
	1.00	0.1975	0.1811	0.2414	0.2127	0.2660	0.4440	66.9
30	0.20	0.0705	0.0569	0.0767	0.0526	0.0762	0.0898	None
	0.40	0.0978	0.0656	0.1046	0.0571	0.1154	0.1593	38.0
	0.60	0.1406	0.0952	0.1435	0.0736	0.1669	0.2550	52.8
	0.80	0.1930	0.1566	0.2069	0.1166	0.2278	0.3870	69.9
	1.00	0.2847	0.2613	0.3505	0.3035	0.3902	0.6015	54.1
40	0.20	0.0819	0.0600	0.0872	0.0574	0.0871	0.1073	23.1
	0.40	0.1189	0.0772	0.1249	0.0668	0.1347	0.1924	42.8
	0.60	0.1664	0.1150	0.1810	0.0932	0.2040	0.3064	50.2
	0.80	0.2387	0.1975	0.2633	0.1520	0.2882	0.4766	65.4
	1.00	0.3656	0.3348	0.4538	0.3886	0.5027	0.7087	41.0
50	0.20	0.0877	0.0638	0.0954	0.0617	0.0958	0.1221	27.5
	0.40	0.1270	0.0799	0.1419	0.0738	0.1568	0.2202	40.4
	0.60	0.1975	0.1407	0.2196	0.1184	0.2443	0.3641	49.0
	0.80	0.2882	0.2475	0.3290	0.1942	0.3547	0.5558	56.7
	1.00	0.4350	0.4038	0.5392	0.4623	0.5930	0.7911	25.0
100	0.20	0.1135	0.0750	0.1261	0.0723	0.1306	0.1687	29.2
	0.40	0.1957	0.1249	0.2325	0.1216	0.2520	0.3374	33.9
	0.60	0.3308	0.2561	0.3886	0.2217	0.4221	0.5635	33.5
	0.80	0.5090	0.4894	0.5982	0.4198	0.6263	0.8020	28.1
	1.00	0.7258	0.7087	0.8414	0.7680	0.8845	0.9660	9.2

Table 4. Simulated power values for the log-normal distribution against the gamma distribution (25,000 repetitions)

n	p	KS	K	CVM	W	AD	W_{AKM}	Increase%
10	0.40	0.0746	0.0576	0.0747	0.0520	0.0730	0.0877	None
	0.60	0.0969	0.0600	0.0899	0.0477	0.0969	0.1440	48.6
	0.80	0.1198	0.0872	0.1076	0.0522	0.1186	0.2144	76.5
	1.00	0.1670	0.1582	0.1967	0.1784	0.2089	0.3458	65.5
20	0.20	0.0686	0.0593	0.0721	0.0554	0.0682	0.0751	None
	0.40	0.0999	0.0622	0.1016	0.0548	0.1081	0.1504	39.1
	0.60	0.1458	0.0914	0.1508	0.0676	0.1662	0.2595	56.1
	0.80	0.2077	0.1639	0.2199	0.1095	0.2375	0.4073	71.1
	1.00	0.3186	0.3086	0.4001	0.3546	0.4379	0.6342	44.8
30	0.20	0.0748	0.0602	0.0805	0.0550	0.0804	0.0937	None
	0.40	0.1213	0.0772	0.1350	0.0674	0.1433	0.2024	41.2
	0.60	0.1945	0.1345	0.2088	0.1019	0.2345	0.3524	50.3
	0.80	0.2996	0.2577	0.3312	0.1850	0.3524	0.5543	57.3
	1.00	0.4598	0.4576	0.5716	0.5106	0.6239	0.8078	29.5
40	0.20	0.0858	0.0650	0.0940	0.0626	0.0955	0.1174	22.9
	0.40	0.1472	0.0895	0.1612	0.0790	0.1720	0.2481	44.2
	0.60	0.2470	0.1726	0.2759	0.1357	0.3025	0.4358	44.1
	0.80	0.3891	0.3470	0.4377	0.2642	0.4616	0.6776	31.9
	1.00	0.5823	0.5837	0.6990	0.6338	0.7531	0.8964	19.0
50	0.20	0.0973	0.0664	0.1054	0.0652	0.1060	0.1386	30.8
	0.40	0.1690	0.1023	0.1890	0.0918	0.2030	0.2853	40.5
	0.60	0.2940	0.2168	0.3360	0.1761	0.3631	0.5132	41.3
	0.80	0.4624	0.4285	0.5279	0.3327	0.5505	0.7606	38.2
	1.00	0.6772	0.6897	0.8019	0.7367	0.8498	0.9464	11.4
100	0.20	0.1294	0.0838	0.1467	0.0820	0.1518	0.2016	32.8
	0.40	0.2802	0.1830	0.3308	0.1735	0.3512	0.4564	30.0
	0.60	0.5054	0.4208	0.5853	0.3645	0.6110	0.7557	23.7
	0.80	0.7504	0.7574	0.8319	0.6569	0.8454	0.9513	12.5
	1.00	0.9383	0.9498	0.9821	0.9627	0.9911	0.9985	None

- The W_{AKM} test outperforms the other tests in terms of power for all censoring ratios, but p=0.20 for sample sizes n of 20 and 30. The same applies to the full sample case. The AD and CVM tests come after the W_{AKM} test in power performance in all cases. In addition,

W_{AKM}, CVM, KS, and AD at n=20 and W_{AKM}, CVM, and AD at n=30 are the most powerful tests for p=0.20. Again, as the sample size increases, the percentage increase in power due to W_{AKM} tends to decrease.

- For sample sizes of 40 and 50 and for all censoring ratios, including the full sample case, W_{AKM} has the greatest power among the tests. The AD and CVM tests follow the W_{AKM} test in terms of power.

- When the sample size n is 100 and for all censoring ratios except the full sample case, the tests with the highest power are, in order, W_{AKM}, AD, and CVM. In the full sample case, all tests have high powers, but the powers of W_{AKM}, AD, and CVM are much closer to 1 compared to the others.

The simulation results for testing the log-normal distribution versus the exponential distribution are given in Table 5. The following comments can be made from Table 5:

- W_{AKM} dominates the other tests in terms of power when n=10 with censoring ratios p=0.60 and 0.80. The same is true for the full sample case. In the case of p=0.40, the W_{AKM}, KS, CVM and AD tests perform best.

- For sample sizes n of 20, 30, and 40, and for all censoring ratios except p=0.20, W_{AKM} has better power than other tests. The AD and CVM tests have highest power performance after the W_{AKM} test. Additionally, in the case of p=0.20, the W_{AKM}, CVM, AD and KS tests perform best.

- For sample sizes of 50 and 100 and for all censoring ratios, including the full sample case, W_{AKM} has the highest power of the tests considered. The second best performance comes from the AD and CVM tests.

There appears to be a general trend of a decrease in the percentage increase in power in Tables 2 through 5 as the sample size increases. This is due to the fact that power tends to increase in magnitude as the sample size increases.

Table 5. Simulated power values for the log-normal distribution against the exponential distribution (25,000 repetitions)

n	p	KS	K	CVM	W	AD	W_{AKM}	Increase%
10	0.40	0.0675	0.0551	0.0669	0.0508	0.0650	0.0743	None
	0.60	0.0790	0.0521	0.0713	0.0427	0.0768	0.1137	43.9
	0.80	0.0912	0.0671	0.0828	0.0449	0.0899	0.1578	73.0
	1.00	0.1145	0.1067	0.1292	0.1180	0.1366	0.2365	73.1
20	0.20	0.0677	0.0595	0.0692	0.0581	0.0669	0.0709	None
	0.40	0.0828	0.0577	0.0846	0.0523	0.0904	0.1219	34.8
	0.60	0.1070	0.0708	0.1068	0.0565	0.1213	0.1902	56.8
	0.80	0.1422	0.1105	0.1435	0.0758	0.1591	0.2932	84.3
	1.00	0.1982	0.1771	0.2391	0.2077	0.2635	0.4356	65.3
30	0.20	0.0721	0.0592	0.0732	0.0532	0.0752	0.0866	None
	0.40	0.0985	0.0679	0.1034	0.0574	0.1149	0.1596	38.9
	0.60	0.1385	0.0932	0.1424	0.0734	0.1654	0.2562	54.9
	0.80	0.1889	0.1560	0.2042	0.1160	0.2266	0.3867	70.7
	1.00	0.2818	0.2579	0.3494	0.2990	0.3908	0.5976	52.9
40	0.20	0.0825	0.0604	0.0861	0.0598	0.0856	0.1060	None
	0.40	0.1124	0.0724	0.1220	0.0648	0.1355	0.1942	43.3
	0.60	0.1654	0.1138	0.1817	0.0920	0.2050	0.3070	49.8
	0.80	0.2398	0.2004	0.2667	0.1594	0.2907	0.4776	64.3
	1.00	0.3626	0.3362	0.4516	0.3910	0.5005	0.7042	40.7
50	0.20	0.0881	0.0611	0.0937	0.0597	0.0949	0.1214	27.9
	0.40	0.1311	0.0832	0.1449	0.0754	0.1574	0.2202	39.9
	0.60	0.1964	0.1388	0.2181	0.1163	0.2432	0.3596	47.9
	0.80	0.2880	0.2490	0.3244	0.1963	0.3486	0.5520	58.3
	1.00	0.4288	0.4046	0.5360	0.4620	0.5940	0.7864	32.4
100	0.20	0.1110	0.0754	0.1258	0.0721	0.1329	0.1700	27.9
	0.40	0.1992	0.1314	0.2328	0.1235	0.2550	0.3396	33.2
	0.60	0.3276	0.2550	0.3848	0.2219	0.4200	0.5613	33.6
	0.80	0.5058	0.4856	0.5886	0.4156	0.6183	0.8021	29.7
	1.00	0.7294	0.7112	0.8434	0.7701	0.8878	0.9685	9.1

Table 6. The goodness of fit of the data sets to the exponential distribution for some tests at the significance level of α=0.05. (0: non-reject; 1: reject)

Data Sets	p	KS	K	CVM	W	AD	W_{AKM}
	0.20	0	0	0	0	0	0
	0.40	1	0	0	0	0	0

Data set 1 by [27]	0.60	0	0	0	1	0	0
	0.80	0	0	0	1	0	0
	1.00	1	0	0	0	0	0
Data set 2 by [29]	0.20	0	0	0	0	0	0
	0.40	0	0	1	0	0	1
	0.60	1	1	1	0	1	1
	0.80	1	1	1	1	1	1
	1.00	1	1	1	1	1	1
Data set 3 by [31]	0.20	0	0	0	0	0	0
	0.40	1	0	1	0	1	1
	0.60	1	1	1	1	1	1
	0.80	1	1	1	1	1	1
	1.00	1	1	1	1	1	1
Data set 4 by [32]	0.20	1	1	1	1	1	1
	0.40	1	1	1	1	1	1
	0.60	1	1	1	1	1	1
	0.80	1	1	1	1	1	1
	1.00	1	1	1	1	1	1

Finally, in order to illustrate the goodness of fit tests, each of data sets presented in Section 2.4 are Type II right censored by using the following censoring rates: p=r/n=0.20,0.40,0.60,0.80, and 1.00. Then their goodness of fit to an exponential distribution is tested at the critical level of α=0.05. The results of these tests are presented in Table 6. From the results of the tests in Table 6., it can be seen that the newly proposed W_{AKM} test produces compatible results with those of the competing goodness-of-fit tests, such as the Anderson-Darling and Cramer-von Mises tests

4. Discussion and Conclusion

Although the newly proposed WAKM test in this paper can be regarded as a modification of the Watson test, its power performance is higher than the well-known goodness-of-fit tests including the Watson test under the conditions of the simulation studies. In particular, W_{AKM} seems to be useful for researchers in life testing and reliability in distinguishing between the log-normal distribution and the exponential, Weibull, or gamma distribution under complete sample or Type II right censoring schemes, especially for small sample sizes.

Furthermore, the notable superior performance of W_{AKM} in testing the exponential and log-normal distributions against the alternative distributions considered suggests that the proposed test may distinguish well between the distribution with a heavy upper tail and the distribution with a light upper tail when the bodies of the distributions are mostly similar. In conclusion, the newly proposed W_{AKM} test seems to be worth considering along with the commonly used goodness-of-fit tests under the conditions of this study. In addition, the Cramer-von Mises and Anderson-Darling tests also appear to be powerful tests that can be recommended.

Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

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Appendix Simulated critical values for exponential and log-normal distributions (100,000 replicates)

Table A1. Simulated critical values of some goodness-of-fit tests for the exponential distribution for α=0.05

n	p	KS	K	CVM	W	AD	W_{AKM}
10	0.20	0.1729	0.1947	0.0178	0.0056	0.1809	0.3852
	0.40	0.2319	0.3066	0.0555	0.0248	0.4410	0.6970
	0.60	0.2730	0.3886	0.1045	0.0577	0.6811	0.9056
	0.80	0.3022	0.4506	0.1574	0.0996	0.9220	1.0405
	1.00	0.3246	0.4887	0.2171	0.1573	1.2707	1.0864
20	0.20	0.1245	0.1538	0.0174	0.0066	0.2290	0.5382
	0.40	0.1684	0.2283	0.0561	0.0274	0.4707	0.9136
	0.60	0.1981	0.2862	0.1074	0.0623	0.7111	1.1769
	0.80	0.2195	0.3300	0.1645	0.1076	0.9676	1.3287
	1.00	0.2345	0.3533	0.2182	0.1573	1.2872	1.4088
30	0.20	0.1024	0.1296	0.0172	0.0069	0.2377	0.6351
	0.40	0.1390	0.1904	0.0561	0.0284	0.4843	1.0798
	0.60	0.1636	0.2374	0.1085	0.0641	0.7237	1.3909
	0.80	0.1806	0.2736	0.1667	0.1111	0.9821	1.5822
	1.00	0.1934	0.2919	0.2205	0.1585	1.3091	1.6604
40	0.20	0.0891	0.1140	0.0170	0.0070	0.2441	0.7097
	0.40	0.1209	0.1667	0.0565	0.0287	0.4874	1.2187
	0.60	0.1424	0.2076	0.1091	0.0650	0.7319	1.5728
	0.80	0.1577	0.2382	0.1675	0.1118	0.9856	1.7810
	1.00	0.1686	0.2537	0.2210	0.1583	1.3112	1.8789
50	0.20	0.0798	0.1032	0.0168	0.0072	0.2457	0.7821
	0.40	0.1083	0.1496	0.0555	0.0287	0.4848	1.3432
	0.60	0.1274	0.1864	0.1087	0.0655	0.7335	1.7340
	0.80	0.1413	0.2147	0.1667	0.1132	0.9873	1.9533
	1.00	0.1509	0.2279	0.2209	0.1580	1.3141	2.0768
100	0.20	0.0569	0.0746	0.0164	0.0073	0.2513	1.0673
	0.40	0.0776	0.1080	0.0563	0.0294	0.4946	1.8403
	0.60	0.0912	0.1337	0.1097	0.0665	0.7431	2.3823
	0.80	0.1008	0.1536	0.1690	0.1146	0.9936	2.6870
	1.00	0.1072	0.1626	0.2191	0.1581	1.3014	2.8745

Table A2. Simulated critical values of some goodness of fit tests for the log-normal distribution via normal distribution for α=0.05

n	p	KS	K	CVM	W	AD	W_{AKM}
10	0.40	0.1813	0.2917	0.0281	0.0235	0.2116	0.3253
	0.60	0.2229	0.3557	0.0586	0.0499	0.3433	0.4396
	0.80	0.2498	0.4046	0.0921	0.0810	0.4946	0.5025
	1.00	0.2654	0.4353	0.1226	0.1140	0.7207	0.4892
20	0.20	0.0949	0.1493	0.0078	0.0062	0.1040	0.2485
	0.40	0.1373	0.2130	0.0300	0.0240	0.2260	0.4519
	0.60	0.1644	0.2613	0.0611	0.0514	0.3584	0.5883
	0.80	0.1822	0.2962	0.0955	0.0842	0.5127	0.6435
	1.00	0.1930	0.3172	0.1249	0.1156	0.7384	0.6082
30	0.20	0.0813	0.1234	0.0082	0.0063	0.1080	0.3072
	0.40	0.1139	0.1764	0.0303	0.0239	0.2288	0.5459
	0.60	0.1364	0.2163	0.0621	0.0521	0.3644	0.6960
	0.80	0.1508	0.2456	0.0963	0.0852	0.5157	0.7624
	1.00	0.1598	0.2623	0.1257	0.1163	0.7457	0.7054
40	0.20	0.0710	0.1080	0.0083	0.0062	0.1093	0.3503
	0.40	0.0998	0.1540	0.0304	0.0240	0.2317	0.6225
	0.60	0.1188	0.1882	0.0616	0.0516	0.3629	0.7956
	0.80	0.1309	0.2139	0.0964	0.0854	0.5160	0.8530
	1.00	0.1390	0.2286	0.1257	0.1162	0.7474	0.8004
50	0.20	0.0644	0.0973	0.0084	0.0062	0.1111	0.3867
	0.40	0.0896	0.1386	0.0305	0.0240	0.2322	0.6915
	0.60	0.1067	0.1695	0.0619	0.0520	0.3641	0.8757
	0.80	0.1180	0.1922	0.0972	0.0859	0.5209	0.9428
	1.00	0.1248	0.2059	0.1257	0.1164	0.7479	0.8838
100	0.20	0.0465	0.0702	0.0084	0.0062	0.1135	0.5479
	0.40	0.0643	0.0996	0.0307	0.0241	0.2358	0.9627
	0.60	0.0764	0.1211	0.0624	0.0521	0.3684	1.2245
	0.80	0.0845	0.1379	0.0975	0.0866	0.5237	1.3020
	1.00	0.0890	0.1470	0.1258	0.1164	0.7509	1.2254