ESKİŞEHİR TECHNICAL UNIVERSITY JOURNAL OF SCIENCE AND TECHNOLOGY
A- APPLIED SCIENCES AND ENGINEERING

Estuscience – Se, 2024, 25 [2] pp. 289-299, DOI: 10.18038/estubtda.1479718

RESEARCH ARTICLE

# ON THE CONTROL AND SIMULATION OF THE THERMAL CONDUCTIVITY IN A HEAT EQUATION

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**Abstract**

This study operates the Gradient Method to control the leading coefficient function of a heat equation and presents a Maplet Application which facilitates the computation of control function. The control is the heat conductivity function and this function is controlled by aiming the desired value approximation of final heat. After mentioning the existence and uniqueness of the control, the application is submitted by MAPLE mathematical software program and the results are tested on a problem.

**Keywords**

Control Problems,
Regularization,
Mathematical Software

**Time Scale of Article**

Received :07 May 2024
Accepted : 22 June 2024
Online date :28 June 2024

## 1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Having a long history, control problems related to heat equations continue to be popular nowadays. In this context, researches are being conducted aiming to control any parameter in the problem for different purposes. There are many studies about the control of the initial condition, the boundary conditions or the external heat source function. Some of them are presented in the studies [1-3]. Besides them, the problem of controlling the heat conductivity function is very important in terms of the location of this function in the equation. In the studies [4-7], these kinds of problems have been investigated with different cost functionals.

Now, we will examine such a problem. Let us suppose that $h(x,t)$ is a function representing the heat value at time $t$ and position $x$ of a thin rod with length $l$. Then $h(x,t)$ is the solution to the following problem:

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(k(x)\frac{\partial h}{\partial x}\right) = f(x,t), \quad (x,t) \in (0,l) \times (0,T) \tag{1}$$

$$h(x,0) = \varphi(x), x \in (0,l) \tag{2}$$

$$-k(0)h_x(0,t) = g_0(t), \ k(l)h_x(l,t) = g_1(t), \ \ t \in (0,T). \tag{3}$$

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Here $k(x)$ is heat conductivity function, $f(x,t)$ is external heat source function, $\varphi(x)$ is initial heat distribution, $g_0(t)$ and $g_1(t)$ are outward fluxes of the heat at the boundaries.

The generalized solution rather than the classical solution of such a problem is more useful in the applications because it also includes discontinuous functions. The generalized solution of the problem (1)-(3) is the function $h \in V_2^{1,0}(\Omega)$ satisfying the equality of

$$\int_0^T \int_0^l (-h\eta_t + k(x)h_x\eta_x)dxdt = \int_0^T \int_0^l f\eta dxdt + \int_0^l \varphi(x)\eta(x,0)dx + \int_0^T g_1(t)\eta(l,t)dt - \int_0^T g_0(t)\eta(0,t)dt \quad (4)$$

for each $\eta \in W_2^{1,1}(\Omega)$ with $\eta(x,T) = 0$, [4,5,6]. For equality (4) to be meaningful the data functions are chosen from the functional spaces

$$k(x) \in L_\infty(0,l),\ f(x,t) \in L_2(\Omega),\ \varphi(x) \in L_2(0,l),\ g_0(t) \in L_2(0,T),\ g_1(t) \in L_2(0,T).$$

The norms on $V_2^{1,0}(\Omega)$ and $W_2^{1,1}(\Omega)$ are defined respectively such as;

$$\|h\|_{V_2^{1,0}(\Omega)} = \max_{0\le t\le T}\|h\|_{L_2(0,l)} + \|h_x\|_{L_2(\Omega)}$$

$$\|\eta\|_{W_2^{1,1}(\Omega)} = \left[\|\eta\|_{L_2(\Omega)}^2 + \|\eta_x\|_{L_2(\Omega)}^2 + \|\eta_t\|_{L_2(\Omega)}^2\right]^{1/2}.$$

Let us now assume that the rod is desired to close a given heat distribution $\mu(x)$ at final time $T$ and let $k(x)$ be the control function that will perform this request. In this case, it is necessary to find the function $k(x)$ which will make the following functional minimum on an admissible set of controls $K$ which is a closed and convex subset of $L_2(0,l)$.

$$I(k) = \int_0^l [h(x,T;k) - \mu(x)]^2 dx. \quad (5)$$

On the other hand, it is well-known that the minimization problem of the functional (5) is ill-posed. In the ill-posedness when a problem is solved numerically it is frequently encountered that $I(k_*) \cong I(l_*) \cong 0$ while $\|k_* - l_*\|_{L_2(0,l)} \gg 0$ for different control functions $k_*$ and $l_*$ generated by different minimizers $\{k_m\}$ and $\{l_m\}$. This means that the solution may be non-unique and small changes in the initial data reveal huge differences in solution when obtaining a numerical solution. To deal with this situation, a penalty term is included in the functional and regularized solution is investigated.

Hence, we consider the new functional

$$J(k) = \int_0^l [h(x,T;k) - \mu(x)]^2 dx + \alpha \int_0^l [k(x) - k^+(x)]^2 dx \quad (6)$$

and solve the problem

$$\min_{k\in K} J(k). \quad (7)$$

Here, control is searched around of a function $k^+$. The function $k^+$ has the role of an initial guess to the control function. Especially for the local results about convergence rates, the choice of $k^+$ is very crucial. Of course, available apriori information about the minimum element of the functional (5) has to enter into the selection of $k^+$ to get successful results. Besides, $\alpha$ in the functional $J(k)$ is the regularization parameter which is used to balance between the two integrals of (6). The further information about the parameter $\alpha$ and function $k^+$ can be found in [8].

The solution of the problem (7) is called $k^+$-minimum norm control and represented by $k_*(x)$. The existence and uniqueness of $k^+$-minimum norm control $k_*(x)$ is based on the following facts;

- The space $L_2(0,l)$ is uniformly convex.
- $K$ is closed and bounded in $L_2(0,l)$.
- $I(k)$ is continuous by the $L_2$ norm on $K$.
- $I(k)$ is bounded from below on $K$.

Then according to the Goebel Theorem [9] there is a dense subset $G$ in $L_2(0,l)$ such that for $k^+ \in G$ and any $\alpha > 0$ the problem (7) has a unique solution. Similar analyzes have been carried out in the studies [4,5,6,7].

Also, it is known that obtaining a numerical solution and symbolic computation of control function in control problems is a very difficult issue. Moreover, there is no computer program or software prepared for such a problem.

This study has been prepared to fill a gap in this area and gives a Maple Application 'HCC (Heat Conductivity Control)' that produces the solution to the problem (7).

The used computer program Maple is a mathematical software that combines the world's most powerful mathematical engine with an interface that makes it extremely easy to analyze, explore, visualize, and solve mathematical problems. Besides, a Maplet Application is a graphical user interface containing windows, textbox regions, and other visual interfaces, which gives a user point-and-click access to the power of Maple. Users can perform calculations, plot functions, or display dialogs without using the worksheet interface [10].

## 2. APPROXIMATE SOLUTION TO THE CONTROL FUNCTION

In this section, we first state the necessary condition for a control function $\boldsymbol{k}_*(\boldsymbol{x})$ to be the solution of the considered problem.

By the theory of calculus of variations, the definition of Frechet differentiability is

$$\boldsymbol{\Delta J}(\boldsymbol{k}) = \langle \boldsymbol{J}'(\boldsymbol{k}), \Delta \boldsymbol{k}\rangle_{\boldsymbol{L_2(0,l)}} + \boldsymbol{o}\left(\|\Delta \boldsymbol{k}\|_{\boldsymbol{L_2(0,l)}}\right)$$

with $\lim\limits_{\|\Delta \boldsymbol{k}\| \to \boldsymbol{0}} \boldsymbol{o}(\|\Delta \boldsymbol{k}\|)/\|\Delta \boldsymbol{k}\| = \boldsymbol{0}$. With this definition, the derivation of the functional (6) after some manipulations is found by

$$\boldsymbol{J}'(\boldsymbol{k}) = -\int_{\boldsymbol{0}}^{\boldsymbol{T}} \boldsymbol{h_x \eta_x dt} + \boldsymbol{2\alpha}(\boldsymbol{k} - \boldsymbol{k}^+). \tag{8}$$

Then according to [11], the necessary condition for a control $\boldsymbol{k}_*(\boldsymbol{x})$ to be the minimum element for the problem is given by the following inequality:

$$\langle \boldsymbol{J}'(\boldsymbol{k}_*), \boldsymbol{k} - \boldsymbol{k}_*\rangle_{\boldsymbol{L_2(0,l)}} = \langle -\int_{\boldsymbol{0}}^{\boldsymbol{T}} (\boldsymbol{h}_*)_{\boldsymbol{x}} (\boldsymbol{\eta}_*)_{\boldsymbol{x}} \boldsymbol{dt} + \boldsymbol{2\alpha}(\boldsymbol{k}_* - \boldsymbol{k}^+), \boldsymbol{k} - \boldsymbol{k}_*\rangle_{\boldsymbol{L_2(0,l)}} \geq \boldsymbol{0} \tag{9}$$

for $\forall \boldsymbol{k} \in \boldsymbol{K}$, Here, $\boldsymbol{h}_*$ is the solutions of the (1)-(3) state problem and $\boldsymbol{\eta}_*$ is the solution of

$$\boldsymbol{\eta_t + (k(x)\eta_x)_x = 0} \tag{10}$$

$$\boldsymbol{\eta(x,T) = 2[h(x,T;k) - \mu(x)]} \tag{11}$$

$$\boldsymbol{k(0)\eta_x(0,t) = 0,\ k(l)\eta_x(l,t) = 0} \tag{12}$$

adjoint problem in the generalized sense, corresponding to $\boldsymbol{k = k_*}$.

Now, we can discuss how to get an approximate solution for a control function. By Galerkin Method, the approximate solution $\boldsymbol{h^N(x,t)}$ with $\boldsymbol{N}$ sum for (1)-(3) heat equation is constituted such as

$$\boldsymbol{h^N = v^N + w = \sum_{k=1}^{N} C_k^N(t)\varphi_k(x) + w}. \tag{13}$$

Here, $\boldsymbol{v^N}$ is the solution of corresponding homogeneous Neumann problem and

$$\boldsymbol{w(x,t) = \frac{x^2}{2}\frac{1}{lk(l)}g_1(t) + \left(\frac{x^2}{2} - xl\right)\frac{1}{lk(0)}g_0(t)}$$

is an auxiliary function.

Also, $\{\boldsymbol{\varphi_k(x)}\}$ is an orthonormal basis of the functional space $\boldsymbol{L_2(0,l)}$. Then $\boldsymbol{C_k^N(t)}$ 's are unknown functions which are the solutions of the following first order differential system;

$$\begin{gathered}\frac{d}{dt}\mathbf{C^N} + \mathbf{KC^N} = \mathbf{F}\\ \mathbf{C^N(0) = A}\end{gathered} \tag{14}$$

for the vector $\mathbf{C^N} = [\boldsymbol{C_1^N(t)} \quad \cdots \quad \boldsymbol{C_N^N(t)}]^{\boldsymbol{T}}$, where the matrix **K** has the entries

$$\boldsymbol{K_{ij} = \int_0^l \left(k(x)\frac{d}{dx}\varphi_j(x)\right)\frac{d}{dx}\varphi_i(x)dx}$$

for $\boldsymbol{i,j = 1,2,\dots,N}$ and the vectors **F**, **A** and **B** have the entries

$$\boldsymbol{F_i = \int_0^l \tilde{f}(x,t)\varphi_i(x)dx\,,\ A_i = \int_0^l \tilde{\varphi}(x)\varphi_i(x)dx} \text{ for } \boldsymbol{i = 1,2,\dots,N}.$$

with $\boldsymbol{\tilde{\varphi}(x) = \varphi(x) - w(x,0)}$ and $\boldsymbol{\tilde{f}(x,t) = f(x,t) - w_t + (k(x)w_x)_x}$.

The problem of finding the vector $\mathbf{C^N}$ from (14) is a Cauchy problem for the system of first order differential equations. The right-hand side of this system is in the class of square integrable functions. This system has a unique solution on the interval $[\boldsymbol{0,T}]$ as known from the theory of ordinary differential equations.

In the same way, secondly, the approximate solution of (10)-(12) adjoint problem is carried out by the finite sum of

$$\boldsymbol{\eta^N = \sum_{k=1}^{N}(C_e)_k^N(t)\varphi_k(x)}.$$

Here $(\boldsymbol{C_e})_{\boldsymbol{k}}^{\boldsymbol{N}}(\boldsymbol{t})$ 's are unknown functions which are the solutions of a second-order system of differential equations obtained by the generalized solution of the adjoint problem.

After then, using these solutions the approximate derivative of the cost functional (6) is obtained by

$$\left(J^N\right)'\left(k(x)\right) = -\int_0^T \left[h^N\left(x,t;k(x)\right)\right]_x \left[\eta^N\left(x,t;k(x)\right)\right]_x dt + 2\alpha[k(x) - k^+(x)]. \tag{15}$$

Once these functions are found, the minimization stages can be carried out. Starting with an initial $\boldsymbol{k_0}(\boldsymbol{x}) \in \boldsymbol{K}$ element, a minimizer $\{\boldsymbol{k_m}(\boldsymbol{x})\}$ is constituted by Gradient Method with the rule

$$k_{m+1}(x) = k_m(x) - \beta\left(J^N\right)'\left(k_m(x)\right),\ \tau_m > 0,\ m = 0, 1, 2, \ldots \tag{16}$$

In each step of $\boldsymbol{m}$, the parameter $\boldsymbol{\beta} > \boldsymbol{0}$ is chosen sufficiently enough such that the minimizing condition

$$J^N(k_{m+1}) < J^N(k_m) \tag{17}$$

holds.
The computations are executed up to the stopping criteria of

$$\left|J^N(k_m) - J^N(k_{m+1})\right| \leq \varepsilon \tag{18}$$

is provided by given small $\boldsymbol{\varepsilon}$ number. The $\boldsymbol{k_{m+1}}(\boldsymbol{x})$ function satisfying this condition is accepted by the approximate control $\boldsymbol{k_*}(\boldsymbol{x})$.

We can summarize this process with the following algorithm.

### 2.1. Algorithm for the Process

Selecting an initial control $\boldsymbol{k_0}(\boldsymbol{x}) \in \boldsymbol{K}$ , $\boldsymbol{k_{m+1}}(\boldsymbol{x})$ is computed by the following scheme if $\boldsymbol{k_m}(\boldsymbol{x})$ is known for $\boldsymbol{m} \geq \boldsymbol{0}$.

1. Solve the system (14) and get approximate function $\boldsymbol{h_m^N}(\boldsymbol{x},\boldsymbol{t})$.

2. Knowing $\boldsymbol{h_m^N}(\boldsymbol{x},\boldsymbol{t})$, solve the adjoint equation (10)-(12) and get $\boldsymbol{\eta_m^N}(\boldsymbol{x},\boldsymbol{t})$.

3. Using (15), set $\boldsymbol{k_{m+1}}(\boldsymbol{x}) = \boldsymbol{k_m}(\boldsymbol{x}) - \boldsymbol{\beta}\left(\boldsymbol{J_m^N}\right)'\left(\boldsymbol{k_m}(\boldsymbol{x})\right)$ by (16) and select the parameter $\boldsymbol{\beta}$ small enough to assure the condition (17) holds.

4. If condition (18) holds, take the control function as $\boldsymbol{k_*}(\boldsymbol{x}) = \boldsymbol{k_{m+1}}(\boldsymbol{x})$ otherwise go to Step 3.

## 3. A MAPLET APPLICATION CALCULATING THE CONTROL FUNCTION AND TESTING THIS APPLICATION

In this section, using the software Maple 17 we submit a Maplet Application named HCC (Heat Conductivity Control). This application produces the heat conductivity function which is control function of the problem if data is entered. This application can be downloaded using the link given by [12].

Running this Maplet executes the following application;

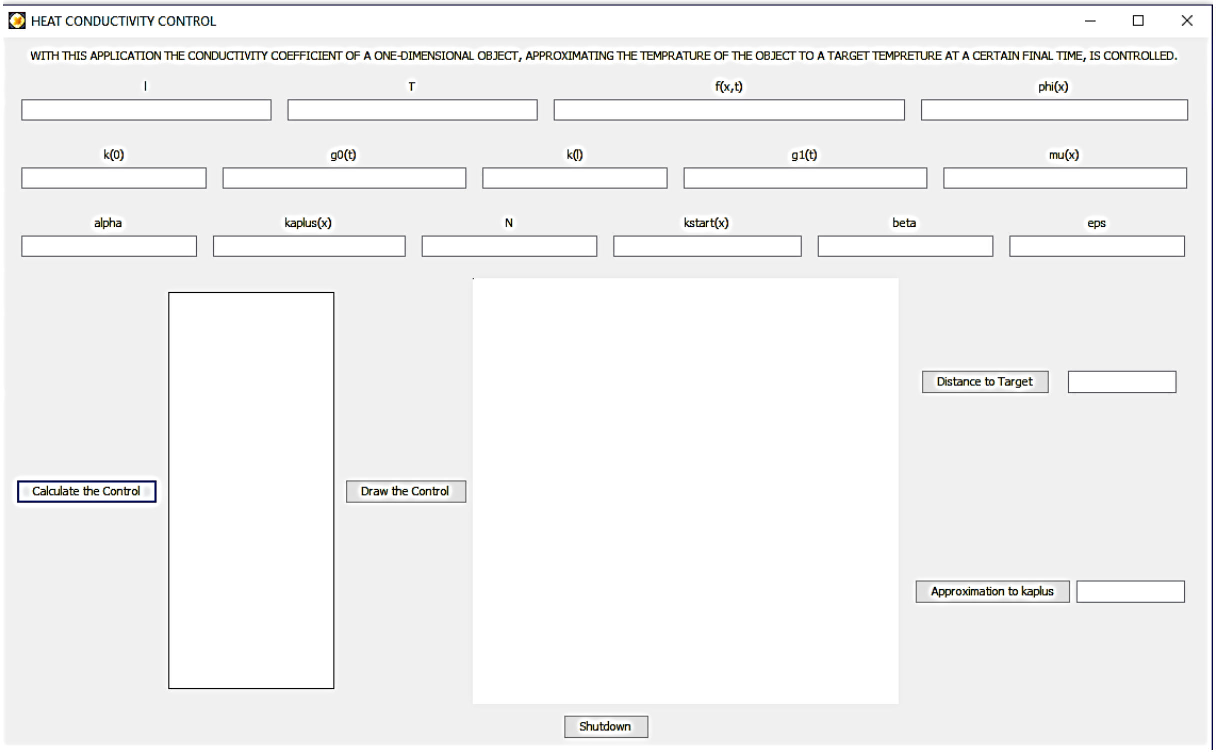


**Figure 1.** Maplet Application Screen

Here the data of the application are such as;

**Table 1.** The data of the application.

| In the Maplet | In the Problem | |
|---|---|---|
| $\boldsymbol{l}$ | $l$ | the length of the interval $(0, l)$ |
| $\boldsymbol{T}$ | $T$ | the length of the interval $(0, T)$ |
| $\boldsymbol{f(x,t)}$ | $f(x,t)$ | external heat source function |
| $\mathbf{phi}(\boldsymbol{x})$ | $\varphi(x)$ | initial heat distribution |
| $\boldsymbol{k}(\mathbf{0})$ | $k(0)$ | coefficient of left heat flux |
| $\boldsymbol{k}(\boldsymbol{l})$ | $k(l)$ | coefficient of left heat flux |
| $\mathbf{g0}(\boldsymbol{t})$ | $g_0(t)$ | left heat flux function |
| $\mathbf{g1}(\boldsymbol{t})$ | $g_1(t)$ | right heat flux function |
| $\mathbf{mu}(\boldsymbol{x})$ | $\mu(x)$ | desired target (final time heat distribution) |
| $\boldsymbol{alpha}$ | $\alpha$ | regularization parameter of the functional $J(k)$ |
| $\mathbf{kaplus}(\boldsymbol{x})$ | $k^+(x)$ | initial guess to the control function |
| $\mathbf{kstart}(\boldsymbol{x})$ | $k_0(x)$ | initial element for the minimizer |
| $\boldsymbol{N}$ | $N$ | the number of finite element used in $h^N$ and $\eta^N$ |
| $\boldsymbol{beta}$ | $\beta$ | the value of parameter used in $k_{m+1}(x)$ |
| $\boldsymbol{eps}$ | $\varepsilon$ | the value of stopping criteria used in $\lvert J^N(k_m) - J^N(k_{m+1}) \rvert \leq \varepsilon$ |

Besides, the outcomes of the application are such as;

**Table 2.** The application results.

| In the Maplet | In the Problem |
|---|---|
| **Draw the Control** | The function $k_*(x) = k_{m+1}(x)$ obeying the stopping criteria |
| **Distance to Target** | Graph of the control function |
| **Approximation to kaplus** | The value of $I(k_*)$ |
| **Draw the Control** | The value of $\|k_* - k^+\|_{L_2(0,l)}$ |

Note that the ability of calculation of the control function depends on the appropriate choice of $N, kstart(x), beta$ and $eps$. If the application does not give the control function these values should be re-arranged.

Now we give a test problem revealing the use and success of given Maplet application.

### 3.1. Problem

Let us consider the following problem on the domain $\Omega = (0,1) \times (0,1)$;

$$\min_{k\in K} \int_0^1 \left[h(x,1;k) - e^{-\pi^2} \sin \pi x\right]^2 dx + \alpha \int_0^1 [k(x) - (1+x)]^2 dx$$

subject to the problem of

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(k(x)\frac{\partial h}{\partial x}\right) = \pi e^{-\pi^2 t}(\pi x \sin \pi x - \cos \pi x), \quad (x,t) \in (0,1) \times (0,1)$$

$$h(x,0) = \sin \pi x \,, x \in (0,1)$$

$$-h_x(0,t) = -\pi e^{-\pi^2}, \; h_x(l,t) = -2\pi e^{-\pi^2}, \; t \in (0,1).$$

Now we will use the given Maplet Application and get the control function. The data given by problem are such as

$$l = 1, T = 1, \mu(x) = e^{-\pi^2} \sin \pi x \,, k^+(x) = 1 + x$$

$$f(x,t) = \pi e^{-\pi^2 t}(\pi x \sin \pi x - \cos \pi x), \varphi(x) = \sin \pi x$$

$$k(0) = 1, g_0(t) = -\pi e^{-\pi^2 t}, k(l) = 2, g_1(t) = -2\pi e^{-\pi^2 t}.$$

Upon this, if we select as

$$\alpha = 0.4, \qquad N = 2, \qquad k_0(x) = 4, \qquad \beta = 0.9, \qquad \varepsilon = 0.0001$$

and enter these into the application then we get the following result;

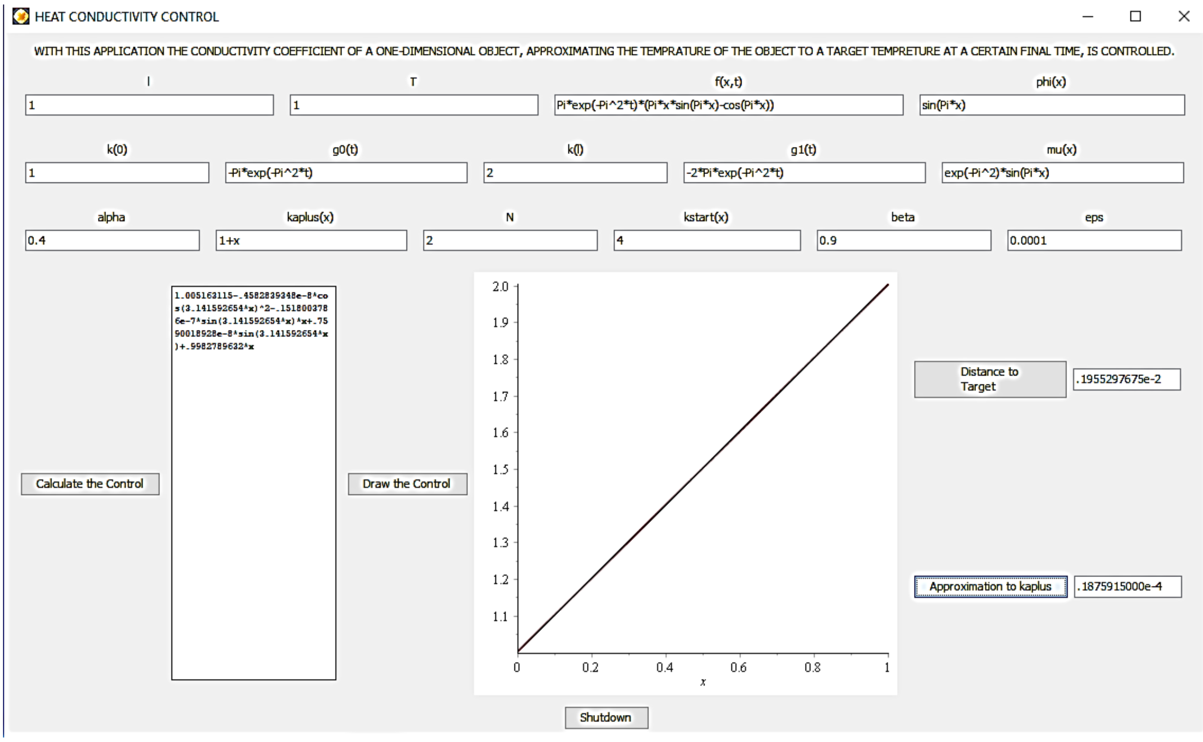


**Figure 2.** Maplet Application Screen for Test Problem

According to this application the control function is

$$k_*(x) = 1.00516311 - 0.458283934(\mathrm{E}-8).\cos^2(3.141592654x) - 0.151800378$$
$$(\mathrm{E}-7).\sin(3.141592654x) + 0.759001892(\mathrm{E}-8).\sin(3.141592654x) + 0.9982789632x$$

Also, the distance to target is $I(k_*) = 0.1955297675(\mathrm{E}-2)$ and approximation to kaplus is $\|k_* - k^+\|_{L_2(0,l)} = 0.1875915000(\mathrm{E}-4)$.

On the other hand, the reason for the choice of $k^+(x) = 1 + x$ is the fact that

$$I(k^+) = \int_0^1 \left[h(x,1;k^+) - e^{-\pi^2}\sin\pi x\right]^2 dx = 0$$

for related heat problem.

Besides, in the following table there are some ‘distance to target’ and ‘approximation to kaplus’ values corresponding some $\alpha$ numbers for this example:

**Table 3.** $I(k_*)$ and $\|k_* - k^+\|^2_{L_2(0,l)}$ values for some regularization parameters.

| $\boldsymbol{\alpha}$ | **Distance to Target** $(\boldsymbol{I(k_*)})$ | **Approximation to kaplus** $(\|\boldsymbol{k_*} - \boldsymbol{k^+}\|^2_{L_2(0,l)})$ |
|---|---|---|
| 0.01 | 0.2027140817(E − 4) | 0.2882505505 |
| 0.015 | 0.3348096495(E − 3) | 0.1548949288 |
| 0.02 | 0.6706345202(E − 3) | 0.8905348472(E − 1) |
| 0.03 | 0.1069109391(E − 2) | 0.4043347441(E − 1) |
| 0.04 | 0.1289383661(E − 2) | 0.2262586246(E − 1) |
| 0.05 | 0.1556482205(E − 2) | 0.1254192760(E − 1) |
| 0.1 | 0.1779405106(E − 2) | 0.2260406516(E − 2) |

The following figure visualizes the approximations in the Table 1.

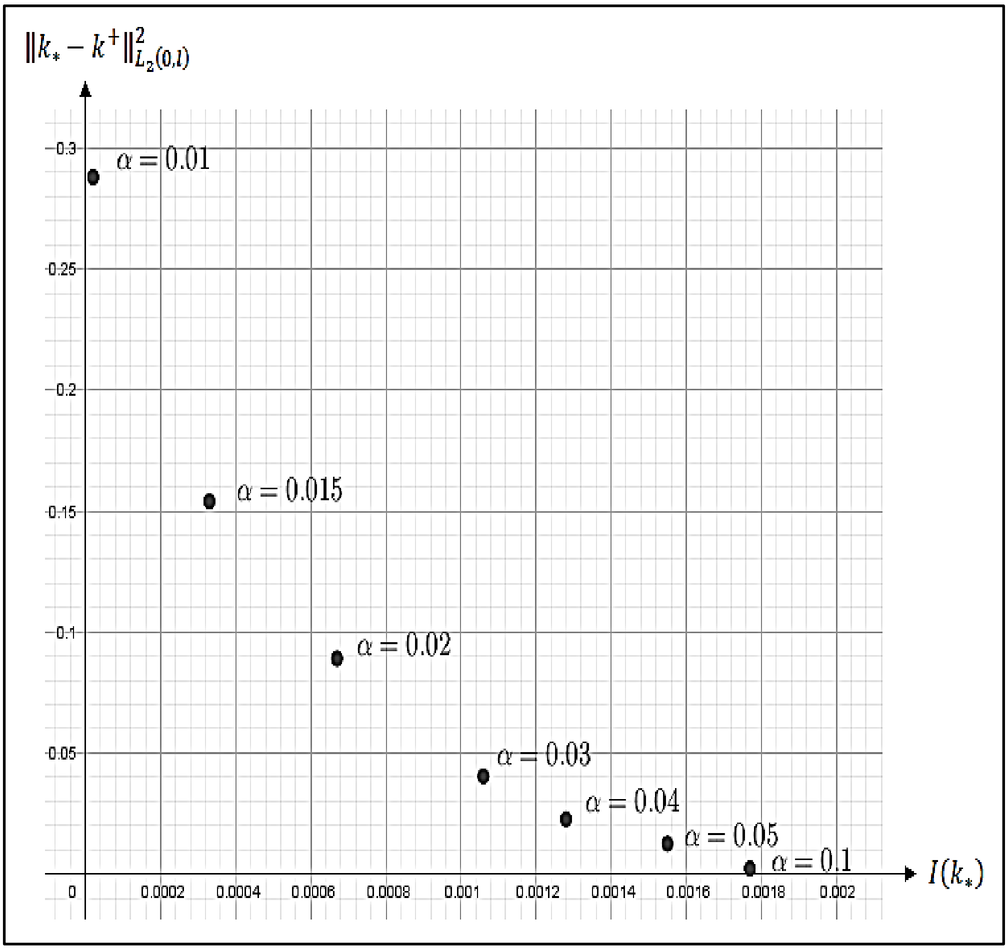


**Figure 3.** Visualization of $I(k_*)$ and $\|k_* - k^+\|^2_{L_2(0,l)}$ by some $\alpha$ values.

Now, using this application, we will examine the ill-posedness for $\alpha = 0$ as we mentioned in section 1.

**Table 4.** Ill-Posedness when $\alpha = 0$.

| $kstart(x)$ | $k_*(x)$ | Distance to Target | Approximation to kaplus |
|---|---|---|---|
| **4** | $4 + 0.1924776275(E-10).\sin^2(3.1415926x)$ $-0.3515450763(E-9).\sin(3.1415926x)\,x$ $+0.1757725382(E-9).\sin(3.1415926x)$ | $0.28959697(E-10)$ | $6.333333333$ |
| **8** | $8 - 0.151513001(E-11).\cos^2(3.1415926x)$ $-0.6456950391(E-10).\sin(3.1415926x)\,x$ $+0.3228475196(E-10).\sin(3.1415926x)$ | $0.11341385(E-10)$ | $42.33333333$ |
| **10** | $10 - 0.71288010(E-12).\cos^2(3.1415926x)$ $-0.3906056418(E-10).\sin(3.1415926x)\,x$ $+0.1953028209(E-10).\sin(3.1415926x)$ | $0.13574858(E-10)$ | $72.33333333$ |

When $\alpha = 0$, the differences at the starting element of the minimizer produce quite different controls which bring the heat closer to the target. In this case, the distances from the obtained controls to the $k^+$ function have been increasing gradually. This situation imposes the numerical ill-posedness.

Let us take the value of $\alpha = 0.4$ and use the application. Then we get the following outcomes;

**Table 5.** The case $\alpha \neq 0$.

| $kstart(x)$ | $k_*(x)$ | Distance to Target | Approximation to kaplus |
|---|---|---|---|
| **4** | $1.005163 - 0.45836(E-8).\cos^2(3.141592x)$ $-0.1518003786(E-7).\sin(3.1415926x)\,x$ $+0.75900189(E-8).\sin(3.1415926x)$ $+0.9982789632x$ | $0.1956610(E-2)$ | $0.1875915(E-4)$ |
| **8** | $1.003373 - 0.47691(E-8).\cos^2(3.141592x)$ $-0.1563177542(E-7).\sin(3.1415926x)x$ $+0.7815887713(E-8).\sin(3.1415926x)$ $+0.9995181097x$ | $0.1955720(E-2)$ | $0.9830589(E-5)$ |
| **10** | $1.004337 - 0.46141(E-8).\cos^2(3.141592x)$ $-0.1525642581(E-7).\sin(3.1415926x)x$ $+0.7628212905(E-8).\sin(3.1415926x)$ $+0.9995181097x$ | $0.1954430(E-2)$ | $0.1679713(E-4)$ |

As seen, the differences in the starting element of the minimizer have no important effect when $\alpha \neq 0$. All the obtained controls are close enough to $k^+$ function and achieve the same degree of convergence to the target. This situation removes the fact of being numerically ill-posedness.

## 4. RESULT AND DISCUSSION

The results of the test problem show that the given Maplet Application HCC works quite efficiently. The outcomes are quite consistent with the control theory. After this point, applications can be prepared for similar types of problems. Thus software that is easy to use for solutions based on a long computation of control problems is developed.

## ACKNOWLEDGEMENTS

The author expresses his gratitude to the editor for providing valuable assistance throughout the publication process.

## CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

## CRediT AUTHOR STATEMENT

**Hakkı Güngör:** Formal analysis, Writing - original draft, Visualization, Conceptualization.

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