Bitlis Eren Üniversitesi Fen Bilimleri Dergisi
BİTLİS EREN UNIVERSITY JOURNAL OF SCIENCE
ISSN: 2147-3129/e-ISSN: 2147-3188
VOLUME: 13 NO: 3 PAGE: 744-749 YEAR: 2024
DOI:10.17798/bitlisfen.1481490

Approximation Solution for Initial Value Problem of Singularly Perturbed Integro-Differential Equation

Derya ARSLAN[1*]

Department of Mathematics, Faculty of Arts and Sciences, Bitlis Eren University, Bitlis, TR
(ORCID: 0000-0001-6138-0607)

Keywords: ADM, Singularly perturbed equation, Volterra-Fredholm integro-differential equation, Approximate solution.

Abstract
Adomian Decomposition Method (ADM) is used to approximately solve the initial value problem of the singularly perturbed Volterra-Fredholm differential equation. With this method, the desired accurate results are obtained in only a few terms. The approach is simple and effective. An example application is made to demonstrate the effectiveness of ADM. The result obtained is compared with the exact solution. Convergence analysis of the method is performed.

1. Introduction

Volterra-Fredholm integro-differential equations are involved in many different fields of science and engineering: Oceanography, fluid mechanics, electromagnetic theory, finance mathematics, plasma physics, population dynamics, artificial neural networks and biological processes are among these fields [1], [2], [28]. It is quite difficult to solve analytically the Volterra-Fredholm integro-differential equations needed in such fields. Therefore, strong numerical methods must be used. Some of them are Adomian decomposition method, spectral collocation method, Legendre wavelet method, 2D block-pulse functions method, finite difference method, Legendre collocation method, Bernstein polynomials method, homotopy perturbation method [3]-[8]. Existence and uniqueness investigations of the solutions of integro-differential equations have also been done [9], [10].

Singularly perturbed problems are characterized by the fact that the coefficient of the highest-order term in the equation is a very small parameter ε. Their approximate solutions have been studied in many articles and books. The mathematical models seen here are population dynamics, fluid dynamics, heat transport problem, nanofluid, neurobiology, mathematical biology, viscoelasticity and simultaneous control systems etc. They can be listed in many applications in fields [11]-[16]. The perturbation parameter ε in the equation produces unlimited derivatives in the solution. Appropriate numerical methods should be preferred to eliminate this situation [14]-[18]. The fact that the problem examined in this study has both singular perturbation and integro-differential equation properties makes it difficult to obtain an analytical solution. Therefore, the Adomian decomposition method was used in the study to overcome these two difficulties. In the literature, there are studies in which different techniques are applied on singularly perturbed Volterra-Fredholm integro-differential equations: Using the Richardson extrapolation, the convergence of the singularly perturbed Volterra integro-differential equations was obtained in [24]. Durmaz and et al., Fredholm created a finite difference scheme for the integro-differential equation [17]. In recent years, many authors have applied different methods such as homotopy analysis method, modified variational iteration method, Adomian decomposition method, modified homotopy perturbation method to obtain approximate analytical solutions for Volterra, Fredholm, Volterra-Fredholm equations and fuzzy Volterra-Fredholm integro-differential equations [17]-[20].

[*] Corresponding autor: *d.arslan@beu.edu.tr* *Received: 09.05.2024, Accepted: 10.08.2024*

In the study, the following singularly perturbed Volterra-Fredholm integro-differential equation with an initial condition are examined [17]:

$$\varepsilon y' + b(x)y + \int_0^x M(x,s)y(s)ds + \mu\int_0^1 N(x,s)y(s)ds = H(x), \quad (1)$$

$$y(0) = A, \quad (2)$$

where $0 < \varepsilon \ll 1$ is a very small perturbation constant, μ and A are real constants, $(x,s) \in (0,1] \times [0,1]$. We assume that $b > 0$; M, H and N are the sufficiently smooth functions.

2. Material and Method

2.1. ADM and Its Convergence Analysis

George Adomian introduced ADM to solve nonlinear functional equations in the 1980s. These solutions are in the form of infinite power series obtained by a simple formula [21]-[23]. Additionally, Cherruault and Adomian [24], [25], [27] obtained convergence analysis of ADM. Al-Kalla [26] offered a different view on the error analysis of ADM.

It is defined as ADM [21]-[23]: Let F be an ordinary or partial differential operator, which is itself nonlinear, containing linear and nonlinear terms, and let the given function be g: The method should be given in detail and clearly in terms of reproducibility of the study. The methods used should be supported by previously published references. In the study, the changes that will contribute to the method should be explained in detail [3], [4].

$$Fy(t) = g(t). \quad (3)$$

Let's take the equation (3). If this equation is written in decomposed form, the following equation is formed:

$$Ly + Ry + Ny = g, \quad (4)$$

L is the highest order derivative of the given differential equation and its inverse is a linear operator that is easily taken. R is the remaining linear part from the linear operator; N is the nonlinear term in the given differential equation.

If the integral operator L^{-1} is applied from the left side to both sides of equation (4), we have

$$L^{-1}Ly + L^{-1}Ry + L^{-1}Ny = L^{-1}g\,,$$

$$\boldsymbol{L^{-1}Ly = L^{-1}g - L^{-1}Ry - L^{-1}Ny}\,. \quad (\mathbf{5})$$

If the differential equation is $\boldsymbol{nth}$-order, linear differential operator for ordinary differential equations is as

$$L(.) = \frac{d^n}{dt^n}(.),$$

the integral operator is as given below:

$$L^{-1}(.) = \int_0^t (.)\,dt,$$

$$L^{-1}(.) = \int_0^t \int_0^t (.)dtdt\,,$$

$$L^{-1}(.) = \int_0^t \int_0^t \ldots \int_0^t (.)\,dtdt \ldots dt.$$

The nonlinear Ny terms in equation (5) are defines as

$$Ny = \sum_{n=0}^{\infty} A_n,$$

where A_n is Adomian polynomials.

$$y = y(0) + L^{-1}g - L^{-1}Ry - L^{-1}Ny. \quad (6)$$

The decomposed series solution function (6) is obtained by using some calculations in equation (5) and the derivative and integral operators above.

The first term y_0 of the series solution function (6) is obtained using the given initial value and integrating of g as following:

$$y_0 = y(0) + L^{-1}g\,,$$

$$y_{n+1} = -L^{-1}R\ y_n - L^{-1}A_n,\ n = 0,1,2,3,\ldots \quad (7)$$

Then, the terms $y_1, y_2, y_3, \ldots$ are obtained with the help of the above recurrence relation (7) with the initial function y_0,

$$y_1 = -L^{-1}R\,y_0 - L^{-1}A_{0,}$$

$$y_2 = -L^{-1}R\,y_1 - L^{-1}A_1.$$

Finally, it has been obtained the following approximate series solution with the ADM

$$\boldsymbol{y(t)} = \sum_{\boldsymbol{n=0}}^{\infty} \boldsymbol{y_n} = \boldsymbol{y_0} + \boldsymbol{y_1} + \boldsymbol{y_2} + \boldsymbol{y_3} + \cdots \qquad (\boldsymbol{8})$$

Convergence analysis of ADM is done with the definition given below:

Definition 2.1.

$\forall\, i \in N$,

$$\alpha_i = \begin{cases} \frac{\|y_{i+1}\|}{\|y_i\|}, & \|y_i\| \neq 0 \\ 0, & \|y_i\| = 0, \end{cases} \qquad (9)$$

is defined [25].

Corollary 2.1. The approximate series solution converges to the exact solution y for as [25]:

$$0 \leq \alpha_i < 1, \qquad \sum_{n=0}^{\infty} y_i.$$

2.2. Application of the ADM

In this section, firstly, the equation (1) is written in operator form with the help of a linear differential operator. Adomian polynomials A_n are used to linearize the nonlinear terms. Then, recurrence relation is obtained. The $y_0, y_1, y_2, ...$ are written in the sum (8) and the series solution is found.

Application 1:

$$\varepsilon y' + y + \int_0^x xy(s)ds + \frac{1}{10}\int_0^1 y(s)ds = \frac{-\varepsilon}{(x+1)^2} + \frac{1}{x+1} + x\varepsilon\left(1 - e^{\frac{-x}{\varepsilon}}\right) + xln(x+1) + \frac{1}{10}\left(1 - e^{\frac{-1}{\varepsilon}} + \ln(2)\right) \qquad (10)$$

$$y(0) = 2, \qquad (11)$$

This problem (10)-(11) has the following exact solution:

$$y(x) = e^{\frac{-x}{\varepsilon}} + \frac{1}{x+1}.$$

$$L(.) = \frac{d}{dt}(.), \qquad L^{-1}(.) = \int_0^x (.)\,dx.$$

Now let's write the equation (10) in the operator form using the above differential operator

$$Ly = -y - \int_0^x xy(s)ds - \frac{1}{10}\int_0^1 y(s)ds - \frac{\varepsilon}{(x+1)^2} + \frac{1}{x+1} + x\varepsilon\left(1 - e^{\frac{-x}{\varepsilon}}\right) + xln(x+1) + \frac{1}{10}\left(1 - e^{\frac{-1}{\varepsilon}} + \ln(2)\right)$$

Let's apply the left-hand integrate operator to both sides of the above equation.

$$L^{-1}Ly = -L^{-1}y - L^{-1}\left(\int_0^x xy(s)ds - \frac{1}{10}\int_0^1 y(s)ds\right) - L^{-1}\left(\frac{\varepsilon}{(x+1)^2}\right) + L^{-1}\left(\frac{1}{x+1}\right) + L^{-1}\left(x\varepsilon\left(1 - e^{\frac{-x}{\varepsilon}}\right) + xln(x+1)\right) + L^{-1}\left(\frac{1}{10} - \frac{e^{\frac{-1}{\varepsilon}} + \ln(2)}{10}\right), \qquad (12)$$

if we find the value of the $L^{-1}Ly$ here and substitute it, we get the following equation:

$$L^{-1}Ly = \int_0^t y'(x)dx = \varepsilon y(x) - 2\varepsilon. \qquad (13)$$

When the equation (13) is written in equation (12), the following decomposition series solution function is obtained.

If the y_n terms of the decomposition series obtained from the reduction relation are substituted in the series solution function, we have series solution as:

$$y(x) = 2.000000000 + \frac{1}{x+1}(1.8.10^{-29}(-6.127188770$$

$$\times 10^{28} x e^{1.111111111x} - 4.5.10^{28} e^{1.111111111x}$$
$$+ 1.234567901 \times 10^{28} x^3 e^{1.111111111x}$$
$$+3.086419752 \times 10^{28} \ln(x+1)\, e^{1.111111111x})$$
$$+ \cdots + 4.500000000 \times 10^{28} + \cdots \qquad (14)$$

Table 1. The values of the exact solution, approximate solution and error for $\varepsilon = 0.9$

Exact solution	Approximate solution	Error
2.0000000000	2.0000000000	0.00000000
1.8039302260	1.8021043650	0.00182584
1.6340707360	1.6314713230	0.00259941
1.4857620800	1.4844007420	0.00136131
1.3554661030	1.3576017060	0.00213552
1.2404200880	1.2478274050	0.00740736
1.1384171190	1.1515967820	0.01317965
1.0476611180	1.0649416730	0.01728048
0.9666678461	0.9831551520	0.01648714
0.8941952300	0.9005210400	0.00632573
0.8291929878	0.8100025118	0.01919066

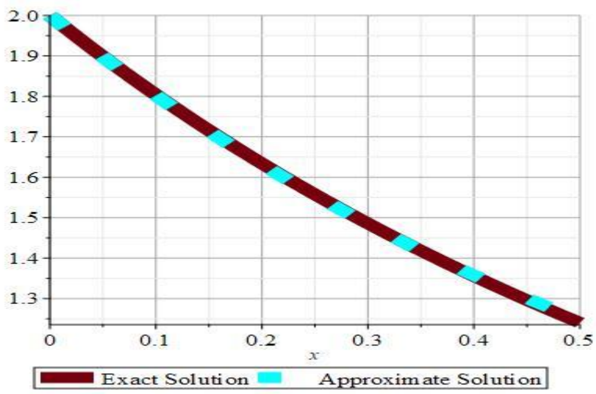

Figure 1. The curves of approximate and exact solution for ε=0.9

An algorithm is built to solve this example using ADM for ε=0.9. This algorithm is solved with a suitable mathematical program. Numerical results were obtained with only 3 iterations. Figure 1 shows that the exact and approximate solution curves overlap. That is, the convergence has achieved. In Table 1, the values of the exact, approximate solution and error are given. Here, it has seen that the error increases as the x values approach towards 1.

To investigate the convergence of Example 1, Definition 2.1 and Corollary 2.1 have been used and this convergence has been showed as follows:

$$\alpha_0 = 0.9831550086 < 1,$$

$$\alpha_1 = 0.9005209752 < 1,$$

$$\alpha_2 = 0.8100025048 < 1.$$
$$\ldots$$

These convergence results are great proof for the accuracy of the approximate solution. This proof shows that ADM is a powerful and well-chosen method in paper.

3. Conclusion and Suggestions

The initial value problem of the Volterra-Fredholm integro-differential equation with singularly perturbed was quickly solved using only three iterations with ADM. Approximate, exact solution and error values are compared in the Table 1 and Figure 1. Convergence analysis was performed. So, the values of α_i, $i = 0,1,2,\ldots$ were found to be less than 1. According to all these results, the method is stable, reliable and useful. In order to contribute to the literature, it can be said that ADM can also be applied to the delayed, fuzzy and fractional types of integral equations.

Conflict of Interest Statement

There is no conflict of interest between the authors.

References

[1] O. Diekmann, "Thresholds and travelling waves for the geographical spread of infection," *J. Math. Biol.*, vol. 6, pp. 109-130, 1978.

[2] E.H. Ouda, S. Shihab and M. Rasheed, "Boubaker wavelet functions for solving higher order integro-differential equations," *J. Southwest Jiaotong Univ.*, vol. 55, pp. 1-12, 2020.

[3] M.K. Kadalbajoo and V. Gupta, “A brief survey on numerical methods for solving singularly perturbed problems,” *Appl. Math. Comput*., vol. 217, pp. 3641-3716, 2010.

[4] M.E. Durmaz, Ö. Yapman, M. Kudu and G. M. Amiraliyev, “An efficient numerical method for a singularly perturbed Volterra-Fredholm integro-differential equation,” *Hacettepe Journal of Mathematics & Statistics*, vol. 52, pp. 326 – 339, 2023.

[5] L.A. Dawood, A.A. Hamoud and N.M. Mohammed, “Laplace discrete decomposition method for solving nonlinear Volterra-Fredholm integro-differential equations,” *J. Math. Computer Sci*., vol. 21, pp. 158-163, 2020.

[6] N.A. Mbroh, S.C. Oukouomi Noutchie and R.Y. M’pika Massoukou, “A second order finite difference scheme for singularly perturbed Volterra integro-differential equation,” *Alex. Eng. J.*, vol. 59, pp. 2441-2447, 2020.

[7] M.S.B. Issa, A.A. Hamoud and K.P. Ghadle, “Numerical solutions of fuzzy integro-differential equations of the second kind,” *J. Math. Computer Sci*., vol. 23, pp. 67-74, 2021.

[8] G. Adomian, “A review of the decomposition method and some recent results for nonlinear equation,” *Math. Comput. Model*., vol. 13, pp. 17, 1992.

[9] G. Adomian and R. Rach, “Analytic solution of nonlinear boundary-value problems in several dimensions by decomposition,” *Journal of Mathematical Analysis and Applications*, vol.174, pp. 118-137, 1993.

[10] G. Adomian, *Solving Frontier problems of physics: The decomposition method*. Kluwer Academic Publishers, Boston: 1994.

[11] Y. Cherruault and G. Adomian, “Decomposition methods a new proof of convergence,” *Mathematical and Computer modelling*, vol. 18, pp. 103- 106, 1993.

[12] Y. Cherruault, “Convergence of Adomian’s method,” *Kybernetes*, vol. 18, pp. 31–38, 1989.

[13] I. El-Kalla, Error analysis of Adomian series solution to a class of non-linear differential equations, *Appl. Math E-Notes*, vol. 7, pp. 214-221, 2007.

[14] M. Cakir and D. Arslan, “The Adomian decomposition method and the differential transform method for numerical solution of multi-pantograph delay differential equations,” *Applied Mathematics*, vol. 6, pp. 1332-1343, 2015.

[15] E. Banifatemi, M. Razzaghi and S. Youse, “Two-dimensional Legendre wavelets method for the mixed Volterra-Fredholm integral equations,” *J. Vib. Control*, vol.13, pp. 1667-1675, 2007.

[16] H. Brunner, "Numerical analysis and computational solution of integro-differential equations." *Contemporary Computational Mathematics*-A Celebration of the 80th Birthday of Ian Sloan, 205-231, 2018.

[17] M. Gülsu, Y. Öztürk and M. Sezer, “A new collocation method for solution of mixed linear integro-differential-difference equations,” *Appl. Math. Comput*., vol. 216, pp. 2183- 2198, 2010.

[18] D.A. Maturi and E.A.M. Simbawa, “The modified decomposition method for solving Volterra Fredholm integro-differential equations using Maple”, *Int. J. GEOMATE*, vol. 18, pp. 84-89, 2020.

[19] B. Raftari, “Numerical solutions of the linear Volterra integro-differential equations: Homotopy perturbation method and finite difference method,” *World Appl. Sci. J*., vol. 9, pp. 7-12, 2010.

[20] A. M. Dalal, “Adomian decomposition method for solving of Fredholm integral equation of the second kind using matlab,” *International Journal of GEOMATE*, vol. 11, pp. 2830-2833, 2016.

[21] A.A. Hamoud and K.P. Ghadle, “Existence and uniqueness of the solution for Volterra-Fredholm integro-differential equations,” *J. Sib. Fed. Univ. - Math. Phys*., vol. 11, pp. 692-701, 2018.

[22] A.H. Mahmood and L.H. Sadoon, “Existence of a solution of a certain Volterra-Fredholm integro differential equations,” *J. Educ. Sci.*, vol. 25, pp. 62-67, 2012.

[23] H.G. Roos, M. Stynes and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations*. Springer-Verlag, Berlin Heidelberg: 2008.

[24] E.R. Doolan, J. J. H. Miller, and W. H. A. Schilders, *Uniform numerical methods for problems with initial and boundary layers*. Dublin: Boole Press, 1980.

[25] G.M. Amiraliyev and I. Amirali, *Nümerik Analiz Teori ve Uygulamalarla*. Ankara: Seçkin Yayıncılık, 2018.

[26] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O’Riordan and G.I. Shishkin, *Robust Computational Techniques for Boundary Layers.* New York: Chapman Hall/CRC, 2000.

[27] R.O. O'Malley, *Singular Perturbations Methods for Ordinary Differential Equations*. New York: Springer-Verlag, 1991.
[28] V. Lakshmikantham, *Theory of Integro-differential Equations*. CRC press, 1995.