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Research Article

Mackey convergence and separation in (L, M)-fuzzy bornological vector spaces

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Abstract

This paper aims to introduce the concepts of Mackey convergence degree for sequences and separation degree for spaces in (L, M)-fuzzy bornological vector spaces. Additionally, the paper presents the concept of bornological closure degree for fuzzy sets. Moreover, the paper discusses various characteristics of these concepts. Furthermore, the paper examines the degree relationships among a Mackey convergence sequence, a separated space, and a bornologically closed fuzzy set. Finally, the paper analyzes the properties of functors ω and ι between M-fuzzifying bornological vector spaces and (L, M)-fuzzy bornological vector spaces in terms of Mackey convergence degree and separation degree.

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1. Introduction

In order to apply the concept of boundedness to a general topological space, Hu[11] originally introduced an axiomatic approach to bornology. In recent years, the theory of general bornological spaces (Hu[12]) has played a key role in research on convergence structures on hyperspaces ([3,6,15]), optimization theory ([4]), and the study of topologies on function spaces ([5, 7, 8, 18]). Nowadays, the theory of bornological spaces is being developed in various directions by many authors.

Zadeh[26] introduced the concept of fuzzy sets, which has since been applied to various branches of mathematics. In 2011, Abel and Šostak[1] generalized the notion of axiomatic bornology to the fuzzy case, which is called L-bornology. In the following years, Paseka et.al.[19] investigated L-bornological vector spaces and demonstrated that for certain complete lattices. After that, Zhang and Zhang[27] introduced the concept of I-bornological vector spaces and discussed two methods for constructing new I-bornological vector spaces. Recently, Jin and Yan[13] proposed L-Mackey convergence and separation

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in L-bornological vector spaces, and discussed an equivalent characterization of separation in terms of L-Mackey convergence.

Šostak[23] presented an alternative approach to the fuzzification of bornology, known as $(L, *)$-valued bornology. Unlike L-bornology, each $(L, *)$-valued bornology on a set X is a mapping from 2^X to L, satisfying L-valued analogues of the axioms of bornology. We refer to this fuzzy bornology as an M-fuzzifying bornology for convenience. Jin and Yan[14] recently introduced the concept of fuzzifying bornological linear spaces and analyzed the necessary and sufficient conditions for compatibility between fuzzifying bornologies and linear structures. In 2023, Liang et.al.[16] introduced the concept of (L, M)-fuzzy bornological vector spaces and demonstrated that the category of M-fuzzifying bornological spaces can be integrated into the category of stratified (L, M)-fuzzy bornological spaces as a coreflective subcategory.

As we have known, Mackey convergence of sequences and separation of spaces are very important notions in the theory of bornological vector spaces. The study of their properties is a recurring theme in the theory of bornological vector spaces. It is natural to see the equivalent notions of Mackey convergence of sequences and separation of spaces in (L, M)-fuzzy bornological vector spaces. This article aims to study the Mackey convergence and separation in the context of (L, M)-bornological vector spaces. This study will contribute to the development of a more systematic theory of (L, M)-bornological vector spaces and explore the potential application of the variational principle in this context.

The paper is structured as follows. Section 2 provides necessary concepts and notations. In section 3, the degrees to which sequences exhibit Mackey convergence and spaces are separated in (L, M)-fuzzy bornological vector spaces are introduced, along with their properties. Additionally, we introduce the degree to which a fuzzy set is bornologically closed and discuss the relationship between separation and bornological closed in (L, M)-fuzzy bornological vector spaces. Section 4 explores the properties of functors ω and ι between M-fuzzifying bornological vector spaces and (L, M)-fuzzy bornological vector spaces on degree of Mackey convergence sequence and degree of separation spaces.

2. Preliminaries

Accoring to the terminology [10], for a and b belonging to a complete lattice L, we say that a is wedge below b [10], denoted as $a \lhd b$, if for any subset $S \subseteq L$, the relation $b \leq \bigvee S$ always implies the existence of $c \in S$ satisfying $a \leq c$. A complete lattice L is called completely distributive if for any $a \in L$, it holds that $b = \bigvee\{a \in L : a \lhd b\}$ [20]. For any $b \in L$, define $\beta(b) = \{a \in L : a \lhd b\}$. It is easy to see that for all $b \in L$, $\beta(b) = \bigcup\limits_{a \lhd b} \beta(a)$. Hence the wedge below relation in a completely distributive lattice has the interpolation property, this means $a \lhd b \Rightarrow \exists c \in L$ such that $a \lhd c \lhd b$. Moreover we can see that $a \lhd \bigvee\limits_{i \in I} b_i$ such that $a \lhd b_i$ for some $i \in I$. Some properties of the map β can be found in [17].

Throughout this paper, L and M always denote completely distributive lattices with an order-reversing involution denoted as $\alpha \mapsto \alpha'$. The smallest and greatest elements of this lattice are denoted as $\perp_L$ and $\top_L$ respectively.

Let X be a non-empty set. Each element in L^X is referred to as an L-fuzzy subset of X. We use $\underline{\lambda}$ to denote an L-fuzzy subset that takes the constant value λ on X. A element λ in L is called prime if the condition $\lambda \geq \alpha \wedge \beta$ implies that $\lambda \geq \alpha$ or $\lambda \geq \beta$, where $\alpha, \beta \in L$. The set of all prime elements in L is denoted by $Pr(L)$. A element λ in L is called co-prime if the condition $\lambda \leq \alpha \vee \beta$ implies that $\lambda \leq \alpha$ or $\lambda \leq \beta$, where $\alpha, \beta \in L$. The set of all nonzero co-prime elements in L is denoted by $J(L)$. The set of all nonzero co-prime elements in L^X is denoted by $J(L^X)$. It is easy to verify that $J(L^X) = \{x_\lambda | x \in X, \lambda \in J(L)\}$, where x_λ represents an L-fuzzy point on X.

In the following, let X be a vector space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and θ denotes the zero element in X.

Definition 2.1 ([2]). Let X be a linear space over $\mathbb{R}$. Consider a fuzzy subset N of $X \times \mathbb{R}$, where N satisfies the condition $\forall x, u \in X, c \in \mathbb{R}$:

(N1) $N(x,t) = 0$ for all $t \leq 0$;

(N2) $x = \theta$ if and only if $N(x,t) = 1$ for all $t > 0$;

(N3) If $c \neq 0$ then $N(cx,t) = N(x, \frac{t}{|c|})$ for all $t \in \mathbb{R}$;

(N4) $N(x+u, s+t) \geq N(x,s) \wedge N(u,t), \forall s,t \in \mathbb{R}$;

(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim\limits_{t\to\infty} N(x,t) = 1$.

Then N is called a fuzzy norm on X and (X, N) called a fuzzy normed space.

Definition 2.2 ([25]). Let $f : X \to Y$ ba a mapping. The L-fuzzification of f, denoted by $f^{\to}$, is defined

$$f^{\to}(A)(y) = \begin{cases} \bigvee\limits_{f(x)=y} A(x), & y \in f(X) \\ 0, & \text{otherwise} \end{cases}, \forall A \in L^X.$$

The L-fuzzification of f is also called Zadeh's type function induced by f and L, it is an order-homomorphism from L^X to L^Y, and

$$f^{\leftarrow}(B)(x) = B(f(x)), \forall B \in L^Y, x \in X.$$

Lemma 2.3 ([25]). *Suppose that $f : X \to Y, A \in L^X, B \in L^Y$, then we have*

(1) $A \leq f^{\leftarrow}(f^{\to}(A))$;

(2) $B \geq f^{\to}(f^{\leftarrow}(B))$;

(3) $f^{\to}(A) \leq B$ *iff* $A \leq f^{\leftarrow}(B)$.

Definition 2.4 ([9]). The addition and scale multiplication operators in L^X are defined as follows, respectively. For $A, B \in L^X$ and $k \in \mathbb{K}$,

$$(A+B)(x) = \bigvee_{s+t=x} (A(s) \wedge B(t));$$

$$(kA)(x) = A(x/k) \text{ whenever } k \neq 0;$$

$$(0A)(x) = \begin{cases} \bigvee\limits_{y\in X} A(y), & x = \theta \\ 0, & x \neq \theta \end{cases}.$$

In particular, for L-fuzzy points, we have $x_\lambda + y_\mu = (x+y)_{\lambda\wedge\mu}$, $kx_\lambda = (kx)_\lambda$.

For all $a \in L$ and $U \in L^X$, we use the following natation: $U^{(a)} = \{x \in X : U(x) \nleq a\}$. Then we have $U = \bigwedge\limits_{a\in L} (\underline{a} \vee \chi_{U^{(a)}})$.

Definition 2.5 ([9, 21]). Let X be a vector space over $\mathbb{K}$. An L-fuzzy set A of L^X is called balanced if $tA \leq A$ for each t with $|t| \leq 1$.

Definition 2.6 ([23,24]). An M-fuzzifying bornology on a set X is a mapping $\mathcal{B} : 2^X \to M$ which satisfies:

(MB1) $\mathcal{B}(\{x\}) = \top_M, \forall x \in X$, (MB2) For each $A, B \in 2^X$, $A \subseteq B \Rightarrow \mathcal{B}(A) \geq \mathcal{B}(B)$,

(MB3) $\mathcal{B}(A \cup B) \geq \mathcal{B}(A) \wedge \mathcal{B}(B), \forall A, B \in 2^X$.

The pair $(X, \mathcal{B})$ is called an M-fuzzifying bornological space. $\mathfrak{B}(A)$ can be interpreted as the degree of boundedness of A.

Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be two M-fuzzifying bornological spaces. A mapping $f : X \to Y$ is called M-fuzzifying bounded provided that $\mathcal{B}_X(A) \leq \mathcal{B}_Y(f(A))$ for all $A \in 2^X$.

Definition 2.7 ([14])**.** An M-fuzzifying bornological vector space is a triple $(X, \mathbb{K}, \mathcal{B})$, where X is a vector space over $\mathbb{K}$, and $(X, \mathcal{B})$ is an M-fuzzifying bornological space such that:

$f : X \times X \to X, (x, y) \mapsto x + y$ is bounded,

$g : \mathbb{K} \times X \to X, (k, x) \mapsto kx$ is bounded,

where $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product M-fuzzifying bornologies $\mathcal{B} \times \mathcal{B}$ and $\mathcal{B}_{\mathbb{K}} \times \mathcal{B}$ (here $\mathcal{B}_{\mathbb{K}}$ is a M-fuzzifying bornology determined by the crisp bornology on $\mathbb{K}$), respectively.

Theorem 2.8 ([14, 22])**.** *Let X be a vector space over $\mathbb{K}$, and $(X, \mathcal{B})$ be an M-fuzzifying bornological space. Then $(X, \mathbb{K}, \mathcal{B})$ is an M-fuzzifying bornological vector space ($\mathcal{B}$ is a linear M-fuzzifying bornology) if and only if $\mathcal{B}$ satisfies the following conditions: $\forall A, B \in 2^X$,*

(MB4) $\mathcal{B}(A) \wedge \mathcal{B}(B) \leq \mathcal{B}(A + B)$,

(MB5) $\mathcal{B}(A) \leq \mathcal{B}(\lambda A), \forall \lambda \in \mathbb{K}$,

(MB6) $\mathcal{B}(A) \leq \mathcal{B}(\bigcup_{|\alpha| \leq 1} \alpha A)$.

Definition 2.9 ([14])**.** *Let $(X, \mathcal{B})$ be an M-fuzzifying bornological space and $\{x^n\}_{n \in \mathbb{N}}$ be a sequence in X. The degree to which x^n is converge bornologically to x is*

$$Borc(x^n, x) = \bigvee_{\substack{B \in Bal(X) \\ s_n \to 0}} \{\mathcal{B}(B) : \forall n \in \mathbb{N}, x^n - x \in s_n B\},$$

where $Bal(X)$ means the family of all balanced sets on X.

Theorem 2.10 ([14])**.** *Suppose that $\{(X_i, \mathcal{B}_i)\}_{i \in I}$ is a family of M-fuzzifying bornological vector spaces, $X = \prod_{i \in I} X_i$ and $p_i : X \to X_i$ is a linear mapping. Define $\mathcal{B} : 2^X \to M$ by $\mathcal{B}(A) = \bigwedge_{i \in I} \{\mathcal{B}_i(A_i) : p_i(A) = A_i\}$. Then $\mathcal{B}$ is a linear M-fuzzifying bornology, which is called the product of $\{\mathcal{B}_i\}_{i \in I}$, denoted by $\mathcal{B} = \prod_{i \in I} \mathcal{B}_i$.*

Theorem 2.11 ([14])**.** *Let $(X, \mathcal{B}_X)$ be an M-fuzzifying bornological vector space and $f : X \to Y$ be a linear mapping. Define $\mathcal{B}_Y : 2^Y \to M$ as follows:*

$$\mathcal{B}_Y(C) = \bigvee_{C \subseteq f(A)} \mathcal{B}_X(A).$$

Then $(X, \mathcal{B}_Y)$ be an M-fuzzifying bornology, denote by $\mathcal{B}_Y = \mathcal{B}_X / f$.

Definition 2.12 ([13, 24])**.** An L-bornological space is a pair $(X, \mathfrak{B})$, where X is a set, and $\mathfrak{B}$ (an L-bornology on X) is a subfamily of L^X (the elements of which are called bounded L-sets), which satisfy the following axioms:

(B1) for every $x \in X$, $\bigvee_{B \in \mathfrak{B}} B(x) = \top_L$;

(B2) given $B \in \mathfrak{B}$ and $D \in L^X$ such that $D \leq B$, it follows that $D \in \mathfrak{B}$;

(B3) if $\mathfrak{S} \subseteq \mathfrak{B}$ is finite, then $\bigvee \mathfrak{S} \in \mathfrak{B}$.

Given L-bornological spaces $(X_1, \mathfrak{B}_1)$ and $(X_2, \mathfrak{B}_2)$, a map $f : X_1 \to X_2$ is called L-bounded provided that $f^{\to}(B_1) \in \mathfrak{B}_2$ for every $B_1 \in \mathfrak{B}_1$.

Definition 2.13 ([13])**.** An L-bornological vector space is a tuple $(X, +, *, \mathfrak{B})$, where $(X, +, *)$ is a vector space over $\mathbb{K}$, and $(X, \mathfrak{B})$ is an L-bornological space such that:

$f : X \times X \to X, (x, y) \mapsto x + y$ is bounded,

$g : \mathbb{K} \times X \to X, (k, x) \mapsto kx$ is bounded,

where $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product L-bornology $\mathfrak{B} \times \mathfrak{B}$ and $\mathfrak{B}_{\mathbb{K}} \times \mathfrak{B}$ (here $\mathfrak{B}_{\mathbb{K}}$ is an L-bornology determined by the crisp bornology on $\mathbb{K}$), respectively.

Theorem 2.14 ([13])**.** *Let $(X, \mathfrak{B})$ be an L-bornological space. Then $(X, \mathfrak{B})$ is an L-bornological vector space ($\mathfrak{B}$ is an L-vector bornology) if and only if $\mathfrak{B}$ satisfies the following conditions:*

(B4) $U, V \in \mathfrak{B} \Rightarrow U + V \in \mathfrak{B}$;

(B5) $\forall t \in \mathbb{K}$, $U \in \mathfrak{B} \Rightarrow tU \in \mathfrak{B}$;

(B6) $U \in \mathfrak{B} \Rightarrow \bigvee_{|t| \leq 1} tU \in \mathfrak{B}$.

Definition 2.15 ([13])**.** Let $(X, \mathfrak{B})$ be an L-bornological vector space, then $(X, \mathfrak{B})$ is separated if and only if $suppM = \{\theta\}$ for all fuzzy vector subspace $M \in \mathfrak{B}$, where $suppM$ is the support set of M.

Theorem 2.16 ([13])**.** *Let $(X, \mathfrak{B})$ be an L-bornological vector space. Then $(X, \iota_\alpha(\mathfrak{B}))$ is a crisp bornological vector space, where $\iota_\alpha(\mathfrak{B}) = \{A^{(\alpha)} : A \in \mathfrak{B}\}$ and $\alpha \in Pr(L)$.*

Lemma 2.17 ([13])**.** *Let $(X, \mathfrak{B})$ be an L-bornological vector space and $M \in \mathfrak{B}$. Then M is a fuzzy vector subspace of X if and only if $M^{(\alpha)}$ is a bounded vector subspace of X for any $\alpha \in Pr(L)$.*

Theorem 2.18 ([13])**.** *Let $(X, \mathfrak{B})$ be an L-bornological vector space, then $(X, \mathfrak{B})$ is separated if and only if $(X, \iota_\alpha(\mathfrak{B}))$ is separated for any $\alpha \in Pr(L)$.*

Definition 2.19 ([16,24])**.** An M-valued L-fuzzy bornology, or an (L, M)-fuzzy bornology for short on a set X is a mapping $\mathscr{B} : L^X \to M$ which satisfies:

(LMB1) $\mathscr{B}(x_{\top_L}) = \top_M$,

(LMB2) For each $A, B \in L^X$, $A \leq B \Rightarrow \mathscr{B}(A) \geq \mathscr{B}(B)$,

(LMB3) $\mathscr{B}(A \vee B) \geq \mathscr{B}(A) \wedge \mathscr{B}(B), \forall A, B \in L^X$.

The pair $(X, \mathscr{B})$ is called an (L, M)-fuzzy bornological space. $\mathscr{B}(A)$ can be interpreted as the degree of boundedness of A.

Let $(X, \mathscr{B}_X)$ and $(Y, \mathscr{B}_Y)$ be two (L, M)-fuzzy bornological spaces. A mapping $f : X \to Y$ is called (L, M)-fuzzy bounded provided that $\mathscr{B}_X(A) \leq \mathscr{B}_Y(f^{\to}(A))$ for all $A \in L^X$.

Theorem 2.20 ([16]). *Let $(X,\mathscr{B})$ be an (L,M)-fuzzy bornological space. Then $\forall a\in Pr(L), \mathscr{B}^{(a)}=\{A\in L^X:\mathscr{B}(A)\nleq a\}$ is an L-bornology on X.*

Definition 2.21 ([16]). An (L,M)-fuzzy bornological vector space is a triple $(X,\mathbb{K},\mathscr{B})$, where X is a vector space over $\mathbb{K}$, and $(X,\mathscr{B})$ is an (L,M)-fuzzy bornological space such that:

$f:X\times X\to X,(x,y)\mapsto x+y$ is (L,M)-fuzzy bounded;

$g:\mathbb{K}\times X\to X,(k,x)\mapsto kx$ is (L,M)-fuzzy bounded,

where $X\times X$ and $\mathbb{K}\times X$ are equipped with the corresponding product (L,M)-fuzzy bornologies $\mathscr{B}\times\mathscr{B}$ and $\mathscr{B}_{\mathbb{K}}\times\mathscr{B}$ (here $\mathscr{B}_{\mathbb{K}}$ is a (L,M)-fuzzy bornology determined by the crisp bornology on $\mathbb{K}$), respectively.

Theorem 2.22 ([16]). *Let X be a vector space over $\mathbb{K}$, and $(X,\mathscr{B})$ be an (L,M)-fuzzy bornological space. Then $(X,\mathbb{K},\mathscr{B})$ is an (L,M)-fuzzy bornological vector space if and only if $\mathscr{B}$ satisfies the following conditions: $\forall A,B\in L^X$,*

(LMB4) $\mathscr{B}(A)\wedge\mathscr{B}(B)\leq\mathscr{B}(A+B)$,

(LMB5) $\mathscr{B}(A)\leq\mathscr{B}(\lambda A),\forall\lambda\in\mathbb{K}$,

(LMB6) $\mathscr{B}(A)\leq\mathscr{B}(\bigvee\limits_{|\alpha|\leq1}\alpha A)$.

Theorem 2.23 ([16]). *Suppose that $\{(X_i,\mathscr{B}_i)\}_{i\in I}$ is a family of (L,M)-fuzzy bornological spaces, $X=\prod\limits_{i\in I}X_i$ and $p_i:X\to X_i$ is the projection. Define $\mathscr{B}:L^X\to M$ by $\mathscr{B}(A)=\bigvee\limits_{A\leq\prod\limits_{i\in I}A_i}\bigwedge\limits_{i\in I}\mathscr{B}_i(A_i),\forall A\in L^X$. Then $(X,\mathscr{B})$ is an (L,M)-fuzzy bornological space, which is called the product space of $\{(X_i,\mathscr{B}_i)\}_{i\in I}$, denoted by $(X,\prod\limits_{i\in I}\mathscr{B}_i)$.*

Remark 2.24. It is easy to know that $\mathscr{B}(A)=\bigwedge\limits_{i\in I}\mathscr{B}_i(p_i(A))$. Since $A(x)\leq\prod\limits_{i\in I}(p_i(A))(x)$, we have $\mathscr{B}(A)=\bigvee\limits_{A\leq\prod\limits_{i\in I}A_i}\bigwedge\limits_{i\in I}\mathscr{B}_i(A_i)\geq\bigwedge\limits_{i\in I}\mathscr{B}_i(p_i(A))$. On the contrary, for every $\prod\limits_{i\in I}A_i\geq A$, by $(\prod\limits_{i\in I}A_i)(x)=\bigwedge\limits_{i\in I}A_i(x_i)$, we know that $(p_i(\prod\limits_{i\in I}(A_i)))(x_i)=\bigvee\limits_{\substack{y\in X\\p_i(y)=x_i}}(\prod\limits_{j\in I}A_j)(y)=\bigvee\limits_{\substack{y\in X\\p_i(y)=x_i}}\bigwedge\limits_{j\in I}A_j(y_j)\leq A_i(x_i)$ for all $i\in I$. It follows that $p_i(A)\leq p_i(\prod\limits_{i\in I}(A_i))\leq A_i$ for all $i\in I$. Then $\bigwedge\limits_{i\in I}\mathscr{B}_i(p_i(A))\geq\bigwedge\limits_{i\in I}\mathscr{B}_i(A_i)$. Hence $\mathscr{B}(A)=\bigvee\limits_{A\leq\prod\limits_{i\in I}A_i}\bigwedge\limits_{i\in I}\mathscr{B}_i(A_i)\leq\bigwedge\limits_{i\in I}\mathscr{B}_i(p_i(A))$.

Theorem 2.25 ([16]). *Let $(X,\mathcal{B})$ be an M-fuzzifying bornological vector space. Define a mapping $\omega(\mathcal{B}):L^X\to M$ by $\omega(\mathcal{B})(A)=\bigwedge\limits_{a\in L}\mathcal{B}(A^{(a)}),\forall A\in L^X$. Then $(X,\omega(\mathcal{B}))$ is an (L,M)-fuzzy bornological vector space.*

Theorem 2.26 ([16]). *Let X be a vector space and $\varphi:2^X\to M$ be a mapping. Define $\mathcal{B}:2^X\to M$ by*

$$\mathcal{B}(A)=\bigwedge\{\mathcal{D}_X(A):\varphi\leq\mathcal{D}_X,(X,\mathbb{K},\mathcal{D}_X)\in\Lambda_X\},\forall A\in 2^X,$$

where Λ_X denotes the family of all M-fuzzifying bornological vector spaces on X. Then $(X,\mathbb{K},\mathcal{B})$ is an M-fuzzifying bornological vector space. In this case, we say φ generates the M-fuzzifying bornological vector space $(X,\mathbb{K},\mathcal{B})$.

Theorem 2.27 ([16]). *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological space. Define $\varphi_{\mathscr{B}} : 2^X \to M$ as follows:*

$$\varphi_{\mathscr{B}}(U) = \bigvee_{a\in L} \bigvee\{\mathscr{B}(A) : A \in L^X, A^{(a)} = U\}, \forall U \in 2^X.$$

Suppose that $\iota(\mathscr{B})$ denotes the M-fuzzifying bornological vector spaces generated by $\varphi_{\mathscr{B}}$. Then $\iota \circ \omega = id$ and $\omega \circ \iota \geq id$.

Example 2.28. Let (X, N) be a fuzzy normed linear space and let A be a fuzzy set of X. The value $\bigwedge_{a\in[0,1)} \bigvee_{t>0} \bigwedge_{x\in A^{(a)}} N(x, t)$ is called the bounded degree of A and is denoted by $Bd(A)$. Then (X, Bd) is an (L, M)-fuzzy bornological vector space.

We need to prove (LMB3), (LMB4), (LMB5) and (LMB6).

(LMB3). For each $\alpha \in [0, 1]$ with $Bd(A) \wedge Bd(B) \geq \alpha$. Then for all $a \in [0, 1)$, exists $t_1, t_2 > 0$ such that $N(x, t_1) \geq \alpha$ and $N(y, t_2) \geq \alpha$ for all $x \in A^{(a)}, y \in B^{(a)}$. Thus, for all $z \in (A \vee B)^{(a)}$, we know $z \in A^{(a)}$ or $z \in B^{(a)}$, which implies that $Bd(A \vee B) \geq \alpha$.

(LMB4). Let $Bd(A) \wedge Bd(B) \geq \alpha$. Then for all $a \in [0, 1)$, exists $t_1, t_2 > 0$ such that $N(x, t_1) > \alpha$ and $N(y, t_2) > \alpha$ for all $x \in A^{(a)}, y \in B^{(a)}$. Since $(A + B)(z) = \bigvee_{x+y=z} A(x) \wedge B(y) \nleq a$, then for all $z \in (A + B)^{(a)}$, there exists x and y with $x + y = z$ such that $x \in A^{(a)}$ and $y \in B^{(a)}$. We obtain that $N(z, t_1 + t_2) \geq N(x, t_1) \wedge N(y, t_2) \geq \alpha$, which implies that $Bd(A + B) \geq \alpha$.

(LMB5). For all $\lambda \in \mathbb{K}$. let $Bd(A) \geq \alpha$. Then for all $a \in [0, 1)$, exists $t > 0$ such that $N(x, t) > \alpha$ for all $x \in A^{(a)}$. For each $y \in (\lambda A)^{(a)}$, we have $\frac{y}{\lambda} \in A^{(a)}$ and $N(y, |\lambda|t) = N(\frac{y}{\lambda}, t) \geq \alpha$ with $\lambda \neq 0$. If $\lambda = 0$, then $Bd(0 \cdot A) = \bigwedge_{a\in[0,1)} \bigvee_{t>0} \bigwedge_{x\in(0\cdot A)^{(a)}} N(x, t) = \bigvee_{t>0} N(\theta, t) = 1$. Hence $Bd(\lambda A) \geq \alpha$ for $\lambda \in \mathbb{K}$.

(LMB6). For all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$. let $Bd(A) \geq \alpha$. Then for all $a \in [0, 1)$, exists $t > 0$ such that $N(x, t) > \alpha$ for all $x \in A^{(a)}$. For each $y \in (\bigvee_{|\lambda|\leq1} \lambda A)^{(a)}$, we have $(\bigvee_{|\lambda|\leq1} \lambda A)(y) \nleq a$. Then exists $|\lambda_0| \leq 1$ such that $(\lambda_0 A)(y) \nleq a$, i.e., $\frac{y}{\lambda_0} \in A^{(a)}$. It follows that $N(y, |\lambda_0|t) = N(\frac{y}{\lambda_0}, t) \geq \alpha$. We get $Bd(\bigvee_{|\lambda|\leq1} \lambda A) \geq \alpha$.

3. Mackey convergence and separation in (L, M)-fuzzy bornological vector spaces

In this section, we will introduce the concepts of Mackey convergence degree for sequences in L^X and separation degree for (L, M)-fuzzy bornological vector spaces. We will study some interesting properties related to these notions. The relationship between Mackey convergence and separation in (L, M)-fuzzy bornological vector spaces will be explored. Additionally, we will define the degree to which an L-fuzzy set is bornologically closed and confirm the close relationship between bornological closure of a space and separation in (L, M)-fuzzy bornological vector spaces.

According to the terminology induced by Liu and Luo [17], a sequence $\{x^n_{\lambda(n)}\}_{n\in\mathbb{N}}$ is called λ-sequence ($\lambda \in J(L)$) if $\lambda = \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n} \lambda(m)$. Where $x^n_{\lambda(n)}$ is defined as follows:

$$x^n_{\lambda(n)}(y) = \begin{cases} \lambda(n), & y = x^n \\ 0, & y \neq x^n \end{cases}, \forall y \in X. \text{ And } \lambda(n) \in J(L) \text{ for every } n \in \mathbb{N}.$$

Definition 3.1. Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space, $\lambda \in J(L)$, $\{x^n_{\lambda(n)}\}_{n\in\mathbb{N}}$ be a λ-sequence in L^X. For every $\alpha \geq \lambda, \alpha \in J(L)$, the degree to which $x^n_{\lambda(n)}$

is Mackey convergence (or converges bornologically) to x_α is

$$Mac(x^n_{\lambda(n)},\ x_\alpha) = \bigvee_{\substack{B\in Bal(L^X)\\ s_n\to 0}} \{\mathscr{B}(B) : \forall n \in \mathbb{N}, (x^n - x)_{\lambda(n)} \not\leq s_n B'\},$$

where $Bal(L^X)$ means the family of all balanced L-fuzzy sets in L^X.

Remark 3.2. If $s < 0$, since B is L-balanced, then $sB'(x) = |s|B'(-x) = |s|B'(x)$ for all $x \in X$. Thus when $s_n < 0$, $(x^n - x)_{\lambda(n)} \not\leq s_n B'$ in the above definition is equivalent to $(x^n - x)_{\lambda(n)} \not\leq |s_n|B'$. Therefore we can only consider the case $s_n \geq 0$ in the following.

Example 3.3. Let $X = l^\infty$ be the space of sequences, $||x|| = \sup_{n\in\mathbb{N}}\{|x_n|\}, ||x||_0 = \sup_{n\in\mathbb{N}}\{\frac{|x_n|}{n}\}$ for all $x = (x_1, x_2, \cdots, x_n, \cdots)$. Then the mapping $N : X \times \mathbb{R} \to [0, 1]$ is defined by

$$N(x,t) = \begin{cases} 0, & t \leq ||x||_0, \\ \frac{1}{2}, & ||x||_0 < t \leq ||x||, \\ 1, & t > ||x||. \end{cases}$$

Then (X, N) is a fuzzy normed vector space. For any fuzzy set A of X, define $Bd(A)$ as Example 2.28. Take $y^n = (0, 0, \cdots, n, 0, \cdots) = ne^n, n \in \mathbb{N}$. Put $U = \{y^n\}_{n\in\mathbb{N}}$, $A = \bigvee_{a\in[0,1)} (\underline{a} \wedge \chi_U)$. It is easy to check $Bd(A) = \bigwedge_{a\in[0,1)} Bd(A^{(a)}) = Bd(U) = \frac{1}{2}$. Since $[e^n_{\lambda(n)} \xrightarrow{b} (0, 0, \cdots, 0, \cdots)_\top] = \bigvee_{\substack{B\in Bal(L^X)\\ s_n\to 0}} \{\mathscr{B}(B) : \forall n \in \mathbb{N}, e^n_{\lambda(n)} \not\leq s_n B'\}$, for all $B \in Bal(L^X)$ and $s_n \to 0$ satisfying $s_n B(e^n) = B(\frac{1}{s_n} e^n) \not\leq (\lambda(n))'$, let $V = \{\frac{1}{s_n} e^n\}_{n\in\mathbb{N}}$. Similarly, we can get $Bd(V) = \frac{1}{2}$. Thus we have $\frac{1}{s_n} e^n \in B^{(\lambda(n))'}$, which implies that $Bd(B) = \bigwedge_{a\in[0,1)} Bd(B^{(a)}) \leq Bd(V) = \frac{1}{2}$. It follows that $Mac(e^n_{\lambda(n)},\ (0, 0, \cdots, 0, \cdots)_\top) \leq \frac{1}{2}$. On the other hand, it is easy to know that $A \in Bal(L^X)$ and $y^n \in A^{(\lambda(n))'}$, i.e., $e^n_{\lambda(n)} \not\leq \frac{1}{n} A'$. So we have $Mac(e^n_{\lambda(n)},\ (0, 0, \cdots, 0, \cdots)_\top) \geq Bd(A) = \frac{1}{2}$. Hence

$$Mac(e^n_{\lambda(n)},\ (0, 0, \cdots, 0, \cdots)_\top) = \tfrac{1}{2}.$$

In the following, we will study some properties of Mackey convergence in (L, M)-fuzzy bornological vector spaces at first.

Theorem 3.4. *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space and $\alpha \in J(L)$, $\{x^n_{\lambda(n)}\}_{n\in\mathbb{N}}$ be a α-sequence in L^X. For any $\lambda \in J(L), \lambda \geq \alpha, t_n, t \in \mathbb{K}$ with $t_n \to t$. The following inequality holds.*

$$Mac(x^n_{\lambda(n)},\ x_\lambda) \leq Mac(t_n x^n_{\lambda(n)},\ tx_\lambda).$$

Proof. For each $a \in M$ with $a \lhd Mac(x^n_{\lambda(n)},\ x_\lambda)$, there exist $B \in Bal(L^X)$ and $s_n \to 0$ satisfying $(x^n - x)_{\lambda(n)} \not\leq s_n B'$ for all $n \in \mathbb{N}$ such that $\mathscr{B}(B) \geq a$. Since $t_n \to t$, there exists $k_0 > 0$ such that $|t_n| < k_0$ for all $n \in \mathbb{N}$. Thus

$$(t_n x^n - tx)_{\lambda(n)} = (t_n x^n - t_n x + t_n x - tx)_{\lambda(n)} \not\leq \max\{|t_n|s_n, t_n - t\}(B + \chi_{\{x\}})'.$$

From $|t_n|B' \geq k_0 B'$, it implies $(t_n x^n - t_n x + t_n x - tx)_{\lambda(n)} \not\leq \max\{k_0 s_n, t_n - t\}(B + \chi_{\{x\}})'$. Since $\mathscr{B}(\chi_{\{x\}}) = \top_M$, we have $\mathscr{B}(B + \chi_{\{x\}}) \geq \mathscr{B}(B) \wedge \mathscr{B}(\chi_{\{x\}}) \geq a$. Therefore $Mac(t_n x^n_{\lambda(n)},\ tx_\lambda) \geq a$. The proof is completed. □

Theorem 3.5. *Let $(X,\mathscr{B})$ be an (L,M)-fuzzy bornological vector space, $\alpha\in J(L)$, $\{x^n_{\lambda(n)}\}_{n\in\mathbb{N}}$ and $\{y^n_{\lambda(n)}\}_{n\in\mathbb{N}}$ be α-sequences in L^X. Then for all $\lambda\geq\alpha, \lambda\in J(L)$,*

$$Mac(x^n_{\lambda(n)},\ x_\lambda)\wedge Mac(y^n_{\lambda(n)},\ y_\lambda)\leq Mac(x^n_{\lambda(n)}+y^n_{\lambda(n)},\ x_\lambda+y_\lambda).$$

Proof. For every $a\in M$ such that $a\lhd Mac(x^n_{\lambda(n)},\ x_\lambda)\wedge Mac(y^n_{\lambda(n)},\ y_\lambda)$, there exist $B_1,B_2\in Bal(L^X)$ and sequences $s_n,t_n\to 0$ such that $(x^n-x)\lambda(n)\nleq s_nB_1'$ and $(y^n-y)\lambda(n)\nleq t_nB_2'$ for all $n\in\mathbb{N}$, satisfying $\mathscr{B}(B_1)\wedge\mathscr{B}(B_2)\geq a$. Since $\lambda(n)\in M(L)$, we obtain $\lambda(n)\nleq s_nB_1'(x^n-x)\vee t_nB_2'(y^n-y)$. Thus, $\lambda(n)\nleq\max\{s_n,t_n\}(B_1+B_2)'(x^n-x+y^n-y)$, which implies $(x^n-x+y^n-y)\lambda(n)\nleq\max\{s_n,t_n\}(B_1+B_2)$. Since $\mathscr{B}(B_1+B_2)\geq\mathscr{B}(B_1)\wedge\mathscr{B}(B_2)\geq a$, we conclude that $Mac(x^n\lambda(n)+y^n_{\lambda(n)},\ x_\lambda+y_\lambda)\geq a$. By considering the arbitrary choice of a, it follows that $Mac(x^n_{\lambda(n)},\ x_\lambda)\wedge Mac(y^n_{\lambda(n)},y_\lambda)\leq Mac(x^n_{\lambda(n)}+y^n_{\lambda(n)},\ x_\lambda+y_\lambda)$.

□

Theorem 3.6. *Let $(X,\mathscr{B})$ be an (L,M)-fuzzy bornological vector space, $\alpha\in J(L)$, $\{x^n_{\lambda(n)}\}_{n\in\mathbb{N}}$ be α-sequence in L^X and a linear mapping $f:X\to Y$ be (L,M)-fuzzy bounded. Then for every $\lambda\in J(L),\lambda\geq\alpha$, $Mac(x^n_{\lambda(n)},\ x_\lambda)\leq Mac(f(x^n)_{\lambda(n)},\ f(x)_\lambda)$.*

Proof. For every $a\in M$ such that $a\lhd Mac(x^n_{\lambda(n)},\ x_\lambda)$, there exists $B\in Bal(L^X)$ and $s_n\to 0$ such that $(x^n-x)_{\lambda(n)}\nleq s_nB'$ for all $n\in\mathbb{N}$. Moreover, $\mathscr{B}(B)\geq a$. Since $f^{\leftarrow}(f^{\rightarrow}(B))\geq B$, it follows that $f^{\leftarrow}((f^{\rightarrow}(B))')=(f^{\leftarrow}(f^{\rightarrow}(B)))'\leq B'$. This implies that $(x^n-x)_{\lambda(n)}\nleq s_nf^{\leftarrow}((f^{\rightarrow}(B))')$. As f is linear, it follows that $(f(x^n)-f(x))_{\lambda(n)}\nleq s_n(f^{\rightarrow}(B))'$. Considering the boundedness of f, we obtain $\mathscr{B}_Y(f^{\rightarrow}(B))\geq\mathscr{B}_X(B)\geq a$. Therefore $Mac(f(x^n)_{\lambda(n)},\ f(x)_\lambda)\geq a$. □

In classical bornological vector space theory, separability is a significant property that is closely associated with Mackey convergence. The separation of a space is sufficient to guarantee the uniqueness of the Mackey convergence limit. Exploring how to draw this conclusion in the fuzzy context is an intriguing and relevant topic. In the following section, we will investigate the relationship between separation and Mackey convergence in (L,M)-fuzzy bornological vector spaces.

Definition 3.7. Let $(X,\mathscr{B})$ be an (L,M)-fuzzy bornological vector space. The degree to which $(X,\mathscr{B})$ is separated is defined as follows:

$$T(X,\mathscr{B})=\bigwedge_{\substack{E^{(\perp)}\neq\{\theta\}\\ E\in Svec(L^X)}}(\mathscr{B}(E))',$$

where $Svec(L^X)$ means the family of all fuzzy vector subspaces in L^X.

Theorem 3.8. *Let $(X,\mathscr{B})$ be an (L,M)-fuzzy bornological vector space. Then $T(X,\mathscr{B})=\bigvee\{\nu\in J(M):\ (X,\mathscr{B}^{(\nu')})$ is a separated L-bornological vector space$\}$.*

Proof. For all $\mu\in J(M)$ with $\mu\lhd T(X,\mathscr{B})$, clearly, $\mu\leq T(X,\mathscr{B})$. Then for each $E^{(\perp)}\neq\{\theta\}$ with $E\in Svec(L^X)$, we have $\mathscr{B}(E)\leq\mu'$. It implies that $(X,\mathscr{B}^{(\mu')})$ is a separated L-bornological vector space. Thus $\mu\in\{\nu\in J(M):\ (X,\mathscr{B}^{(\nu')})$ is a separated L-bornological vector space$\}$. So we obtain $T(X,\mathscr{B})\leq\bigvee\{\nu\in J(M):\ (X,\mathscr{B}^{(\nu')})$ is a separated L-bornological vector space$\}$.

Conversely, if $T(X,\mathscr{B})\ngeq a=\bigvee\{\mu\in J(M):\mu\in\beta(a)\}$, then there exists $\mu\in J(M)$(with $\mu\leq a$) such taht $T(X,\mathscr{B})\ngeq\mu$. Consequently, there exists $E^{(\perp)}\neq\{\theta\}$ with $E\in Svec(L^X)$ such that $\mathscr{B}(E)\nleq\mu'$. This implies that $(X,\mathscr{B}^{(\mu')})$ is not a separated L-bornological vector space. This deduces that $a\nleq\bigvee\{\nu\in J(M):\ (X,\mathscr{B}^{(\nu')})$ is a separated L-bornological vector space$\}$. Otherwise, if $a \leq \bigvee\{\nu \in J(M) : (X, \mathscr{B}^{(\nu')})$ is a separated L-bornological vector space$\}$, then for the aforementioned μ, there exists $\nu \in J(M)$ with $\mu \leq \nu$ such that $(X, \mathscr{B}^{(\nu')})$ is a separated L-bornological vector space. For the above $E^{(\perp)} \neq \{\theta\}$ with $E \in Svec(L^X)$, we have $\mathscr{B}(E) \nleq \mu'$. It follows that $\mathscr{B}(E) \nleq \nu'$, i.e., $(X, \mathscr{B}^{(\nu')})$ is not a separated L-bornological vector space. This leads to a contradiction. Therefore $T(X, \mathscr{B}) \geq \bigvee\{\nu \in J(M) : (X, \mathscr{B}^{(\nu')})$ is a separated L-bornological vector space$\}$. The proof is completed. □

Theorem 3.9. *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space and the linear mappings $f_i : \prod_{i\in I} X_i \to X_i$ be (L, M)-fuzzy bounded for all $i \in I$, where $X = \prod_{i\in I} X_i$.*

(1) If for every $x \in X$ with $x \neq \theta$, there exists $j \in I$ such that $f_j(x) \neq \theta_j$. Then $\bigwedge_{i\in I} T(X_i, \mathscr{B}_i) \leq T(X, \mathscr{B})$.

(2) If there exists $x_0 \in \prod_{i\in I} X_i$ with $x_0 \neq \theta$ such that $f_i(x_0) = \theta_i$ for all $i \in I$. Then $\bigwedge_{i\in I} T(X_i, \mathscr{B}_i) \wedge T(X, \mathscr{B}) = \perp_M$

Proof. (1) By the Definition 3.7, it suffices to prove the following:

$$\bigvee_{\substack{E^{(\perp)}\neq\{\theta\}\\ E\in Svec(L^X)}} \mathscr{B}(E) \leq \bigvee_{i\in I} \bigvee_{\substack{F_i^{(\perp)}\neq\{\theta_i\}\\ F_i\in Svec(L^{X_i})}} \mathscr{B}_i(F_i).$$

For each $a \in M$ with $a \lhd \bigvee_{E^{(\perp)}\neq\theta\ E\in Svec(L^X)} \mathscr{B}(E)$, there exists an $E^{(\perp)} \neq \theta$ with $E \in Svec(L^X)$ such that $\mathscr{B}(E) \geq a$. For all $x \in E^{(\perp)}$ with $x \neq \theta$, there exists an $i \in I$ such that $f_i(x) \neq \theta_i$. Let $F_i = f_i^{\rightarrow}(E)$. It is evident that $f_i^{\rightarrow}(E) \in Svec(L^{X_i})$. Since $f_i^{\rightarrow}(E)(f_i(x)) \geq E(x)$, it follows that $f_i^{\rightarrow}(E)(f_i(x)) \nleq \perp_L$. Consequently, $F_i^{(\perp)} \neq \theta_i$. According to the boundedness of f_i, we have $\mathscr{B}_i(F_i) \geq \mathscr{B}(E) \geq a$. Hence, $\bigvee_{i\in I} \bigvee_{\substack{F_i^{(\perp)}\neq\theta_i\\ F_i\in Svec(L^{X_i})}} \mathscr{B}_i(F_i) \geq a$.

(2) If $\bigwedge_{i\in I} T(X_i, \mathscr{B}_i) \wedge T(X, \mathscr{B}) \neq\perp_M$, then $\bigwedge_{i\in I} T(X_i, \mathscr{B}_i) \neq\perp_M$ and $T(X, \mathscr{B}) \neq\perp_M$. For each $E^{(\perp)} \neq \theta$ with $E^{(\perp)} \in Svec(L^X)$, we have $\mathscr{B}(E) \neq\perp_M$, which implies $\mathscr{B}(E) = \bigwedge_{i\in I} \mathscr{B}_i(f_i^{\rightarrow}(E)) \neq \top_M$. There exists an $i \in I$ such that $\mathscr{B}_i(f_i^{\rightarrow}(E)) \neq \top_M$. Take $E = \mathrm{span}\{x_0\}_{\top_L}$ (where $\mathrm{span}\{x_0\}$ is a space spanned by x_0, i.e., $U = hx_0$, for $h \in \mathbb{K}$). Then, there exists an $i \in I$ such that $\mathscr{B}_i(f_i^{\rightarrow}(E)) = \mathscr{B}_i((f_i(\mathrm{span}\{x_0\}))_{\top_L}) = \mathscr{B}_i((\theta_i)_{\top_L}) \neq \top_M$. This contradicts $\forall i \in I, \mathscr{B}i((\theta_i)_{\top_L}) = \top_M$. □

The above theorem we investigate how the property of an (L, M)-fuzzy bornology being separated behaves with respect to the fundamental constructions described.

Theorem 3.10. *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space, X_1 be a subspace of X and $\mathscr{B}_1(A_1) = \bigvee_{A\cap X_1=A_1} \mathscr{B}(A)$ for all $A_1 \in L^{X_1}$. Then $T(X, \mathscr{B}) \leq T(X_1, \mathscr{B}_1)$.*

Proof. First, we prove $(X_1, \mathscr{B}_1)$ is an (L, M)-fuzzy bornological vector space. It only need to prove $(LMB4)$ and $(LMB5)$.

$(LMB4)$. For each $\alpha \in M$ with $a \lhd \mathscr{B}_1(A_1) \wedge \mathscr{B}_1(B_1)$, then exist $A, B \in L^X$ with $A \cap X_1 = A_1$ and $B \cap X_1 = B_1$ such that $\mathscr{B}(A) \wedge \mathscr{B}(B) \geq a$. Obviously, $(A + B) \cap X_1 = (A \cap X_1) + (B \cap X_1) = A_1 + B_1$. Since $\mathscr{B}(A + B) \geq \mathscr{B}(A) \wedge \mathscr{B}(B) \geq a$, it follows that $\mathscr{B}_1(A_1 + B_1) \geq a$. Thus $\mathscr{B}_1(A_1) \wedge \mathscr{B}_1(B_1) \leq \mathscr{B}_1(A_1 + B_1)$.

$(LMB5)$. For all $\lambda \in \mathbb{K}, A_1 \in L^{X_1}$, clearly,

$$\begin{aligned}\mathscr{B}_1(A_1) = \bigvee_{A\cap X_1 = A_1} \mathscr{B}(A) &\leq \bigvee_{A\cap X_1=A_1} \mathscr{B}(\lambda A)\\ &\leq \bigvee_{(\lambda A)\cap X_1=\lambda A_1} \mathscr{B}(\lambda A) = \mathscr{B}_1(\lambda A_1).\end{aligned}$$

Hence $(X_1, \mathscr{B}_1)$ is an (L, M)-fuzzy bornological vector space. For each $\nu \in Pr(M)$ with $\nu \geq T(X_1, \mathscr{B}_1)$, since

$$T(X_1, \mathscr{B}_1) = \bigwedge_{\substack{C_1^{(\perp)} \neq \{\theta\}\\ C_1 \in Svec(L^{X_1})}} (\mathscr{B}_1(C_1))',$$

there exists $C_1^{(\perp)} \neq \{\theta\}$ with $C_1 \in Svec(L^{X_1})$ such that $(\mathscr{B}_1(C_1))' \leq \nu$, i.e., $\nu' \leq \mathscr{B}_1(C_1) = \bigvee_{C\cap X_1=C_1} \mathscr{B}(C)$. Then there exists $C \in L^X$ with $C \cap X_1 = C_1$ Let $D(x) = \begin{cases} C(x), & x \in X_1 \\ \perp_L, & x \notin X_1 \end{cases}$. It is easy to find $D \in Svce(L^X)$ and $D^{(\perp)} \neq \{\theta\}$. In addition, $\mathscr{B}(D) \geq \mathscr{B}(C) \geq \nu'$. So $T(X, \mathscr{B}) \leq \nu$. By the arbitrariness of ν, we have $T(X, \mathscr{B}) \leq T(X_1, \mathscr{B}_1)$. □

Theorem 3.11. *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space. Then*

$$\bigwedge_{\substack{\{x^n_{\lambda(n)}\}\subseteq L^X, \lambda\geq\alpha, \lambda,\alpha\in J(L)\\ \alpha=\bigwedge_{n\in\mathbb{N}}\bigvee_{m\geq n}\lambda(m), x\neq y}} \Big(Mac(x^n_{\lambda(n)},\ x_\lambda) \wedge Mac(x^n_{\lambda(n)},\ y_\lambda)\Big)' \leq T(X, \mathscr{B}).$$

Proof. The above result is equivalent to

$$\bigvee_{\substack{E^{(\perp)}\neq\{\theta\}\\ E\in Svec(L^X)}} \mathscr{B}(E) \leq \bigvee_{x\neq y} \bigvee_{\substack{\{x^n_{\lambda(n)}\}\subseteq L^X, \lambda\geq\alpha, \lambda,\alpha\in J(L)\\ \alpha=\bigwedge_{n\in\mathbb{N}}\bigvee_{m\geq n}\lambda(m)}} \bigvee_{\substack{B_1,B_2\in Bal(L^X)\\ s_n,t_n\to 0}} \Big\{\mathscr{B}(B_1) \wedge \mathscr{B}(B_2) : \forall n \in \mathbb{N},\ (x^n - x)_{\lambda(n)} \not\leq s_n B_1', (x^n - y)_{\lambda(n)} \not\leq t_n B_2'\Big\}.$$

For each $a \in M$ with $a \lhd \bigvee_{\substack{E^{(\perp)}\neq\{\theta\}\\ E\in Svec(L^X)}} \mathscr{B}(E)$. Then there exists $E^{(\perp)} \neq \{\theta\}$ with $E \in Svec(L^X)$ such that $\mathscr{B}(E) \geq a$. Fixed $x \in E^{(\perp)}, x \neq \theta$, there exists $\mu \in Pr(L)$ such that $x \in E^{(\mu)}$, i.e., $x_{\mu'} \not\leq E'$. Let $x^n_{\lambda(n)} = x_{\mu'}$ and $s_n = \frac{1}{n}$, it is clear $\theta_{\top_L} \in Bal(L^X)$ and $(x^n - x)_{\lambda(n)} \not\leq (s_n\theta_{\top_L})'$ for all $n \in \mathbb{N}$, we have $\mathscr{B}(\theta_{\top_L}) = \top_M$. On the other hand, let $B_2 = E$, then $x^n_{\lambda(n)} = x_{\mu'} \not\leq E' = \frac{1}{n}E' = B_2'$, which implies that $\mathscr{B}(\theta_{\top_L}) \wedge \mathscr{B}(B_2) \geq a$. Then the inequality is established. □

Theorem 3.12. *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space, $\alpha \in J(L)$, $\{x^n_{\lambda(n)}\}_{n\in\mathbb{N}}$ be α-sequence in L^X and L be a chain. Then for all $\lambda \in J(L), \lambda \geq \alpha$ and $x \neq y$, the following equality holds:*

$$Mac(x^n_{\lambda(n)},\ x_\lambda) \wedge Mac(x^n_{\lambda(n)},\ y_\lambda) \wedge T(X, \mathscr{B}) = \perp_M .$$

Proof. It needs to prove that if $T(X, \mathscr{B}) = \top_M$, the equality

$$Mac(x^n_{\lambda(n)},\ x_\lambda) \wedge Mac(x^n_{\lambda(n)},\ y_\lambda) = \perp_M \text{ holds.}$$

If $Mac(x^n_{\lambda(n)},\ x_\lambda) \wedge Mac(x^n_{\lambda(n)},\ y_\lambda) \neq \perp_M$, there exists $a \in J(M)$ such that $a \lhd Mac(x^n_{\lambda(n)},\ x_\lambda) \wedge Mac(x^n_{\lambda(n)},\ y_\lambda)$. Furthermore, there exist $B_1, B_2 \in Bal(L^X)$ and

$s_n, t_n \to 0$ with $(x^n - x)_{\lambda(n)} \not\leq s_n B_1'$ and $(x^n - y)_{\lambda(n)} \not\leq t_n B_2'$ for all $n \in \mathbb{N}$ such that $\mathscr{B}(B_1) \wedge \mathscr{B}(B_2) \geq a$. Hence, we have $\lambda(n) \not\leq s_n B_1'(x^n - x) \vee t_n B_2'(x^n - y)$. Thus, $\lambda(n) \not\leq \max\{s_n, t_n\}(B_1 + B_2)'(y - x)$, i.e., $(y - x)_{\lambda(n)} \not\leq \max\{s_n, t_n\}(B_1 + B_2)'$. Since $\lambda \geq \alpha = \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n} \lambda(m)$, for all $\mu \in J(L)$ with $\lambda \lhd \mu$, there exists $n_0 \in \mathbb{N}$ such that $\lambda(m) \leq \mu$ for each $m \geq n$. Given that $\max\{s_n, t_n\} \to 0$, there is $n_1 \geq n_0$ such that $(B_1 + B_2) \geq \max\{s_n, t_n\}(B_1 + B_2)$ whenever $n \geq n_1$. Thus, $y - x \in (B_1 + B_2)^{(\mu')}$ and $\max\{s_n, t_n\}(B_1 + B_2)(y - x) \not\leq \mu'$ for all $n \geq n_1$. Denote $U = \mathrm{span}\{y - x\}$ ($\mathrm{span}\{y - x\}$ is a space spanned by $y - x$, i.e., $U = h(y - x)$, for $h \in \mathbb{K}$), we claim that $U \subseteq (B_1 + B_2)^{(\mu')}$. Indeed, for all $k \in \mathbb{K}$, if $|k| \leq 1$, since $B_1 + B_2$ is L-balance set, then we have $(B_1+B_2)(k(y-x)) = \frac{1}{k}(B_1+B_2)(y-x) \geq (B_1+B_2)(y-x)$ and $(B_1+B_2)(y-x) \not\leq \mu'$, hence $k(y-x) \in (B_1+B_2)^{(\mu')}$. If $|k| > 1$, then exists $n_2 \geq n_1$ such that $\max\{s_{n_2}, t_{n_2}\} < |\frac{1}{k}|$. We get $(B_1 + B_2)(k(y - x)) = \frac{1}{k}(B_1 + B_2)(y - x) \geq \max\{s_{n_2}, t_{n_2}\}(B_1 + B_2)(y - x)$ and $\max\{s_{n_2}, t_{n_2}\}(B_1 + B_2)(y - x) \not\leq \mu'$, hence $k(y - x) \in (B_1 + B_2)^{(\mu')}$. So, the inclusion relation $U \subseteq (B_1 + B_2)^{(\mu')}$ holds. Since L is a chain, we have $A = \underline{\mu'} \wedge \chi_U \leq \bigvee_{\mu' \in J(L)} \underline{\mu'} \wedge \chi_{(B_1+B_2)^{(\mu')}} = B_1 + B_2$. It is clear that $A \in Svec(L^X)$. Since $T(X, \mathscr{B}) = \top_M$, we have $\mathscr{B}(A) = \bot_M$. This implies that $\mathscr{B}(B_1 + B_2) \leq \mathscr{B}(A) = \bot_M$. However, $\mathscr{B}(B_1 + B_2) \geq \mathscr{B}(B_1) \wedge \mathscr{B}(B_2) \geq a \neq \bot_M$. This leads to a contradiction. □

The relationship between bornological closure of a space and its separation is widely recognized in mathematical research. In the following, we will explore this relationship in the fuzzy cases.

Definition 3.13. Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space. Then the degree to which A is bornologically closed is defined as follows:

$$BC(A) = \bigwedge_{\substack{x^n_{\lambda(n)} \leq A \\ x_\beta \not\leq A, \beta = \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n} \lambda(m)}} \bigwedge_{\substack{B \in Bal(L^X) \\ s_n \to 0}} \{(\mathscr{B}(B))' : \forall n \in \mathbb{N}, (x^n - x)_{\lambda(n)} \not\leq s_n B'\}.$$

Theorem 3.14. *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space and L be a chain. Then $BC(\theta_{\top_L}) = T(X, \mathscr{B})$.*

Proof. For each $a \in M$ with $a \leq T(X, \mathscr{B})$, and for all $E^{(\bot)} \neq \{\theta\}$ with $E^{(\bot)} \in Svec(L^X)$, we have $\mathscr{B}(E)' \geq a$. For each $x^n_{\lambda(n)} \leq \theta_\top, x_\beta \not\leq \{\theta_{\top_L}\}, B \in Bal(L^X), s_n \to 0$ with $(x^n - x)_{\lambda(n)} \not\leq s_n B'$, we know $\lambda(n) \not\leq s_n B'(x^n - x) = s_n B'(-x)$ and $B(-x) \geq s_n B(-x) \not\leq (\lambda(n))'$. Since $\beta = \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n} \lambda(m)$, then $x \neq \theta$ and for each $\mu \in J(L)$ with $\beta \lhd \mu$, there exists $n_0 \in \mathbb{N}$, for each $m \geq n$ such that $\lambda(m) \leq \mu$. Then we get $B(-x) \not\leq \mu'$, i.e., $-x \in B^{(\mu')}$. It follows that $span\{x\} \subseteq B^{(\mu')}$. Take $A = \underline{\mu'} \wedge \chi_{span\{x\}}$. We obtain $A^{(\bot)} = span\{x\}, A \in Svec(L^X)$. From the fact L is a chain, we have $A = \underline{\mu'} \wedge \chi_{span\{x\}} \leq \underline{\mu'} \wedge \chi_{B^{(\mu')}} \leq \bigvee_{\mu' \in J(L)} \underline{\mu'} \wedge \chi_{B^{(\mu')}} = B$. It follows that $(\mathscr{B}(B))' \geq (\mathscr{B}(A))' \geq a$. So, $T(X, \mathscr{B}) \leq BC(\theta_{\top_L})$.

In addition, let

$$a = \bigvee_{x \neq y} \bigvee_{\substack{\{x^n_{\lambda(n)}\} \subseteq L^X, \lambda \geq \alpha, \lambda, \alpha \in J(L) \\ \alpha = \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n} \lambda(m)}} \bigvee_{\substack{B_1, B_2 \in Bal(L^X) \\ s_n, t_n \to 0}} \Big\{\mathscr{B}(B_1) \wedge \mathscr{B}(B_2) : \forall n \in \mathbb{N},$$
$$(x^n - x)_{\lambda(n)} \not\leq s_n B_1', (x^n - y)_{\lambda(n)} \not\leq t_n B_2'\Big\}.$$

Then for each $\gamma \in \beta(a)$, there exist ν-sequence $\{x^n_{\lambda(n)}\} \subseteq L^X$, $x \neq y$, $s_n, t_n \to 0$, $\lambda \in J(L), \lambda \geq \nu$, and L-fuzzy set $A, B \in Bal(L^X)$ with $(x^n - x)_{\lambda(n)} \nleq s_n A', (x^n - y)_{\lambda(n)} \nleq t_n B'$ for all $n \in \mathbb{N}$ such that $\mathscr{B}(A) \geq \gamma, \mathscr{B}(B) \geq \gamma$. Clearly, $\{x^n - x^n\}_{\lambda(n)} \subseteq \{\theta_{\top_L}\}$, $\nu = \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n} \lambda(m)$ and $(x-y)_\nu \nleq \theta_{\top_L}$. Moreover,

$$(x-y)_{\lambda(n)} = (x^n - x^n + x - y)_{\lambda(n)} = (x^n - y - (x^n - x))_{\lambda(n)} \nleq \max\{s_n, t_n\}(A+B)'.$$

Since $\mathscr{B}(A+B) \geq \mathscr{B}(A) \wedge \mathscr{B}(B) \geq \gamma$. Thus $(BC(\theta_\top))' \geq \gamma$. It follows that $a \leq BC(\theta_{\top_L})'$. So

$$\begin{aligned} BC(\theta_{\top_L}) &\leq \bigwedge_{\substack{\{x^n_{\lambda(n)}\}\subseteq L^X, \lambda\geq\alpha, \lambda,\alpha\in J(L)\\ \alpha=\bigwedge_{n\in\mathbb{N}}\bigvee_{m\geq n}\lambda(m), x\neq y}} \Big(Mac(x^n_{\lambda(n)},\ x_\lambda) \wedge Mac(x^n_{\lambda(n)},\ y_\lambda)\Big)' \\ &\leq T(X,\mathscr{B}). \end{aligned}$$

The proof is completed. □

Theorem 3.15. *Let $(X,\mathscr{B})$ be an (L,M)-fuzzy bornological vector space and E be a subspace of X. If $\mathscr{B}^{X/E}$ is a quotient (L,M)-fuzzy bornological vector space on X/E and L is a chain, then $BC(\chi_E) = T(X/E, \mathscr{B}^{X/E})$.*

Proof. By Definition 3.13 and Theorem 3.14, it follows that

$$\begin{aligned} T(X/E,\mathscr{B}^{X/E}) &= BC(\hat{\theta}_{\top_L}) \\ &= \bigwedge_{\substack{\hat{x}^n_{\lambda(n)}\leq\hat{\theta}_{\top_L}\\ \hat{x}_\mu\nleq\hat{\theta}_{\top_L}}} \bigwedge_{\substack{B\in Bal(L^{X/E})\\ s_n\to 0}} \{(\mathscr{B}^{X/E}(B))' : \forall n\in\mathbb{N}, (\widehat{x^n - x})_{\lambda(n)} \nleq s_n B', \\ &\quad \mu = \bigwedge_{n\in\mathbb{N}}\bigvee_{m\geq n}\lambda(m)\} \end{aligned}$$

and

$$\begin{aligned} BC(\chi_E) &= \bigwedge_{\substack{x^n_{\lambda(n)}\leq\chi_E\\ x_\mu\nleq\chi_E}} \bigwedge_{\substack{C\in Bal(L^X)\\ t_n\to 0}} \{(\mathscr{B}(C))' : \forall n\in\mathbb{N}, (x^n - x)_{\lambda(n)} \nleq t_n C', \\ &\quad \mu = \bigwedge_{n\in\mathbb{N}}\bigvee_{m\geq n}\lambda(m)\}. \end{aligned}$$

For each $a \lhd (BC(\theta_{\top_L}))'$, there exist $\hat{x}^n_{\lambda(n)} \leq \hat{\theta}_{\top_L}, \hat{x}_\mu \nleq \hat{\theta}_{\top_L}$ and $\widehat{B} \in Bal(L^{X/E})$, $s_n \to 0$ with $(\widehat{x^n - x})_{\lambda(n)} \nleq s_n \widehat{B}'$ such that $\mathscr{B}^{X/E}(\widehat{B}) \geq a$. Since $f : X \to X/E$ with $E \mapsto \hat{\theta}$, then $E = f^{-1}(\hat{\theta})$, we know $f(x_\mu) \nleq \hat{\theta}_\top \Leftrightarrow x_\mu \nleq \chi_E$, which implies that $x^n_{\lambda(n)} \leq \chi_E, x_\mu \nleq \chi_E$. Clearly, $f^{\leftarrow}(\widehat{B}) \in Bal(L^X)$ and $\lambda(n) \nleq s_n \widehat{B}'(\widehat{x^n - x}) = s_n \widehat{B}' f(x^n - x) = s_n (f^{\leftarrow}(\widehat{B}))'(x^n - x)$, i.e., $(x^n - x)_{\lambda(n)} \nleq s_n (f^{\leftarrow}(\widehat{B}))'$. In addition, $\mathscr{B}(f^{\leftarrow}(\widehat{B})) = \mathscr{B}^{X/E}(\widehat{B}) \geq a$. So, $a \leq (BC(\chi_E))'$. It follows that $BC(\chi_E) \leq BC(\theta_{\top_L})$.

Conversely, suppose $a \lhd (BC(\chi_E))'$, there exist $x^n_{\lambda(n)} \leq \chi_E, x_\mu \nleq \chi_E$ and $C \in Bal(L^X), t_n \to 0$ with $(x^n - x)_{\lambda(n)} \nleq t_n C'$ for all $n \in \mathbb{N}$ such that $\mathscr{B}(C) \geq a$, which implies that $\hat{x}^n_{\lambda(n)} \leq \hat{\theta}_\top, \hat{x}_\mu \nleq \hat{\theta}_\top$ and $f^{\rightarrow}(C) \in Bal(L^{X/E})$ with $(\widehat{x^n - x})_{\lambda(n)} \nleq t_n (f^{\rightarrow}(C))'$. It follows that $\mathscr{B}^{X/E}(f^{\rightarrow}(C)) \geq \mathscr{B}(C) \geq a$. Thus we have $(BC(\theta_{\top_L}))' \geq a$. Hence $BC(\chi_E) \geq BC(\theta_{\top_L})$. This completes the proof. □

4. Some further properties of the functors ω and ι

As described by Liang and Shi in their work [16], two functors ω and ι were introduced to establish connections between the category of M-fuzzifying bornological vector spaces (**MFBV**) and the category of (L,M)-fuzzy bornological vector spaces ((L,M)-**FBV**). Their research demonstrated that **MFBV** can be embedded in (L,M)-**FBV** as a reflective subcategory. This section aims to further investigate the properties of the functors ω and ι. The paper provides proof that the functor ω preserves the product and quotient spaces. Moreover, it delves into relationships between Mackey convergence and separation, which are relevant to both functors ω and ι.

Theorem 4.1. *Consider a family $(X_i,\mathcal{B}_i)_{i\in I}$ of M-fuzzifying bornological vector spaces, and let $X=\prod_{i\in I}X_i$. If we assume that $\omega(\mathcal{B})$ is a mapping defined by Theorem 2.25, we have $\omega(\prod_{i\in I}\mathcal{B}i)=\prod_{i\in I}\omega(\mathcal{B}_i)$.*

Proof. Let $p_i: X\to X_i$ be a projection. For all $A\in L^X$, we know that

$$\omega(\prod_{i\in I}\mathcal{B}_i)(A)=\bigwedge_{a\in L}(\prod_{i\in I}\mathcal{B}_i)(A^{(a)})=\bigwedge_{a\in L}\bigwedge_{i\in I}\mathcal{B}_i(p_i^{\rightarrow}(A^{(a)}))$$

and

$$\begin{aligned}\prod_{i\in I}(\omega(\mathfrak{B}_i))(A)&=\bigvee_{A\leq\prod_{i\in I}A_i}\bigwedge_{i\in I}\omega(\mathfrak{B}_i)(A_i)\\&=\bigwedge_{i\in I}\omega(\mathfrak{B}_i)(p_i^{\rightarrow}(A))=\bigwedge_{i\in I}\bigwedge_{a\in L}\mathfrak{B}_i((p_i^{\rightarrow}(A))^{(a)})\\&=\bigwedge_{a\in L}\bigwedge_{i\in I}\mathfrak{B}_i(p_i^{\rightarrow}(A^{(a)})).\end{aligned}$$

Hence $\omega(\prod_{i\in I}\mathcal{B}_i)=\prod_{i\in I}\omega(\mathcal{B}_i)$. □

Theorem 4.2. *Let $(X,\mathscr{B}_X)$ be an (L,M)-fuzzy bornological vector space and $f: X\to Y$ be a linear mapping. Define $\mathscr{B}_Y: L^Y\to M$ by*

$$\mathscr{B}_Y(C)=\bigvee_{C\leq f^{\rightarrow}(A)}\mathscr{B}_X(A).$$

Then $(X,\mathscr{B}_Y)$ is an (L,M)-fuzzy bornological vector space, denoted by $\mathscr{B}_Y=\mathscr{B}_X/f$.

Proof. Our first task is to establish that $\mathscr{B}_Y$ fulfills conditions (LMB1)-(LMB6). It can be readily observed that (LMB1), (LMB2), and (LMB5) are satisfied.

(LMB3) For all $a\lhd\mathscr{B}_Y(U)\wedge\mathscr{B}_Y(V)$, there exist $A,B\in L^X$ such that $U\leq f^{\rightarrow}(A), V\leq f^{\rightarrow}(B)$ and $\mathscr{B}_X(A)\geq a$ and $\mathscr{B}_X(B)\geq a$. Thus, we have $U\vee V\leq f^{\rightarrow}(A\vee B)$ and $\mathscr{B}_X(A\vee B)\geq\mathscr{B}_X(A)\wedge\mathscr{B}_X(B)\geq a$. It is clear that $\mathscr{B}_Y(U\vee V)\geq a$ and $\mathscr{B}_Y(U\vee V)\geq\mathscr{B}_Y(U)\wedge\mathscr{B}_Y(V)$.

(LMB4) For each $a\lhd\mathscr{B}_Y(U)\wedge\mathscr{B}_Y(V)$, there exist $A,B\in L^X$ such that $U\leq f^{\rightarrow}(A), V\leq f^{\rightarrow}(B)$ and $\mathscr{B}_X(A)\geq a, \mathscr{B}_X(B)\geq a$. Thus $U+V\leq f^{\rightarrow}(A)+f^{\rightarrow}(B)=f^{\rightarrow}(A+B)$ and $\mathscr{B}_X(A+B)\geq\mathscr{B}_X(A)\wedge\mathscr{B}_X(B)\geq a$. It is obvious that $\mathscr{B}_Y(U+V)\geq a$ and $\mathscr{B}_Y(U+V)\geq\mathscr{B}_Y(U)\wedge\mathscr{B}_Y(V)$.

(LMB6) For each $a\lhd\mathscr{B}_Y(U)$, there exists $A\in L^X$ with $U\leq f^{\rightarrow}(A)$ such that $\mathscr{B}_X(A)\geq a$. It is clear that $(\bigvee_{|t|\leq 1}tU)\leq f^{\rightarrow}(\bigvee_{|t|\leq 1}tA)$ and $\mathscr{B}_X(\bigvee_{|t|\leq 1}tA)\geq\mathscr{B}_X(A)\geq a$. Hence, we have $\mathscr{B}_Y(\bigvee_{|t|\leq 1}tU)\geq\mathscr{B}_Y(U)$.

Therefore $(X,\mathscr{B}_Y)$ is an (L,M)-fuzzy bornological vector space. □

Theorem 4.3. *Consider $(X,\mathcal{B})$ as an M-fuzzifying bornological space with $f:X\to Y$ as a linear mapping. If $\omega(\mathcal{B})$ is a mapping defined by Theorem* 2.25*, we have $\omega(\mathcal{B}/f)=\omega(\mathcal{B})/f$.*

Proof. For all $A\in L^X$, we have $\omega(\mathcal{B}/f)(A)=\bigwedge\limits_{a\in L}(\mathcal{B}/f)(A^{(a)})=\bigwedge\limits_{a\in L}\bigvee\limits_{A^{(a)}\subseteq f^{\to}(U)}\mathcal{B}(U)$ and

$$(\omega(\mathcal{B})/f)(A)=\bigvee_{A\leq f^{\to}(V)}\omega(\mathcal{B})(V)=\bigvee_{A\leq f^{\to}(V)}\bigwedge_{a\in L}\mathcal{B}(V^{(a)}).$$

For each $\mu\in L$ with $\mu\lhd\ \omega(\mathcal{B}/f)(A)$, then for all $a\in L$, there exists $U\subseteq X$ with $A^{(a)}\subseteq f^{\to}(U)$ such that $\mathcal{B}(U)\geq\mu$. Put $V=\bigvee\limits_{a\in L}\underline{a}\wedge\chi_U$. Then we have

$$f^{\to}(V)=\bigvee_{a\in L}f^{\to}(\underline{a}\wedge\chi_U)=\bigvee_{a\in L}(\underline{a}\wedge\chi_{f^{\to}(U)})\geq\bigvee_{a\in L}(\underline{a}\wedge\chi_{A^{(a)}})=A.$$

For all $x\in V^{(a)}$, we have $(\bigvee\limits_{\nu\in L}\underline{\nu}\wedge\chi_U)(x)\nleq a$, which implies that $x\in U$, thus $V^{(a)}\subseteq U$. It follows that $\mathcal{B}(V^{(a)})\geq\mathcal{B}(U)\geq\mu$. Hence $(\omega(\mathcal{B})/f)(A)\geq\mu$. By the arbitrariness of μ, we have $\omega(\mathcal{B}/f)(A)\leq(\omega(\mathcal{B})/f)(A)$.

On the contrary, let $\mu\lhd(\omega(\mathcal{B})/f)(A)$, there exists V_1 with $A\leq f^{\to}(V_1)$, for all $a\in L$ such that $\mathcal{B}(V_1^{(a)})\geq\mu$. Since $f^{\to}(V_1^{(a)})=(f^{\to}(V_1))^{(a)}$ for all $a\in L$, we have $A^{(a)}\subseteq f^{\to}(V_1^{(a)})$. It implies that $\omega(\mathcal{B}/f)(A)=\bigwedge\limits_{a\in L}\bigvee\limits_{A^{(a)}\subseteq f^{\to}(U)}\mathcal{B}(U)\geq\bigwedge\limits_{a\in L}\mathcal{B}(V_1^{(a)})\geq\mu$. It follows that $(\omega(\mathcal{B})/f)(A)\leq\omega(\mathcal{B}/f)(A)$. This completes the proof. □

Theorem 4.4. *Let $(X,\mathcal{B})$ be an M-fuzzifying bornological space, $\{x^n\}_{n\in\mathbb{N}}$ be a sequence in X and $\omega(\mathcal{B})$ be a mapping defined by Theorem* 2.25*. Then $Mac(x^n_\top,\ x_\top)=Borc(x_n,\ x)$.*

Proof. It is known that

$Mac(x^n_\top,\ x_\top)=\bigvee\limits_{\substack{B\in Bal(L^X)\\ s_n\to 0}}\{\omega(\mathcal{B})(B):\forall n\in\mathbb{N},(x^n-x)_{\top(n)}\nleq s_nB'\}$. For any $\mu\lhd Mac(x^n_\top,\ x_\top)$, there exist $s_n\to 0,\ B\in Bal(L^X)$ with $(x^n-x)_\top\nleq s_nB'$ such that $\omega(\mathcal{B})(B)\geq\mu$. It follows that $x^n-x\in s_nB^{(\perp)}$ and $B^{(\perp)}\in Bal(X)$. In addition, $\mu\leq\omega(\mathcal{B})(B)=\bigwedge\limits_{a\in L}\mathcal{B}(B^{(a)})\leq\mathcal{B}(B^{(\perp)})$. Thus $\mu\leq Borc(x_n,\ x)$.

On the other hand, for every $\nu\lhd Borc(x_n,\ x)$, there exist $t_n\to 0,\ B\in Bal(X)$ with $(x^n-x)\in t_nB$ such that $\mathcal{B}(B)\geq\nu$. Clearly, $\chi_B\in Bal(L^X)$, $(x^n-x)_\top\nleq t_n(\chi_B)'$ and $\omega(\mathcal{B})(\chi_B)=\bigwedge\limits_{a\in L}\mathcal{B}((\chi_B)^{(a)})=\mathcal{B}((\chi_B)^{(\perp)})\geq\nu$. Then $Mac(x^n_\top,\ x_\top)\geq\nu$. Hence $Borc(x^n,\ x)\leq Mac(x^n_\top,\ x_\top)$. □

Theorem 4.5. *Let $(X,\mathcal{B})$ be an M-fuzzifying bornological space and $\omega(\mathcal{B})$ be a mapping defined by Theorem* 2.25*. Then $T(X,\mathcal{B})=T(X,\omega(\mathcal{B}))$.*

Proof. It is known that $T(X,\mathcal{B})=\bigwedge\limits_{\substack{E\neq\{\theta\}\\ E\in Svec(X)}}(\mathcal{B}(E))'$ and

$$T(X,\omega(\mathcal{B}))=\bigwedge_{\substack{F^{(\perp)}\neq\{\theta\}\\ F\in Svec(L^X)}}(\omega(\mathcal{B})(F))'=\bigwedge_{\substack{F^{(\perp)}\neq\{\theta\}\\ F\in Svec(L^X)}}\bigvee_{a\in L}(\mathcal{B}(F^{(a)}))'.$$

First, we prove that $\mathcal{B}(M^{(\perp)})=\bigwedge\limits_{a\in L}\mathcal{B}(M^{(a)})$. It is easy to know $\mathcal{B}(M^{(\perp)})\geq\bigwedge\limits_{a\in L}\mathcal{B}(M^{(a)})$. From $M^{(a)}\subseteq M^{(\perp)}$, we have $\mathcal{B}(M^{(\perp)})\leq\bigwedge\limits_{a\in L}\mathcal{B}(M^{(a)})$. For each $a\leq T(X,\mathcal{B})$ and any

$E \neq \{\theta\}$ with $E \in Svec(X)$, we have $(\mathcal{B}(E))' \geq a$. Then for each $F^{(\perp)} \neq \{\theta\}$ with $F \in Svec(L^X)$, we have $\bigvee_{a\in L} (\mathcal{B}(F^{(a)}))' = (\mathcal{B}(F^{(\perp)}))' \geq a$. Hence $T(X, \omega(\mathcal{B})) \geq a$.

On the contrary, for any $a \lhd T(X, \omega(\mathcal{B}))$, it deduces that $\bigvee_{a\in L} (\mathcal{B}(E^{(a)}))' \geq a$ for all $E^{(\perp)} \neq \{\theta\}$ with $E \in Svec(L^X)$. For each $F \neq \{\theta\}$ with $F \in Svec(X)$, take $E_1 = \chi_F$. It follows that $\mathcal{B}(F)' = \mathcal{B}(E_1^{(\perp)})' = \bigvee_{\mu\in L} (\mathcal{B}(E_1^{(\mu)}))' \geq a$. Hence $T(X, \mathcal{B}) = T(X, \omega(\mathcal{B}))$. □

Theorem 4.6. *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space and $\iota(\mathscr{B})$ be a mapping defined by Theorem 2.27. Then $T(X, \mathscr{B}) = T(X, \iota(\mathscr{B}))$.*

Proof. It is known that $T(X, \mathscr{B}) = \bigwedge_{\substack{E^{(\perp)} \neq \{\theta\} \\ E \in Svec(L^X)}} (\mathscr{B}(E))'$

and

$$\begin{aligned} T(X, \iota(\mathscr{B})) &= \bigwedge_{\substack{F \neq \{\theta\} \\ F \in Svec(X)}} (\iota(\mathscr{B})(F))' \\ &= \bigwedge_{\substack{F \neq \{\theta\} \\ F \in Svec(X)}} \bigvee \{(\mathcal{D}_X(F))' : \varphi_{\mathscr{B}} \leq \mathcal{D}_X, (X, \mathbb{K}, \mathcal{D}_X) \in \Lambda_X\} \\ &= \bigwedge_{\substack{F \neq \{\theta\} \\ F \in Svec(X)}} (\varphi_{\mathscr{B}}(F))' \\ &= \bigwedge_{\substack{F \neq \{\theta\} \\ F \in Svec(X)}} \bigwedge_{\mu \in L} \bigwedge \{(\mathscr{B}(A))' : A^{(\mu)} = F\}. \end{aligned}$$

For each $a \leq T(X, \mathscr{B})$ and for all $F \neq \{\theta\}, F \in Svec(X)$, let $A = \chi_F$, it follows that $A^{(\mu)} = F$ for all $\mu \in L \setminus \{\top\}$ and $A \in Svec(L^X), A^{(\perp)} = F \neq \{\theta\}$. Then we have $(\mathscr{B}(A))' \geq a$. Thus $a \leq T(X, \iota(\mathscr{B}))$. So $T(X, \mathscr{B}) \leq T(X, \iota(\mathscr{B}))$. Moreover, for each $\nu \lhd (T(X, \mathscr{B}))'$, there exists $E \in Svec(L^X)$ with $E^{(\perp)} \neq \{\theta\}$ such that $\mathscr{B}(E) \geq \nu$. Then $\varphi_{\mathscr{B}}(E^{(\perp)}) = \bigvee_{\mu\in L} \bigvee \{\mathscr{B}(A) : A \in L^X, A^{(\mu)} = E^{(\perp)}\} \geq \mathscr{B}(E) \geq \nu$. Hence

$$(T(X, \iota(\mathscr{B})))' = \bigvee_{\substack{F \neq \{\theta\} \\ F \in Svec(X)}} \varphi_{\mathscr{B}}(F) \geq \varphi_{\mathscr{B}}(E^{(\perp)}) \geq \nu.$$

It follows that $T(X, \iota(\mathscr{B})) \leq T(X, \mathscr{B})$. This completes the proof. □

5. Conclusions and future work

Based on the research by Liang and Shi [16], this paper introduces the concepts of the degree of Mackey convergence for sequences and the degree of separation for spaces in (L, M)-fuzzy bornological vector spaces. Additionally, it presents the notion of bornological closure degree for fuzzy sets. The paper discusses several properties associated with these concepts, including the relationships among the degree of Mackey convergence for sequences, the degree of separation for spaces, and the degree of bornological closure for fuzzy sets. Furthermore, the paper explores the properties of the functors ω and ι defined by Liang and Shi [16], demonstrating that the functor ω preserves product and quotient spaces. The paper concludes by discussing the properties of the functors ω and ι in relation to Mackey convergence sequences and separation spaces.

A potential direction for future research is to establish a more comprehensive and systematic theory of (L, M)-bornological vector spaces. Additionally, studying the connections between the category of (L, M)-fuzzy topological vector spaces and the category of (L, M)-fuzzy bornological vector spaces would be beneficial.

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