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On Absolute Tribonacci Series Spaces and Some Matrix Operators

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Abstract
In this article, the absolute Tribonacci space $|T_\theta|_q$ is introduced as the domain of the Tribonacci matrix on ℓ_q. First, certain algebraic and topological structures such as $BK-$space, isomorphism, duals, and Schauder basis are studied. Then, some characterizations of compact and matrix operators on this space are given their norms, and Hausdorff measures of noncompactness are determined.

Keywords: *Absolute summability, Compact operator, Hausdorff measure of noncompactness, Matrix transformations, Tribonacci matrix*

AMS Subject Classification (2020): *40C05; 46B45; 40F05; 46A45*

1. Introduction

By ω, ℓ_∞, c, ℓ_q $(q>1)$ and ℓ, we stand for the set of all sequences of complex numbers, the sequence space of all bounded, convergent sequences and also for the spaces of all q-absolutely convergent series and absolutely convergent series, respectively. Also, throughout the paper, the abbreviations HM and HMN will be used instead of "Hausdorff measure" and "Hausdorff measure of noncompactness" for brevity and $\mathbb{N}=\{0,1,2,3,...\}$. Let $\Lambda=(\lambda_{nv})$ be an arbitrary infinite matrix of complex components and U, V be two subspaces of ω. If the series

$$\Lambda_n(u)=\sum_{v=0}^{\infty}\lambda_{nv}u_v,$$

converges for all $n\in\mathbb{N}$, then, we define the Λ-transform of the sequence $u=(u_v)$ by $\Lambda(u)=(\Lambda_n(u))$. Also, it is said that Λ defines a matrix transformation from the space U into the space V, and denote it by $\Lambda\in(U,V)$ or $\Lambda:U\to V$ if $\Lambda u=(\Lambda_n(u))\in V$ for every $u\in U$. On the other hand, the $\alpha-,\beta-,\gamma-$ duals of U are defined by

$$U^\alpha=\left\{\epsilon\in\omega:\forall u\in U,\sum_n|\epsilon_n u_n|<\infty\right\},$$

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$$U^{\beta}=\left\{\epsilon\in\omega:\forall u\in U,(\sum_{i=0}^{n}\epsilon_n u_n)\in c\right\},$$

$$U^{\gamma}=\left\{\epsilon\in\omega:\forall u\in U,(\sum_{i=0}^{n}\epsilon_n u_n)\in \ell_\infty\right\}$$

respectively, and the domain of the matrix Λ in U is defined by

$$U_\Lambda=\{u=(u_n)\in\omega:\Lambda(u)\in U\}. \tag{1.1}$$

Further, if U is a complete normed space with continuous coordinates $r_m:U\to\mathbb{C}$ described by $r_m(u)=u_m$ for each $m\in\mathbb{N}$, then it is said that U is a BK-space. If there exists unique sequence of coefficients (u_k) such that, for each $u\in U$,

$$\left\|u-\sum_{k=0}^{p}u_k b_k\right\|\to 0,\ \ p\to\infty$$

then, the sequence (b_k) is called the Schauder basis for U, and it can be written $u=\sum\limits_{k=0}^{\infty}u_k b_k$.

Assume that $1\le q<\infty$ and $\theta=(\theta_n)$ is a sequence of positive terms and also take $\sum u_v$ as an infinite series with its nth partial sum s_n. Then, the series $\sum u_v$ is said to be summable $|\Lambda,\theta_n|_q$, $q\ge 1$, if

$$\sum_{n=0}^{\infty}\theta_n^{q-1}|\Delta\Lambda_n(s)|^q<\infty,$$

where $\Delta\Lambda_n(s)=\Lambda_n(s)-\Lambda_{n-1}(s),\ \ \Lambda_{-1}(s)=0$ (see [1]).

It is clear that this method includes a good number of well known methods for special selections. We refer to reader [2–6]. Recently, the literature of summability theory has expanded in many respects, with many studies using both the summability methods and the absolute summability methods (see [7–19]).

On the other hand, Tribonacci numbers are the sequence of integers identified by the third order recurrence relation with initial conditions $t_0=1,t_1=1,t_2=2$,

$$t_j=t_{j-1}+t_{j-2}+t_{j-3}$$

$$t_{-j}=0, j\ge 1$$

[20]. So, some of the first Tribonacci numbers can be written as follows:

$$1,1,2,4,7,13,24,44,...$$

Besides, Tribonacci numbers have the following useful properties:

$$\sum_{j=0}^{m}t_j=\frac{t_{m+2}+t_m-1}{2}, m\ge 0,$$

$$\sum_{j=0}^{m}t_{2j}=\frac{t_{2m+1}+t_{2m}-1}{2}, m\ge 0,$$

$$\lim_{m\to\infty}\frac{t_m}{t_{m+1}}=0.54368901...$$

Tribonacci matrix $T=(t_{mj})$ has recently been defined by Yaying and Hazarika [19] as follows:

$$t_{mj}=\begin{cases}\frac{2t_j}{t_{m+2}+t_m-1}, & 0\le j\le m\\ 0, & j>m\end{cases}$$

where t_m be the mth Tribonacci number for all $m\in\mathbb{N}$.

Throughout the whole paper, q^* is the conjugate of q, i.e., $1/q+1/q^*=1$ for $q>1$, and $1/q^*=0$ for $q=1$.

Now, let remind certain lemmas which are used in the proof of our theorems.

Lemma 1.1. *[21]* $\Lambda \in (\ell_q, \ell)$ *iff*

$$\|\Lambda\|_{(\ell_q,\ell)} = \sup_{S\in\mathfrak{T}} \left\{ \sum_{k=0}^{\infty} \left| \sum_{n\in S}^{\infty} \lambda_{nk} \right|^{q^*} \right\}^{1/q^*}.$$

where $1 < q < \infty$ *and* $\mathfrak{T}$ *is defined as the collection of all the finite subsets of* $\mathbb{N}$.

While Lemma 1.1 introduces a condition that is very difficult to implement in applications, the following lemma, which gives the equivalent condition, will be more useful in a lot of cases.

Lemma 1.2. *[22]* $\Lambda \in (\ell_q, \ell)$ *iff*

$$\|\Lambda\|'_{(\ell_q,\ell)} = \left\{ \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} |\lambda_{nk}| \right)^{q^*} \right\}^{1/q^*} < \infty,$$

where $1 < q < \infty$. *Moreover since*

$$\|\Lambda\|_{(\ell_q,\ell)} \le \|\Lambda\|'_{(\ell_q,\ell)} \le 4 \|\Lambda\|_{(\ell_q,\ell)},$$

there exists $1 \le \eta \le 4$ *such that* $\|\Lambda\|'_{(\ell_q,\ell)} = \eta \|\Lambda\|_{(\ell_q,\ell)}$.

Lemma 1.3. *[23]* $\Lambda \in (\ell, \ell_q)$ *iff*

$$\|\Lambda\|_{(\ell,\ell_q)} = \sup_k \left\{ \sum_{n=0}^{\infty} |\lambda_{nk}|^q \right\}^{\frac{1}{q}},$$

where $1 \le q < \infty$.

Lemma 1.4. *[21]*

1. $\Lambda \in (\ell, c) \Leftrightarrow$ $(i)\ \lim_n \lambda_{nk}$ *exists for* $k \ge 0$, $(ii)\ \sup_{n,k} |\lambda_{nk}| < \infty$,
2. $\Lambda \in (\ell, \ell_\infty) \Leftrightarrow$ *(ii) holds,*
3. *If* $1 < q < \infty, \Lambda \in (\ell_q, c) \Leftrightarrow$ (i) *holds,* $(iii)\ \sup_n \sum_{k=0}^{\infty} |\lambda_{nk}|^{q^*} < \infty$,
4. *If* $1 < q < \infty, \Lambda \in (\ell_q, \ell_\infty) \Leftrightarrow (iii)$ *holds.*

Let (U, d) be a metric space and $B, P \subset U$. For every $p \in P$, if there exists an $b \in B$ such that $d(p, b) < \varepsilon$ then, B is called an ε-net of P; if B is finite, then the ε-net B of P is called a finite ε-net of P. Assume that U and V are Banach spaces. If domain of a linear operator $\mathcal{S}$ is all of U and, for every bounded sequence (u_n) in U, the sequence $(\mathcal{S}(u_n))$ has a convergent subsequence in V, then the operator $\mathcal{S} : U \to V$ is called compact. $\mathcal{C}(U, V)$ determines the class of all such operators. Assume that Q defines a bounded subset of U. The HMN of Q is described by the number

$$\chi(Q) = inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon - \text{net in } U\},$$

where χ is the HMN.

Lemma 1.5. *[24] Assume that* Q *is a bounded subset of the normed space* U *where* $U = \ell_q$ *for* $1 \le q < \infty$ *or* $U = c_0$. *If* $R_n : U \to U$ *is the operator defined by* $R_n(u) = (u_0, u_1, ...u_n, 0, 0, ...)$ *for all* $u \in U$, *then*

$$\chi(Q) = \lim_{r\to\infty} \left(\sup_{U\in Q} \|(I - R_r)(u)\| \right).$$

Assume that U, V are Banach spaces; χ_1 χ_2 are the HM on these spaces, respectively and $\mathcal{S}: U \to V$ is a linear operator. If $\mathcal{S}(Q) \subset V$ is a bounded set and there exists a positive constant ζ such that $\chi_2(\mathcal{S}(Q)) \leq \zeta\chi_1(\mathcal{S}(Q))$ for all bounded subset $Q \subset U$, then the linear operator $\mathcal{S}: U \to V$ is called (χ_1, χ_2)- bounded. If an operator $\mathcal{S}$ is (χ_1, χ_2)- bounded, then the number

$$\|\mathcal{S}\|_{(\chi_1,\chi_2)} = inf\left\{\zeta > 0 : \chi_2(\mathcal{S}(Q)) \leq \zeta\chi_1(\mathcal{S}(Q)) \text{ for all bounded set } Q \subset U\right\}$$

is called the (χ_1, χ_2)-measure noncompactness of $\mathcal{S}$. Specifically, if $\chi_1 = \chi_2 = \chi$ then we get $\|\mathcal{S}\|_{(\chi,\chi)} = \|\mathcal{S}\|_{\chi}$.

Lemma 1.6. *[25] Assume that U, V are Banach spaces, $\mathcal{S} \in \mathcal{B}(U, V)$ and $S_u = \{u \in U : \|u\| \leq 1\}$ is the unit sphere in U. Then,*

$$\|\mathcal{S}\|_{\chi} = \chi(\mathcal{S}(S_u))$$

and

$$\mathcal{S} \in \mathcal{C}(U, V) \Leftrightarrow \|\mathcal{S}\|_{\chi} = 0.$$

Lemma 1.7. *[26] Assume that U is a normed sequence space, $T = (t_{nv})$ is an infinite triangle, χ_T and χ denote the HMN on M_{U_T} and M_U, the collections of all bounded sets in U_T and U, respectively. Then, for all $Q \in M_{U_T}$,*

$$\chi_T(Q) = \chi(T(Q)).$$

The main purpose of the paper is to establish the space $|T_\theta|_q$ combining Tribonacci matrix given by Yaying and Hazarika [19] and the concept of absolute summability. After the introduction of the space, some inclusion relations are expressed, the $\alpha-, \beta-, \gamma-$ duals and basis of the space are constructed, and also characterizing certain matrix operators related to the space, their norms and HMN are determined.

2. Absolute Tribonacci space $|T_\theta|_q$

In this part of the paper, firstly, the absolute Tribonacci space $|T_\theta|_q$ is introduced and then, some inclusion relations, algebraic and topological structures of the space are investigated.

If we choose the Tribonacci matrix instead of Λ in (1.1), then the summabilty method $|\Lambda, \theta_n|_q$ is reduced to the absolute Tribonacci summability. To put it more clearly, take (s_n) which is a sequence of partial sum of $\sum u_v$. So, we have

$$\Lambda_n(s) = \sum_{j=0}^{n} t_{nj}s_j = \sum_{v=0}^{n} u_v \sum_{j=v}^{n} t_{nj} = \sum_{v=0}^{n} u_v \sum_{j=v}^{n} \frac{2t_j}{t_{n+2}+t_n-1}$$

and so, with a few calculations, we get

$$\begin{aligned} \Delta\Lambda_n(s) &= \sum_{j=0}^{n} u_j \sum_{k=j}^{n} \frac{2t_k}{t_{n+2}+t_n-1} - \sum_{j=0}^{n-1} u_j \sum_{k=j}^{n-1} \frac{2t_k}{t_{n+1}+t_{n-1}-1} \\ &= \frac{2t_n}{t_{n+2}+t_n-1}u_n + \sum_{j=0}^{n-1} u_j \left(\frac{2t_n}{t_{n+2}+t_n-1} + \Delta\sigma_n \sum_{k=j}^{n-1} 2t_k \right) \\ &= \sum_{j=0}^{n} \phi_{nj}u_j \end{aligned}$$

where

$$\sigma_n = \frac{1}{t_{n+2}+t_n-1},$$

$$\phi_{nj} = \begin{cases} \frac{2t_n}{t_{n+2}+t_n-1}, & j = n \\ \frac{2t_n}{t_{n+2}+t_n-1} + \Delta\sigma_n \sum_{k=j}^{n-1} 2t_k, & 0 \leq j \leq n-1 \\ 0, & j > n. \end{cases}$$

Now, we are ready to present the absolute Tribonacci space:

$$|T_\theta|_q = \left\{ u \in \omega : \sum_{n=0}^{\infty} \theta_n^{q-1} \left| \sum_{j=0}^{n} \phi_{nj}u_j \right|^q < \infty \right\}.$$

Besides, it is seen immediately that

$$(F^{(q)} \circ \tilde{T})_n(u) = \theta_n^{1/q^*}(\tilde{T}_n(u) - \tilde{T}_{n-1}(u))$$

where

$$\tilde{t}_{nj} = \begin{cases} \sigma_n \sum_{v=j}^{n} 2t_v, & 0 \leq j \leq n \\ 0, & j > n, \end{cases} \tag{2.1}$$

$$f_{nj}^{(q)} = \begin{cases} \theta_n^{1/q^*}, & j = n \\ -\theta_n^{1/q^*}, & j = n-1 \\ 0, & j \neq n, n-1. \end{cases} \tag{2.2}$$

Taking into account the matrices $\tilde{T} = (\tilde{t}_{nk})$ and $F^{(q)} = (f_{nk}^{(q)})$ and the notation of domain, the space may be written that

$$|T_\theta|_q = (\ell_q)_{F^{(q)} \circ \tilde{T}}.$$

Also, it is known that there exists a unique inverse matrix which also is a triangle for every triangle matrix [27]. So, the matrices $\tilde{T}$ and $F^{(q)}$ have unique inverse matrices $\tilde{T}^{-1} = (\tilde{t}_{nk}^{-1})$ and $(\tilde{F}^{(q)})^{-1} = ((\tilde{f}_{nk}^{(q)})^{-1})$ given by

$$\tilde{t}_{nk}^{-1} = \begin{cases} \frac{1}{2\sigma_n t_n}, & k = n \\ -\frac{1}{2\sigma_{n-1} t_n} - \frac{1}{2\sigma_{n-1} t_{n-1}}, & k = n-1 \\ \frac{1}{2\sigma_{n-2} t_{n-1}}, & k = n-2 \\ 0, & k > n \end{cases} \tag{2.3}$$

$$(\tilde{f}_{nk}^{(q)})^{-1} = \begin{cases} \theta_k^{-1/q^*}, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \tag{2.4}$$

respectively.

Now, to explain a relation between the natural norm of the spaces ℓ_q and the norm of $|T_\theta|_q$, we express the following theorem.

Theorem 2.1. *$|T_\theta|_q$ is BK-space with respect to the norm*

$$\|u\|_{|T_\theta|_q} = \left\| F^{(q)} \circ \tilde{T}(u) \right\|_{\ell_q},$$

where $1 \leq q < \infty$.

Proof. Let $1 \leq q < \infty$. It is known that ℓ_q is a BK-space. Also, since $F^{(q)} \circ \tilde{T}$ is a triangle, it is obtained immediately from Theorem 4.3.2 in [27], $|T_\theta|_q = (\ell_q)_{F^{(q)} \circ \tilde{T}}$ is a BK-space. □

Theorem 2.2. *The sequence $b^{(i)} = (b_n^{(i)})$ is a Schauder basis for the space $|T_\theta|_q$ where*

$$b_n^{(i)} = \begin{cases} \theta_i^{-1/q^*}\left(\frac{1}{2\sigma_n t_n} - \frac{1}{2\sigma_{n-1} t_n} - \frac{1}{2\sigma_{n-1} t_{n-1}} - \frac{1}{2\sigma_{n-2} t_{n-1}}\right), & i \leq n-2 \\ \theta_i^{-1/q^*}\left(\frac{1}{2\sigma_n t_n} - \frac{1}{2\sigma_{n-1} t_n} - \frac{1}{2\sigma_{n-1} t_{n-1}}\right), & i = n-1 \\ \theta_i^{-1/q^*} \frac{1}{2\sigma_n t_n}, & i = n \\ 0, & i > n \end{cases}$$

$1 \leq q < \infty$.

Proof. Let remind that $(e^{(i)})$ is the Schauder basis of ℓ_q. So, it is obtained from Theorem 2.3 in [28] that $b^{(i)} = (\tilde{T}_n^{-1}((F^{(q)})^{-1}(e^{(i)})))$ is a Schauder basis of the absolute space $|T_\theta|_q$. □

Theorem 2.3. *Let $1 \leq q \leq s < \infty$. If there exists a constant $C > 0$ such that $\theta_n \leq C$ for all $n \in \mathbb{N}$, then $|T_\theta|_q \subset |T_\theta|_s$.*

Proof. Take $u \in |T_\theta|_q$. Since $\ell_q \subset \ell_s$, then $\left(\theta_n^{\frac{1}{q^*}} \sum_{j=0}^{n} \phi_{nj} u_j\right) \in \ell_s$ and also, since $\theta_n \le C$ for all $n \in \mathbb{N}$,

$$C^{\frac{s}{q^*}-\frac{s}{s^*}} \left|\theta_n^{\frac{1}{s^*}} \sum_{j=0}^{n} \phi_{nj} u_j\right|^s \le \left|\theta_n^{1/q^*} \sum_{j=0}^{n} \phi_{nj} u_j\right|^s$$

where q^* and s^* are the conjugate of q and s, respectively. Hence, we get that $u \in |T_\theta|_s$. This concludes the proof. □

Theorem 2.4. *The space $|T_\theta|_q$ is isomorphic to the space ℓ_q i.e., $|T_\theta|_q \cong \ell_q$ where $1 \le q < \infty$.*

Proof. To prove the theorem, it should be shown that there exists a linear bijection between the spaces $|T_\theta|_q$ and ℓ_q where $1 \le q < \infty$. Taking into account the transformations $\tilde{T} : |T_\theta|_q \to (\ell_q)_{F^{(q)}}, F^{(q)} : (\ell_q)_{F^{(q)}} \to \ell_q$ and the matrices corresponding to them given in (2.1) and (2.2). Since the matrices $F^{(q)}$ and $\tilde{T}$ are triangles, it is obvious that $\tilde{T}$ and $F^{(q)}$ are linear bijections and also the composite function $F^{(q)} \circ \tilde{T}$ is a linear bijective operator. Moreover,

$$\|u\|_{|T_\theta|_q} = \left\|F^{(q)} \circ \tilde{T}(u)\right\|_q,$$

i.e., the norm is preserved and so the proof is concluded. □

We define

$$D_1 = \left\{\epsilon \in \omega : \theta_v^{-1/k^*} \sum_{j=v+2}^{\infty} \epsilon_j \left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right) \text{ exist for all } v\right\},$$

$$D_2 = \left\{\epsilon \in \omega : \sup_m \left(\frac{1}{\theta_m}\left|\frac{\epsilon_m}{2\sigma_m t_m}\right|^{q^*} + \frac{1}{\theta_{m-1}}|\xi_{m-1}|^{q^*} + \sum_{v=0}^{m-2} \frac{1}{\theta_v}\left|\xi_v + \sum_{j=v+2}^{m} \epsilon_j \left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right|^{q^*}\right) < \infty\right\},$$

$$D_3 = \left\{\epsilon \in \omega : \sup_{m,v}\left\{\left|\frac{\epsilon_m}{\sigma_m t_m}\right| + |\xi_{m-1}| + \left|\xi_v + \sum_{j=v+2}^{m} \epsilon_j \left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right|\right\} < \infty\right\},$$

$$D_4 = \left\{\epsilon \in \omega : \sum_{v=0}^{\infty} \frac{1}{\theta_v}\left(\sum_{j=v+2}^{\infty}\left|\epsilon_j\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right| + \left|\frac{\epsilon_{v+1}}{2\sigma_{v+1}t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_v}\right| + \left|\frac{\epsilon_v}{2\sigma_v t_v}\right|\right)^{q^*} < \infty\right\},$$

$$D_5 = \left\{\epsilon \in \omega : \sup_v \left\{\sum_{j=v+2}^{\infty}\left|\epsilon_j\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right| + \left|\frac{\epsilon_{v+1}}{2\sigma_{v+1}t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_v}\right| + \left|\frac{\epsilon_v}{2\sigma_v t_v}\right|\right\} < \infty\right\},$$

$$\xi_v = \frac{\epsilon_v}{2\sigma_v t_v} + \frac{\epsilon_{v+1}}{2\sigma_{v+1}t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_v}.$$

Similarly, when λ_{jv} is used instead of ϵ_v in above equation, the notation $\xi_v^{(j)}$ will be used instead of ξ_v.

Theorem 2.5. *Let $\theta = (\theta_n)$ be a sequence of positive numbers and $1 < q < \infty$. Then,*

(i) $\{|T_\theta|\}^\alpha = D_5$, $\left\{|T_\theta|_q\right\}^\alpha = D_4$,

(ii) $\{|T_\theta|\}^\beta = D_1 \cap D_3$, $\left\{|T_\theta|_q\right\}^\beta = D_1 \cap D_2$,

(iii) $\{|T_\theta|\}^\gamma = D_3$, $\left\{|T_\theta|_q\right\}^\gamma = D_2$.

Proof. We give the proof of (ii).

(ii) It's known that $\epsilon \in \left\{|T_\theta|_q\right\}^\beta$ iff $(\sum_{j=0}^{m} \epsilon_j u_j) \in c$ for all $u \in |T_\theta|_q$. By the inverse transformations of $\tilde{T}, F^{(q)}$, we get

$$\begin{aligned}
\sum_{j=0}^{m} \epsilon_j u_j &= \sum_{j=0}^{m} \epsilon_j \left(\frac{y_j}{2\sigma_j t_j} - \frac{y_{j-1}}{2\sigma_{j-1} t_j} - \frac{y_{j-1}}{2\sigma_{j-1} t_{j-1}} - \frac{y_{j-2}}{2\sigma_{j-2} t_{j-1}} \right) \\
&= \sum_{v=0}^{m} \sum_{j=v}^{m} \frac{\theta_v^{-1/q^*} \epsilon_j}{2\sigma_j t_j} z_v - \sum_{v=0}^{m-1} \sum_{j=v+1}^{m} \frac{\theta_v^{-1/q^*} \epsilon_j}{2\sigma_{j-1} t_j} z_v - \sum_{v=0}^{m-1} \sum_{j=v+1}^{m} \frac{\theta_v^{-1/q^*} \epsilon_j}{2\sigma_{j-1} t_{j-1}} z_v + \sum_{v=0}^{m-2} \sum_{j=v+2}^{m} \frac{\theta_v^{-1/q^*} \epsilon_j}{2\sigma_{j-2} t_{j-1}} z_v \\
&= \frac{\theta_m^{-1/q^*} \epsilon_m}{2\sigma_m t_m} z_m + \theta_{m-1}^{-1/q^*} \left(\frac{\epsilon_{m-1}}{2\sigma_{m-1} t_{m-1}} + \frac{\epsilon_m}{2\sigma_m t_m} - \frac{\epsilon_m}{2\sigma_{m-1} t_m} - \frac{\epsilon_m}{2\sigma_{m-1} t_{m-1}} \right) z_{m-1} \\
&+ \sum_{v=0}^{m-2} \theta_v^{-1/q^*} \left(\xi_v + \sum_{j=v+2}^{m} \epsilon_j \left(\frac{1}{2t_j} \Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}} \Delta(\frac{1}{\sigma_{j-1}}) \right) \right) z_v \\
&= \sum_{v=0}^{m} d_{mv} z_v \ \ (y = \tilde{T}(u), z = F^{(q)}(y))
\end{aligned}$$

where $D = (d_{mv})$ is defined by

$$d_{mv} = \begin{cases} \theta_v^{-1/q^*} \left(\xi_v + \sum_{j=v+2}^{m} \epsilon_j \left(\frac{1}{2t_j} \Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}} \Delta(\frac{1}{\sigma_{j-1}}) \right) \right), & 0 \le v \le m-2 \\ \theta_{m-1}^{-1/q^*} \xi_{m-1}, & v = m-1 \\ \frac{\theta_m^{-1/q^*} \epsilon_m}{2\sigma_m t_m}, & v = m \\ 0, & v > m. \end{cases}$$

Therefore, $\epsilon \in \left\{|T_\theta|_q\right\}^\beta \Leftrightarrow D \in (\ell_q, c)$. Now, applying Lemma 1.4 to the matrix D, it is obtained that $\left\{|T_\theta|_q\right\}^\beta = D_1 \cap D_2$, which concludes the proof.
The proofs of other parts can be similarly verified, so there is no need for this. □

3. Matrix transformations

In this part of the paper, certain characterizations of matrix and compact operators on the absolute Tribonacci space $|T_\theta|_q$ are investigated and also their norms and HMN are computed.

Theorem 3.1. *Let $1 \le q < \infty$, $\Lambda = (\lambda_{nj})$ be an infinite matrix of complex components for each $n, j \in \mathbb{N}$ and identify the matrix $H^{(n)} = \left(h_{mv}^{(n)}\right)$ by*

$$h_{mv}^{(n)} = \begin{cases} \xi_v^{(n)} + \sum_{j=v+2}^{m} \lambda_{nj} \left(\frac{1}{2t_j} \Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}} \Delta(\frac{1}{\sigma_{j-1}}) \right), & 0 \le v \le m-2 \\ \xi_{m-1}^{(n)}, & v = m-1 \\ \frac{\lambda_{nm}}{2\sigma_m t_m}, & v = m \\ 0, & v > m. \end{cases}$$

Moreover, let $\bar{H} = (\bar{h}_{nv})$ be a matrix whose terms is given by $\bar{h}_{nv} = \lim_m h_{mv}^{(n)}$ and $\widetilde{H} = F^{(q)} \circ \tilde{T} \circ \bar{H}$. Then, $\Lambda \in \left(|T_\theta|, |T_\theta|_q\right)$ if and only if

$$\sum_{j=v+2}^{\infty} \lambda_{nj} \left(\frac{1}{2t_j} \Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}} \Delta(\frac{1}{\sigma_{j-1}}) \right) \text{ exists for all } v \tag{3.1}$$

$$\sup_{m,v} \left\{ \left| \frac{\lambda_{nm}}{2\sigma_m t_m} \right| + \left| \xi_{m-1}^{(n)} \right| + \left| \xi_v^{(n)} + \sum_{j=v+2}^{m} \left(\frac{1}{2t_j} \Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}} \Delta(\frac{1}{\sigma_{j-1}}) \right) \lambda_{nj} \right| \right\} < \infty, \tag{3.2}$$

$$\sup_v \sum_{n=0}^{\infty} \left| \widetilde{h}_{nv} \right|^q < \infty. \tag{3.3}$$

If $\Lambda \in \left(|T_\theta|, |T_\theta|_q \right)$*, then* Λ *is a bounded linear operator,*

$$\|\Lambda\|_{\left(|T_\theta|,|T_\theta|_q\right)} = \left\| \widetilde{H} \right\|_{(l,l_q)},$$

and

$$\|\Lambda\|_\chi = \lim_{v\to\infty} \left\{ \sup_r \left(\sum_{n=v+1}^{\infty} \left| \widetilde{h}_{nr} \right| \right)^q \right\}^{\frac{1}{q}}.$$

Proof. $\Lambda \in \left(|T_\theta|, |T_\theta|_q \right)$ equals to $(\lambda_{nv})_{v=0}^{\infty} \in \{|T_\theta|\}^\beta$ and $\Lambda(u) \in |T_\theta|_q$ for all $u \in |T_\theta|$. It is easy to see from Theorem 2.5 that $(\lambda_{nv})_{v=0}^{\infty} \in \{|T_\theta|\}^\beta$ if and only if the conditions (3.1) and (3.2) hold. In addition to this, if a matrix $S = (s_{nv}) \in (\ell, c)$, then the series $S_n(u) = \sum_{v=0}^{\infty} s_{nv}u_v$ is uniformly convergent in n, because, the remaining term of the series is uniformly tending to zero in n, since

$$\left| \sum_{v=p}^{\infty} s_{nv}u_v \right| \leq \sup_v |s_{nv}| \sum_{v=p}^{\infty} |u_v| \to 0 \ \ (p \to \infty).$$

So we get

$$\lim_n S_n(u) = \sum_{v=0}^{\infty} \lim_n s_{nv}u_v. \tag{3.4}$$

Considering (2.3), (2.4) and (3.4) we get immediately

$$\Lambda_n(u) = \lim_m \sum_{k=0}^{m} \lambda_{nk}u_k = \lim_m \sum_{r=0}^{m} h_{mr}^{(n)} z_r = \sum_{r=0}^{\infty} \bar{h}_{nr} z_r.$$

Moreover, according to Theorem 2.4, since there exists a linear isomorphism between $|T_\theta|_q, \ell_q$ for $1 \leq q < \infty$, it is written that $\Lambda(u) \in |T_\theta|_q$ for all $u \in |T_\theta|$ iff $\bar{H} \in \left(\ell, |T_\theta|_q \right)$, or equivalently, since $|T_\theta|_q = (\ell_q)_{F^{(q)} \circ \tilde{T}}$, $\widetilde{H} \in (\ell, \ell_q)$. Here, the terms of matrix $\hat{H}$ and $\widetilde{H}$ can be stated as

$$\hat{h}_{nr} = \sum_{v=0}^{n} \tilde{t}_{nv}\bar{h}_{vr} = \sum_{v=0}^{n} \sigma_n \sum_{j=v}^{n} 2t_j \bar{h}_{vr},$$

$$\widetilde{h}_{nr} = \theta_r^{1/q^*} \left(\hat{h}_{nr} - \hat{h}_{n-1,r} \right), \ \ n \geq 1 \text{ and } \widetilde{h}_{0r} = \bar{h}_{0r}.$$

So, if we apply Lemma 1.3 to the matrix $\widetilde{H}$, then, we get immediately the condition (3.3), and this concludes the first part of the proof.

Also, if $\Lambda \in \left(|T_\theta|, |T_\theta|_q \right)$, then, since the spaces $|T_\theta|_q$ and $|T_\theta|$ are BK-spaces, Λ determines a bounded operator. For the determination of the operator norm of Λ, take into account the isomorphisms $T : |T_\theta|_q \to (\ell_q)_{F^{(q)}}, F^{(q)} : (\ell_q)_{F^{(q)}} \to \ell_q$ defined as in Theorem 2.4. Then, it can be seen easily that $\Lambda = \widetilde{T}^{-1} \circ \left(F^{(q)}\right)^{-1} \circ \widetilde{H} \circ F^{(1)} \circ T$ and so,

$$\begin{aligned} \|\Lambda\|_{\left(|T_\theta|,|T_\theta|_q\right)} &= \sup_{u\neq 0} \frac{\|\Lambda(u)\|_{|T_\theta|_q}}{\|u\|_{|T_\theta|}} = \sup_{u\neq 0} \frac{\left\|\tilde{T}^{-1}\circ\left(F^{(q)}\right)^{-1}\circ\widetilde{H}\circ F^{(1)}\circ\tilde{T}(u)\right\|_{|T_\theta|_q}}{\|u\|_{|T_\theta|}} \\ &= \sup_{z\neq 0} \frac{\left\|\widetilde{H}(z)\right\|_{\ell_q}}{\|z\|_\ell} = \left\|\widetilde{H}\right\|_{(\ell,\ell_q)} \ (z = F^{(1)} \circ \tilde{T}(u)). \end{aligned}$$

Finally, let Q be a unique ball in $|T_\theta|$. Since $F^{(q)} \circ \tilde{T} \circ \Lambda Q = \widetilde{H} \circ F^{(1)} \circ \tilde{T} Q$, it is written that

$$\begin{aligned}\|\Lambda\|_\chi = \chi(\Lambda Q) &= \chi\left(F^{(q)} \circ \tilde{T} \circ \Lambda Q\right) = \chi\left(\widetilde{H} \circ F^{(1)} \circ \tilde{T} Q\right) \\ &= \lim_{v\to\infty}\left(\sup_{z\in F^{(1)}(\tilde{T}(Q))}\left\|(I-R_v)\left(\widetilde{H}(z)\right)\right\|\right) \\ &= \lim_{v\to\infty}\left\{\sup_r\left(\sum_{n=v+1}^{\infty}\left|\widetilde{h}_{nr}\right|\right)^q\right\}^{\frac{1}{q}}.\end{aligned}$$

□

This completes the proof.

The compact operators in this class are characterized by Theorem 3.1 and Lemma 1.6. Corollary 3.1 gives us the condition:

Corollary 3.1. *Under the hypothesis of Theorem 3.1*

$$\Lambda \in \left(|T_\theta|, |T_\theta|_q\right) \text{ is compact} \Leftrightarrow \lim_{v\to\infty}\left\{\sup_r\left(\sum_{n=v+1}^{\infty}\left|\widetilde{h}_{nr}\right|\right)^q\right\}^{\frac{1}{q}} = 0.$$

Theorem 3.2. *Let $1 < q < \infty$, $\lambda = (\lambda_{nj})$ be an infinite matrix of complex components for each $n, j \in \mathbb{N}$ and $H^{(n)} = \left(h_{mv}^{(n)}\right)$ be as in Theorem 3.1. Also, describe $\bar{E} = (\bar{e}_{nv})$ by $\bar{e}_{nv} = \lim_m \theta_v^{-1/q^*} h_{mv}^{(n)}$ and $\widetilde{E} = F^{(1)} \circ \tilde{T} \circ \bar{E}$. Then, $\Lambda \in \left(|T_\theta|_q, |T_\theta|\right)$ if and only if*

$$\sum_{j=r+2}^{\infty}\lambda_{nj}\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right) \text{ exist for all } r,$$

$$\sup_m\left\{\frac{1}{\theta_m}\left|\frac{\lambda_{nm}}{2\sigma_m t_m}\right|^{q^*} + \frac{1}{\theta_{m-1}}\left|\xi_n^{(m-1)}\right|^{q^*} + \sum_{v=0}^{m-2}\frac{1}{\theta_v}\left|\xi_n^{(v)} + \sum_{j=v+2}^{m}\lambda_{nj}\sum_{v=r+2}^{m}\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right|^{q^*}\right\} < \infty,$$

$$\sum_{r=0}^{\infty}\left(\sum_{n=0}^{\infty}|\widetilde{e}_{nr}|\right)^{q^*} < \infty.$$

Moreover, if $\Lambda \in \left(|T_\theta|_q, |T_\theta|\right)$, then Λ is a bounded linear operator,

$$\|\Lambda\|_{\left(|T_\theta|_q, |T_\theta|\right)} = \left\|\widetilde{E}\right\|_{(\ell_q, \ell)}$$

and

$$\|\Lambda\|_\chi = \frac{1}{\eta}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}|\widetilde{e}_{nr}|\right)^{q^*}\right\}^{\frac{1}{q^*}}$$

where $1 \leq \eta \leq 4$.

Corollary 3.2 gives us the characterization of compact operators with together Lemma 1.6 and Theorem 3.2.

Corollary 3.2. *Under the conditions of Theorem 3.2*

$$\Lambda \in \mathcal{C}\left(|T_\theta|_q, |T_\theta|\right) \Leftrightarrow \frac{1}{\eta}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}|\widetilde{e}_{nr}|\right)^{q^*}\right\}^{\frac{1}{q^*}} = 0.$$

4. Conclusion

Recently, in addition to the studies on sequence spaces obtained as the domain of some special matrices and matrix transformations related to them, new sequence spaces obtained by using the concept of absolute summability method have been introduced in the literature. In this study, the absolute Tribonacci space $|T_\theta|_q$ has been introduced as the domain of the Tribonacci matrix on l_q. Then, some algebraic and topological structure have been studied, certain characterizations of compact and matrix operators on these spaces with their norms and Hausdorff meausures of noncompactness have been given. A different perspective has been generated by including the Tribonacci sequence, which is an interesting number sequence, in the subject.

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