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RESEARCH ARTICLE

ON THE FUZZIFICATION OF GREEK PLANES OF KLEIN QUADRIC

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Abstract

A projective space of dimension 3 over a finite Galois field $GF(q)$ is denoted as $PG(3, q)$. It is defined as the set of all one-dimensional subspaces of 4-dimensional vector space over this Galois field. Klein transformation maps a projective plane of $PG(3,2)$ to a Greek plane of the Klein quadric. This paper introduces the fuzzification of Greek planes passing through the base point, any point on the base line different from the base point, and any point not on the base line of the base plane of 5-dimensional fuzzy projective space.

Keywords

Klein quadric,
Projective spaces,
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1. INTRODUCTION

Zadeh introduced fuzzy sets in [11]. A fuzzy set is a function $\lambda: X \rightarrow [0,1]$. Fuzzy set theory provides a convenient method that is easy to implement in real-time applications, and also allows designers and operators to transfer their knowledge to the geometry. In the study [9], Lubczonok gave fuzzy vector spaces. Fuzzy point, fuzzy projective plane and fuzzy projective spaces from the fuzzy vector spaces were defined by Kuijken and Van Maldeghem in [8]. Akça et all showed the fuzzy line spreads of Fano projective plane in [1]. And then Akça et all gave the classifying fuzzy projective lines and fuzzy projective planes in [2, 3, 5]. Fibered harmonic conjugates, and the condition of Reidemeister in the fibered projective plane are determined [4].

Fuzzification is the method of transforming a crisp projective subspace into a fuzzy projective space. Formal tools for mathematical representation and efficient processing of any information can be provided by fuzzy sets.

This research investigates a fuzzification of the β-planes passing the base point of the Klein quadric.

Step 1. Determine the Greek planes of the Klein quadric passing through the base point of $PG(5,2)$.

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Step 2. In order to obtain a maximum flag, the coordinates of any point in these planes must be selected.

Step 3. The membership degrees of points in each subspace are assigned the appropriate real numbers such that the membership degrees in the 5-dimensional fuzzy projective space are the same.

Step 4. Fuzzification is completed by defining a fuzzy set.

2. PRELIMINARIES

The following definitions and theorem are taken from [6].

Definition 2.1 A projective plane in $PG(3,K)$ is the set of all points (x_1,x_2,x_3,x_4) satisfying a linear equation $a_1x_1+a_2x_2+a_3x_3+a_4x_4=0$. This projective space is denoted by $[a_1,a_2,a_3,a_4]$.

Definition 2.2 A three-dimensional projective space $PG(3,K)$ satisfies the following axioms:
S1-) Any two distinct points are contained in a unique line,
S2-) Every line contains at least three points,
S3-) Three non-collinear points define a unique plane,
S4-) There are four points, no three of which are collinear and not all of which lie in the same plane.
S5-) Any two distinct coplanar lines intersect in a unique point.
S6-) Any line not on a given plane intersects the plane in a unique point.

Theorem 2.3 The points of $PG(3,q)$ over the Galois field of order q have a unique form which is $(1,0,0,0)$, $(x_1,1,0,0)$, $(x_1,x_2,1,0)$, $(x_1,x_2,x_3,1)$ for all $x_1,x_2,x_3\in K$, and the planes of $PG(3,q)$ over the Galois field of order q have a unique form which is $[1,0,0,0]$, $[a_1,1,0,0]$, $[a_1,a_2,1,0]$, $[a_1,a_2,a_3,1]$ for all $a_1,a_2,a_3\in GF(q)$.

Let V be 6-dimensional vector space associated projective space $PG(5,q)$. Now, we will describe a certain connection between the lines of $PG(3,q)$ and the certain points of $PG(5,q)$.

Definition 2.4

Let $M=\begin{bmatrix} O_3 & \frac{1}{2}I_3 \\ \frac{1}{2}I_3 & O_3 \end{bmatrix}$ be the matrix of the Klein quadric**.** Where $X=\begin{bmatrix} x_0\\ x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix}\in PG(5,q)$,

$$\mathrm{H}^5:\ X^TMX=x_0x_3+x_1x_4+x_2x_5=0$$

is the equation of the Klein quadric H^5 in $PG(5,2)$.
Klein transformation maps a line in $PG(3,2)$ to the point $(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12})$ of $PG(5,2)$.
Klein transformation maps 15 points of $PG(3,2)$ into 15 Latin planes of H^5 and 15 planes of $PG(3,2)$ into 15 Greek planes of H^5.
Some results about Latin and Greek planes:

- The points of $PG(3,q)$ are the Latin planes in H^5.
- The points of H^5 are obtained from the lines in PG(3,q) by Klein transformation.
- Greek planes of H^5 are obtained from the planes in $PG(3,q)$ by Klein transformation.
- In H^5, two different Latin planes intersect a point.
- In H^5, two different Greek planes intersect a point.
- In H^5, a Latin and a Greek plane are either disjoint or meet in a line.
- In H^5, there are (q + 1) Latin planes and (q + 1) Greek planes.
- Each line in H^5 is the intersection of the Latin and Greek planes.
- In H^5, every point is contained in 2(q+1) planes such thet half of these planes are Latin and half are Greek planes.

In this section, some basic concepts about fuzzy logic are given.

There is no systematic approach to solving a particular problem. The rules in fuzzy logic control are very dependent on human experience.

There is no specific method for selecting membership functions. The most suitable function is found by trial method.

Definition 2.5 (See [11]) A fuzzy set λ on a set X is mapping $\lambda : X \to [0, 1]: x \to \lambda(x)$. The number $\lambda(x)$ is called the degree of membership of the point x in λ. The intersection of two fuzzy sets λ and λ' on X is given by the fuzzy set $\lambda \wedge \lambda' : X \to [0, 1]: x \to \lambda(x) \wedge \lambda'(x)$, where $\wedge$ denotes the minimum operator.

Let λ be a fuzzy subset of a set X and λ' be a fuzzy subset of a set X', then the fuzzy union of the fuzzy sets λ and λ' is defined as a function $\lambda \cup \lambda' : X \cup X' \to [0,1]$ given by

$$(\lambda \cup \lambda')(x) = \begin{Bmatrix} \max\{\lambda(x), \lambda'(x)\} & if\ x \in X \cap X' \\ \lambda(x) & if\ x \in X\ and\ x \notin X' \\ \lambda'(x) & if\ x \in X'\ and\ x \notin X \end{Bmatrix}.$$

Definition 2.6 Suppose V is an 2-dimensional vector space and let $\lambda: V \to [0, 1]$ be a fuzzy set on V. Then we call λ a fuzzy vector space on V if and only if
$\lambda(a\bar{u} + b\bar{v}) \geq \lambda(\overline{u}) \wedge \lambda(\bar{v}), \forall \bar{u}, \bar{v} \in V$ and $\forall a, b \in K$, [9].
Theorem 2.7 If $\lambda: L \to [0,1]$ is a fuzzy vector line on L, then $\lambda(\bar{u}) = \lambda(\bar{v})$,
$\forall \bar{u}, \bar{v} \in L \backslash \{\bar{0}\}$ and $\lambda(\bar{0}) \geq \lambda(\bar{u}), \forall \bar{u} \in L$.
Theorem 2.8 If $\lambda: V \to [0,1]$ is a fuzzy vector plane on V, then there exists a vector line L of V and real numbers $a_0 \geq a_1 \geq a_2 \in [0,1]$ such that λ is of the following form:

$$\begin{aligned} &\lambda: V \to [0,1] \\ &0 \to a_0 \\ &\bar{u} \to a_1 \text{ for } \bar{u} \in L\backslash\{0\} \\ &\bar{u} \to a_2 \text{ for } \bar{u} \in V\backslash L. \end{aligned}$$

Definition 2.9 Suppose V is an n-dimensional vector space. A flag in V is a sequence of distinct non-trivial subspaces $(U_0, U_1, \dots, U_m)$ such that $U_j \subset U_i$ for all $j < i < n-1$. The rank of a flag is the number of subspaces it contains. A maximal flag in V is a flag of length n.
Theorem 2.10 Let P be a 5-dimensional projective space over a finite field. The fuzzy projective space $[P, \lambda']$, $\lambda' : PG(5,2) \to [0,1]$, with $(q, U_1', U_2', \dots, PG(5,2))$ is a maximal flag in $PG(5,2)$ and $a_1 \geq a_2 \geq \dots \geq a_6$ are reals in $[0,1]$, then the fuzzy 5−dimensional projective space is

$$
\begin{aligned}
\lambda' :\quad & PG(5,2) \to [0,1] \\
& q \to a_1 \\
& p \to a_2 \qquad for\ \ p \in U_1' \setminus \{q\} \\
& p \to a_3 \qquad for\ \ p \in U_2' \setminus U_1' \\
& p \to a_4 \qquad for\ \ p \in U_3' \setminus U_2' \\
& p \to a_5 \qquad for\ \ p \in U_4' \setminus U_3' \\
& p \to a_6 \qquad for\ \ p \in PG(5,2) \setminus U_4'.
\end{aligned}
$$

and the fuzzy Klein quadric $[H^5, \lambda'']$ is

$$
\begin{aligned}
\lambda'' :\quad & H^5 \to [0,1] \\
& q \to a_1 \\
& p \to a_2 \qquad for\ p \in U_1'' \setminus \{q\} \\
& p \to a_3 \qquad for\ \ p \in U_2'' \setminus U_1'' \\
& p \to a_4 \qquad for\ \ p \in U_3'' \setminus U_2'' \\
& p \to a_5 \qquad for\ \ p \in U_4'' \setminus U_3'' \\
& p \to a_6 \qquad for\ \ p \in H^5 \setminus U_4''
\end{aligned}
$$

where $(q, U_1'', U_2'', \ldots, H^5)$ is a chain of subspaces of H^5, in [3].

3. THE FUZZIFICATION OF β_i -PLANES OF KLEIN QUADRIC

Some results on the images under the Klein mapping of the projective 3-space of order 4 and the fuzzification of the Klein quadric in 5-dimensional projective space are given in [3].

Since $PG(3,2)$ has 15 points, 35 lines and 15 planes, the Klein quadric has 35 points, 15 the Latin planes α_i, $i = 1,2,\ldots,15$ and 15 the Greek planes β_i, $i = 1,2,\ldots,15$ in the projective space $PG(5,2)$.

In this section, we will construct the fuzzification of Greek planes passing through the base point, any point on the base line different from the base point, and any point not on the base line of the base plane of $[P, \lambda']$.

Theorem 3.1 Let ϕ be the projective plane in $PG(3,2)$, and the image of this plane under the Klein map be the Greek projective plane β_1 of $PG(5,2)$. If β_1 is the base plane of $[P, \lambda']$ then the fuzzy Greek plane λ_{β_1} is of the following form:

$$
\begin{aligned}
\lambda_{\beta_1} :\quad & \beta_1 \to [0,1] \\
& q \to a_1 \\
& p \to a_2 \qquad for\ p \in U_1'' \setminus \{q\} \\
& p \to a_3, \qquad for\ p \in U_2'' \setminus U_1''.
\end{aligned}
$$

Where (q, U_1'', U_2'') is a chain of subspaces of H^5, and $a_1 \geq a_2 \geq a_3$ are reals in $[0,1]$.

Proof. Let the image of the plane
$\phi: [0,0,0,1] = \{(1,0,0,0),\ (0,1,0,0), (0,0,1,0), (1,1,0,0), (1,0,1,0), (0,1,1,0), (1,1,1,0)\}$ in $PG(3,2)$ be the plane β_1. Since the plane β_1 is also a projective plane, it has seven points and seven lines. The distribution of points forming the maximal flag is obtained as follows:

Let the base point be $q = P_1 =$ (1,0,0,0,0,0), the base line be $L = U_1'' = \{P_1 =$ (1,0,0,0,0,0), $P_2 =$ (1,0,0,0,0,1), $P_3 =$ (0,0,0,0,0,1)}, and the base plane of $[P, \lambda']$ be

$U_2'' = \{P_1 = (1,0,0,0,0,0),\ P_2 = (1,0,0,0,0,1),\ P_3 = (0,0,0,0,0,1),\ P_4 = (1,1,0,0,0,0),\ P_5 = (1,1,0,0,0,1),\ P_6 = (0,1,0,0,0,1),\ P_7 = (0,1,0,0,0,0)\}$.
Since the image of the plane ϕ under the Klein map be the Greek projective plane $U_2'' = \beta_1$ of $PG(5,2)$, then the fuzzy Greek plane λ_{β_1} is of the following form:

$$\begin{aligned} \lambda_{\beta_1}:\ & \beta_1 \to [0,1] \\ & q \to a_1 \\ & p \to a_2 \qquad \textit{for } p \in U_1'' \setminus \{q\} \\ & p \to a_3, \qquad \textit{for } p \in U_2'' \setminus U_1'', \end{aligned}$$

where (q, U_1'', U_2'') is a chain of subspaces of H^5, and $a_1 \geq a_2 \geq a_3$ are reals in $[0,1]$.

Theorem 3.2 Let ϕ , φ and γ be three different projective planes passing through a line in $PG(3,2)$, and the image of these planes under the Klein map be the Greek projective planes β_1, β_2, and β_3 of $PG(5,2)$. If the Greek projective planes β_2, and β_3 intersect the base point of the base plane β_1 of $[P, \lambda']$then the fuzzy Greek planes λ_{β_2} and λ_{β_3} are of the following form:

$$\begin{aligned} \lambda_{\beta_2}:\ & \beta_2 \to [0,1] \\ & q \to a_1 \\ & p \to a_4 \qquad \textit{for } p \in U_3'' \setminus U_2'' \\ & p \to a_5, \qquad \textit{for } p \in U_4'' \setminus U_3'' \end{aligned} \quad \text{and} \quad \begin{aligned} \lambda_{\beta_3}:\ & \beta_3 \to [0,1] \\ & q \to a_1 \\ & p \to a_4 \qquad \textit{for } p \in U_3'' \setminus U_2'' \\ & p \to a_5, \qquad \textit{for } p \in U_4'' \setminus U_3'', \end{aligned}$$

where $(q, U_1'', U_2'', \ldots, H^5)$ is a chain of subspaces of H^5, in [3].

Proof. Let ϕ, φ and γ be three projective planes passing through a line in $PG(3,2)$. Three Greek planes β_1, β_2 and β_3 intersect at the base point $P_1 = (1,0,0,0,0,0)$ of $[P, \lambda'']$.
From Theorem 3.1, the base point $q = P_1 = (1,0,0,0,0,0)$, the base lines $L_1 = \{P_1 =$ (1,0,0,0,0,0), $P_2 =$ (1,0,0,0,0,1), $P_3 =$ (0,0,0,0,0,1)}}, the base plane
$\beta_1 = \{(1,0,0,0,0,1), (1,0,0,0,0,0), (0,0,0,0,0,1), (1,1,0,0,0,1), (1,1,0,0,0,0), (0,1,0,0,0,1), (0,1,0,0,0,0)\}$.
If we chose a point (0,0,0,0,1,0) that is outside the plane β_1 and in the other plane
$\beta_2 = \{(1,0,0,0,0,0), (0,0,0,0,1,0), (1,0,0,0,1,0), (1,0,1,0,0,0), (0,0,1,0,0,0), (1,0,1,0,1,0), (0,0,1,0,1,0)\}$, then 3-dimensional projective subspace that satisfies the following equations is obtained:

$$x_5 = 0,\ x_1 + x_6 = 0, x_1 + x_2 = 0, x_1 + x_2 + x_6 = 0, x_1 = 0, x_2 + x_6 = 0, x_2 = 0, x_6 = 0.$$

That is, the equation of the 3- dimensional projective space obtained from these equations is $U_3'' = [0,0,x_3,x_4,0,0]$.

Also, Let a point in β_3, and outside the planes β_1 and β_2 in the other plane be $(0,1,1,0,0,0)$. Since β_3 is {(1,0,0,0,0,0),(0,1,1,0,1,1),(1,1,1,0,1,1),(0,1,1,0,0,0),(1,1,1,0,0,0),(1,0,0,0,1,1),(0,0,0,0,1,1)}, 4-dimensional projective subspace of $PG(5,2)$ has the equation $U_2'' = [0,0,0,x_4,0,,0]$.

Then the fuzzy Greek planes λ_{β_2} and λ_{β_3} are of the following form:

$$\begin{aligned}\lambda_{\beta_2}: \quad & \beta_2 \to [0,1] \\ & q \to a_1 \\ & p \to a_4 \qquad for\ p \in U_3'' \setminus U_2'' \\ & p \to a_5, \qquad for\ p \in U_4'' \setminus U_3''\end{aligned} \qquad \text{and} \qquad \begin{aligned}\lambda_{\beta_3}: \quad & \beta_3 \to [0,1] \\ & q \to a_1 \\ & p \to a_4 \qquad for\ p \in U_3'' \setminus U_2'' \\ & p \to a_5, \qquad for\ p \in U_4'' \setminus U_3'',\end{aligned}$$

where $(q,U_1'',U_2'',U_3'',U_4'')$ is a chain of subspaces of H^5, and $a_1 \geq a_2 \geq a_3, \geq a_4$ are reals in $[0,1]$.

Corollary 3.3 Let ϕ, φ' and γ' be three projective planes passing through a line in $PG(3,2)$. If the images of three planes under the Klein map are three Greek planes β_1, β_2', and β_3' intersect at the point on the base line different from the base point of $[P,\lambda'']$, then fuzzy Greek plane λ_{β_1} is

$$\begin{aligned}\lambda_{\beta_1}: \quad & \beta_1 \to [0,1] \\ & q \to a_1 \\ & p \to a_2 \qquad for\ p \in L \setminus \{q\} \\ & p \to a_3, \qquad for\ p \in \beta_1 \setminus L\end{aligned}$$

and the fuzzification of each the other two Greek planes of Klein quadric passing through this point is either in the form

$$\begin{aligned}\lambda'_{\beta_2}: \quad & \beta_2' \to [0,1] \\ & q \to a_2 \\ & p \to a_4 \qquad for\ p \in U_3'' \setminus U_2'' \\ & p \to a_5, \qquad for\ p \in U_4'' \setminus U_3'',\end{aligned}$$

or

$$\begin{aligned}\lambda'_{\beta_2}: \quad & \beta_2' \to [0,1] \\ & q \to a_2 \\ & p \to a_4 \qquad for\ p \in U_3'' \setminus U_2'' \\ & p \to a_6, \qquad for\ p \in U_5'' \setminus U_4'',\end{aligned}$$

or

$$\begin{aligned} \lambda'_{\beta_2}: \quad & \beta'_2 \to [0,1] \\ & q \to a_2 \\ & p \to a_5 \qquad \text{for } p \in U''_5 \setminus U''_4 \\ & p \to a_6, \qquad \text{for } p \in H^5 \setminus U''_5, \end{aligned}$$

where $(q, U''_1, U''_2, \dots, H^5)$ is a maximal flag in H^5 and $a_1 \geq a_2 \geq \dots \geq a_6$ are reals in $[0,1]$.

Corollary 3.4 Let ϕ, φ'' and γ'' be three projective planes passing through a line in $PG(3,2)$. If the images of three planes under the Klein map are three Greek planes β_1, β''_2, and β''_3 intersect at the point not on the base line of $[P, \lambda']$, then the fuzzy Greek plane λ_{β_1} is

$$\begin{aligned} \lambda_{\beta_1}: \quad & \beta_1 \to [0,1] \\ & q \to a_1 \\ & p \to a_2 \qquad \text{for } p \in L \setminus \{q\} \\ & p \to a_3, \qquad \text{for } p \in \beta_1 \setminus L \end{aligned}$$

and the fuzzification of each the other two Greek planes of Klein quadric passing through this point is either in the form

$$\begin{aligned} \lambda''_{\beta_2}: \quad & \beta''_2 \to [0,1] \\ & q \to a_3 \qquad \text{for } p \in \beta_1 \setminus L \\ & p \to a_4 \qquad \text{for } p \in U''_3 \setminus U''_2 \\ & p \to a_5, \qquad \text{for } p \in U''_4 \setminus U''_3, \end{aligned}$$

or

$$\begin{aligned} \lambda''_{\beta_2}: \quad & \beta''_2 \to [0,1] \\ & p \to a_3 \qquad \text{for } p \in \beta_1 \setminus L \\ & p \to a_4 \qquad \text{for } p \in U''_3 \setminus U''_2 \\ & p \to a_6, \qquad \text{for } p \in U''_5 \setminus U''_4, \end{aligned}$$

or

$$\begin{aligned} \lambda''_{\beta_2}: \quad & \beta''_2 \to [0,1] \\ & q \to a_3 \qquad \text{for } p \in \beta_1 \setminus L \\ & p \to a_5 \qquad \text{for } p \in U''_5 \setminus U''_4 \\ & p \to a_6, \qquad \text{for } p \in H^5 \setminus U''_5, \end{aligned}$$

where $(q, U''_1, U''_2, \dots, H^5)$ is a maximal flag in H^5 and $a_1 \geq a_2 \geq \dots \geq a_6$ are reals in $[0,1]$.

5. CONCLUSION

Klein transformation maps a point, a line, and a projective plane of $PG(3,2)$ to appoint, a Latin plane, and a Greek plane of the Klein quadric, respectively. We give the fuzzification of Greek planes passing through the base point, any point on the base line different from the base point, and any point not on the base line of the base plane of the Klein quadric in $PG(5,2)$.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Münevvere Mine Karakaya: Formal analysis, Writing - original draft, Visualization, Conceptualization. **Ziya Akça:** Supervision, Visualization, Conceptualization**.**

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