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A Kantorovich Type Generalization of the Szàsz Operators via Two Variable Hermite Polynomials

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Abstract

The purpose of this paper is to give the Kantorovich generalization of the operators via two variable Hermite polynomials which are introduced by Krech [1] and to research approximating features with help of the classical modulus of continuity, the class of Lipschitz functions, Voronovskaya type asymptotic formula, second modulus of continuity and Peetre's K-functional for these operators.

1. INTRODUCTION

Heretofore, many authors has studied on linear positive operators and properties of their approximation, see for example [2, 6-11, 17, 18]. In addition to fact that authors working on the approximation theory with help of linear positive operators have been given linear positive operators via some orthogonal polynomials, see for example [1, 3, 4, 5, 12]. Therefore, we are going to define the Kantorovich type of the operators being made up of one of orthogonal polynomials.

Firstly, we recall H_k which is two variable Hermite polynomial (see [13]) defined by

$$H_k(n,\alpha)=k!\sum_{s=0}^{\left[\frac{k}{2}\right]}\frac{n^{k-2s}\alpha^s}{(k-2s)!s!}. \tag{1.1}$$

Furthermore, the generating function of two variable Hermite polynomials is the as follows (see [13])

$$\sum_{k=0}^{\infty}H_k(n,\alpha)\frac{t^k}{k!}=e^{nt+\alpha t^2}. \tag{1.2}$$

Secondly, Krech has presented the Szász operators including two variable Hermite polynomials

(see [1]) as

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$$G_n^{\alpha}(f;x) := e^{-(nx+\alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n,\alpha) f\left(\frac{k}{n}\right), \tag{1.3}$$

where $n = 1,2,3,...$, $\alpha \geq 0$ and $x \in [0,\infty)$.

Now, we introduce a Kantorovich type generalization of G_n^{α}.

2. KANTOROVICH TYPE GENERALIZATION OF OPERATORS G_n^{α}

In this section, the Kantorovich type generalization of G_n^{α} has been defined by

$$S_n^{\alpha}(f;x) := ne^{-(nx+\alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n,\alpha) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \tag{2.1}$$

where $n = 1,2,3,...$, $\alpha \geq 0$, $x \in [0,\infty)$ and $f \in C[0,\infty)$ for which the corresponding series is convergent, in here $C[0,\infty)$ is the space of continuous functions on $[0,\infty)$.

Lemma 1. The operators given by (2.1) yield the following equalities.

i. $S_n^{\alpha}(1;x) = 1$,

ii. $S_n^{\alpha}(t;x) = x + \dfrac{4\alpha x^2 + 1}{2n}$,

iii. $S_n^{\alpha}(t^2;x) = x^2 + \dfrac{4\alpha x^3 + 2x}{n} + \dfrac{12\alpha^2 x^4 + 18\alpha x^2 + 1}{3n^2}$,

iv. $S_n^{\alpha}(t^3;x) = x^3 + \dfrac{12\alpha x^4 + 9x^2}{2n} + \dfrac{24\alpha^2 x^5 + 48\alpha x^3 + 7x}{2n^2} + \dfrac{120\alpha^2 x^4 + 32\alpha^3 x^6 + 64\alpha x^2 + 1}{4n^3}$,

v. $$S_n^{\alpha}(t^4;x) = x^4 + \frac{8\alpha x^5 + 8x^3}{n} + \frac{24\alpha^2 x^6 + 60\alpha x^4 + 15x^2}{n^2} + \frac{32\alpha^3 x^7 + 144\alpha^2 x^5 + 108\alpha x^3 + 6x}{n^3}$$
$$+ \frac{840\alpha^2 x^4 + 560\alpha^3 x^6 + 80\alpha^4 x^8 + 210\alpha x^2 + 1}{5n^4}.$$

Observe that the operators are well-defined for all test function $e_i(\mathrm{t}) = \mathrm{t}^i$ for $i = 0,1,2,3,4.$

Lemma 2. The operators given by (2.1) yield the following equalities.

i. $\psi_1 = S_n^{\alpha}((t-x)^1;x) = \dfrac{4\alpha x^2 + 1}{2n}$,

ii. $\psi_2 = S_n^{\alpha}((t-x)^2;x) = \dfrac{x}{n} + \dfrac{12\alpha^2 x^4 + 18\alpha x^2 + 1}{3n^2}$,

iii. $$\psi_3 = S_n^{\alpha}((t-x)^3;x) = \frac{12\alpha x^3 + 5x}{2n^2} + \frac{120\alpha^2 x^4 + 32\alpha^3 x^6 + 64\alpha x^2 + 1}{4n^3},$$

iv. $$\psi_4 = S_n^{\alpha}((t-x)^4;x) = \frac{3x^2}{n^2} + \frac{24\alpha^2 x^5 + 44\alpha x^3 + 5x}{n^3} + \frac{840\alpha^2 x^4 + 560\alpha^3 x^6 + 80\alpha^4 x^8 + 210\alpha x^2 + 1}{5n^4}.$$

Now, we can give Theorem 1 for approximation properties of the operators S_n^{α} using the well known Korovkin theorem with the help of Lemma 1.

Theorem 1. Let the operator defined by S_n^{α} in (2.1) and $f \in C_B[0,\infty)$. So, $S_n^{\alpha}(f;x)$ is uniformly convergent to $f(x)$ on $[0,b]$ where $C_B[0,\infty)$ is the space of uniformly continuous and bounded functions on $[0,\infty)$.

Proof. Follow the standart procedure in [16], see also [14,15].

3. APPROXIMATION PROPERTIES OF OPERATORS S_n^{α}

In this section, we present the rate of convergence of the operators with the help of the usual and second modulus of continuity, Lipschitz class functions, Peetre's K - functional and Voronovskaya type formula. Firstly, we remind some definitions as follows.

Let $Lip_M(\beta)$ be Lipschitz class of order β. If $f \in Lip_M(\beta)$, the inequality

$$|f(t) - f(u)| \le M|t-u|^{\beta} \tag{3.1}$$

holds, where $t,u \in [0,\infty)$, $0 < \beta \le 1$ and $M > 0$. The classical modulus of continuity of $f \in C_B[0,\infty)$ is denoted by

$$\omega(f;\delta) = \sup_{|h|\le\delta}\{|f(x+h) - f(x)| : x \in [0,\infty)\}, \tag{3.2}$$

where $\delta > 0$.

Furthermore, a vector space, $C_B{}^2[0,\infty) = \{f \in C_B[0,\infty) : f', f'' \in f \in C_B[0,\infty)\}$, is normed space with following norm that

$$\|f\|_{C_B{}^2[0,\infty)} = \|f\|_{C_B[0,\infty)} + \|f'\|_{C_B[0,\infty)} + \|f''\|_{C_B[0,\infty)} \tag{3.3}$$

for every $f \in C_B{}^2[0,\infty)$. We can remind Peetre's K-functional of the function $f \in C_B[0,\infty)$ that is as follows

$$K(f;\delta) = \inf_{g \in C_B{}^2[0,\infty)}\left\{\|f-g\|_{C_B[0,\infty)} + \delta\|g\|_{C_B{}^2[0,\infty)}\right\} \tag{3.4}$$

for $\delta > 0$. We define the second-order modulus of smoothness of function $f \in C_B[0,\infty)$ by

$$\omega_2(f;\delta) = \sup_{0<h\le\delta} \left\{ \left| f(x+2h) - 2f(x+h) + f(x) \right| : x \in [0,\infty) \right\} \tag{3.5}$$

for $\delta > 0$. Moreover, we have the inequality that is relation between Peetre's K-functional and ω_2 as following that

$$K(f;\delta) \le M \left\{ \omega_2(f;\sqrt{\delta}) + \min(1,\delta) \|f\|_{C_B[0,\infty)} \right\} \tag{3.6}$$

for all $\delta > 0$ and M is positive constant.

Theorem 2. The operators S_n^α defined in (2.1) verify the following inequality

$$\left| S_n^\alpha(f;x) - f(x) \right| \le M \omega\left(f;\delta_n\right), \tag{3.7}$$

where $f \in C_B[0,\infty)$, $x \in [0,b]$, M is a constant and $\delta_n = \frac{1}{\sqrt{n}}$.

Proof. We know that modulus of continuity of function $f \in C_B[0,\infty)$ verifies the following inequality

$$|f(t) - f(x)| \le \omega(f;\delta)\left(\frac{|t-x|}{\delta} + 1 \right). \tag{3.8}$$

Using (3.8), Cauchy-Schwarz inequality and Lemma 2, we have

$$\begin{aligned} \left| S_n^\alpha(f;x) - f(x) \right| &\le S_n^\alpha(|f(t) - f(x)|;x) \\ &\le \omega(f;\delta)\left(1 + \frac{1}{\delta} S_n^\alpha(|t-x|;x) \right) \\ &\le \omega(f;\delta)\left(1 + \frac{1}{\delta} \sqrt{S_n^\alpha((t-x)^2;x)} \right) \\ &\le \omega(f;\delta)\left(1 + \frac{1}{\delta} \sqrt{\psi_2} \right) \\ &\le M\omega\left(f;\delta_n\right), \end{aligned}$$

where $M = 1 + \sqrt{b + 12\alpha^2 b^4 + 18\alpha b^2 + 1}$ and $\delta_n = \frac{1}{\sqrt{n}}$.

Theorem 3. If $f \in Lip_M(\beta)$, then we have

$$\left|S_n^{\alpha}(f;x)-f(x)\right| \le M^*(\delta_n)^{\frac{\beta}{2}}, \tag{3.9}$$

where $x \in [0,b]$, M^* is constant and $\delta_n = \frac{1}{n}$.

Proof. From $f \in Lip_M(\beta)$ and linearity property of S_n^{α}, we obtain

$$\left|S_n^{\alpha}(f;x)-f(x)\right| \le S_n^{\alpha}(\left|f(t)-f(x)\right|;x)$$

$$\le MS_n^{\alpha}(\left|t-x\right|^{\beta};x).$$

On the basis of Lemma 2 and Hölder's inequality firstly for integral and then for sum via

p=$\frac{\beta}{2}$, $q=\frac{2-\beta}{2}$, we obtain

$$\left|S_n^{\alpha}(f;x)-f(x)\right| \le MS_n^{\alpha}(\left|t-x\right|^{\beta};x)$$

$$\le M(S_n^{\alpha}((t-x)^2;x))^{\frac{\beta}{2}}$$

$$\le M(\psi_2)^{\frac{\beta}{2}}$$

$$\le M^*(\delta_n)^{\frac{\beta}{2}},$$

where $M^* = M\left(b+12\alpha^2 b^4+18\alpha b^2+1\right)^{\frac{\beta}{2}}$ and $\delta_n = \frac{1}{n}$.

Theorem 4. Let K be Peetre's K-functional. The operators S_n^{α} defined in (2.1) verify the following inequality

$$\left|S_n^{\alpha}(f;x)-f(x)\right| \le 2K(f,\delta_n), \tag{3.10}$$

where $f \in C_B[0,\infty), x \in [0,b]$ and $\delta_n = \frac{b}{n}+\frac{4\alpha b^2+1}{2n}+\frac{12\alpha^2 b^4+18\alpha b^2+1}{3n^2}$.

Proof . From the Taylor's series expansion of the function $g \in C_B{}^2[0,\infty)$, we have

$$g(t)=g(x)+g'(x)(t-x)+g''(c)\frac{(t-x)^2}{2},\ c \in (x,t).$$

When we apply the operators S_n^α to both sides of the aforementioned equality and recall the linearity property of the operators S_n^α, we obtain

$$S_n^\alpha(g;x)-g(x)=g'(x)S_n^\alpha((t-x);x)+\frac{g''(c)}{2}S_n^\alpha\left((t-x)^2;x\right).$$

By Lemma 2, we have

$$\begin{aligned}\left|S_n^\alpha(g;x)-g(x)\right| &\le g'(x)\psi_1+\frac{g''(c)}{2}\psi_2\\ &\le g'(x)\frac{4\alpha x^2+1}{2n}+\frac{g''(c)}{2}\left(\frac{x}{n}+\frac{12\alpha^2x^4+18\alpha x^2+1}{3n^2}\right)\\ &\le \|g'\|_{C_B[0,\infty)}\frac{4\alpha x^2+1}{2n}+\frac{\|g''\|_{C_B[0,\infty)}}{2}\left(\frac{x}{n}+\frac{12\alpha^2x^4+18\alpha x^2+1}{3n^2}\right)\\ &\le \left(\frac{x}{n}+\frac{4\alpha x^2+1}{2n}+\frac{12\alpha^2x^4+18\alpha x^2+1}{3n^2}\right)\left(\|g'\|_{C_B[0,\infty)}+\frac{\|g''\|_{C_B[0,\infty)}}{2}\right)\\ &\le \left(\frac{x}{n}+\frac{4\alpha x^2+1}{2n}+\frac{12\alpha^2x^4+18\alpha x^2+1}{3n^2}\right)\|g\|_{C_B^2[0,\infty)}\\ &\le \left(\frac{b}{n}+\frac{4\alpha b^2+1}{2n}+\frac{12\alpha^2b^4+18\alpha b^2+1}{3n^2}\right)\|g\|_{C_B^2[0,\infty)}.\end{aligned}$$

Now, let $f\in C_B[0,\infty)$. We use the above inequality as follow

$$\begin{aligned}\left|S_n^\alpha(f;x)-f(x)\right| &= \left|S_n^\alpha(f;x)-S_n^\alpha(g;x)+S_n^\alpha(g;x)-g(x)+g(x)-f(x)\right|\\ &\le S_n^\alpha(|f-g|;x)+|f(x)-g(x)|+\left|S_n^\alpha(g;x)-g(x)\right|\\ &\le 2\|f-g\|_{C_B[0,\infty)}+2\|g\|_{C_B^2[0,\infty)}\left(\frac{b}{n}+\frac{4\alpha b^2+1}{2n}+\frac{12\alpha^2b^4+18\alpha b^2+1}{3n^2}\right).\end{aligned}$$

By applying infimum both sides of this inequality for $g\in C_B{}^2[0,\infty)$, we have

$$\left|S_n^\alpha(f;x)-f(x)\right|\le 2K(f,\delta_n),$$

where $\delta_n=\frac{b}{n}+\frac{4\alpha b^2+1}{2n}+\frac{12\alpha^2b^4+18\alpha b^2+1}{3n^2}$.

Theorem 5. For the operators (2.1), the following inequality holds

$$\left|S_n^{\alpha}(f;x)-f(x)\right| \le 2M\left\{\omega_2\left(f,\sqrt{\lambda_n}\right)+\min\left(1,\lambda_n\right)\|f\|\right\}, \tag{3.11}$$

where $f \in C_B[0,\infty), x \in [0,b]$, M is a positive constant that is independent of n and $\lambda_n = \frac{b}{n} + \frac{4\alpha b^2+1}{2n} + \frac{12\alpha^2 b^4 + 18\alpha b^2 + 1}{3n^2}$.

Proof. By using Theorem 4, we obtain $\left|S_n^{\alpha}(f;x)-f(x)\right| \le 2K(f,\lambda_n)$. The proof is completed by choosing $\delta = \lambda_n$ in (3.6).

Theorem 6. Let $f \in C_B{}^2[0,\infty)$ and $x \in [0,\infty)$ is a fixed point. Then, we have

$$\lim_{n\to\infty} n\left[S_n^{\alpha}(f;x)-f(x)\right] = \frac{1}{2}\left[(4\alpha x^2+1)f'(x)+xf''(x)\right]. \tag{3.12}$$

Proof. By Taylor formula for the function f, we get

$$f(t) = f(x) + f'(x)(t-x) + f''(x)\frac{(t-x)^2}{2} + (t-x)^2\mu(t,x),$$

where $\mu(.,x) \in C_B[0,\infty)$ and $\lim_{t\to x}\mu(t,x) = 0$.

When we apply the operators S_n^{α} to both sides of the aforementioned equality and recall the linearity property of the operators S_n^{α}, we obtain

$$S_n^{\alpha}(f;x)-f(x) = f'(x)S_n^{\alpha}((t-x);x) + \frac{f''(x)}{2}S_n^{\alpha}((t-x)^2;x) + S_n^{\alpha}\left((t-x)^2\mu(t,x);x\right).$$

By Lemma 2, we get

$$S_n^{\alpha}(f;x)-f(x) = f'(x)\psi_1 + \frac{f''(x)}{2}\psi_2 + S_n^{\alpha}\left((t-x)^2\mu(t,x);x\right). \tag{3.13}$$

By Cauchy-Schwarz inequality, we have

$$nS_n^{\alpha}\left((t-x)^2\mu(t,x);x\right) \le \left(n^2 S_n^{\alpha}\left((t-x)^4;x\right)\right)^{\frac{1}{2}}\left(S_n^{\alpha}\left(\mu^2(t,x);x\right)\right)^{\frac{1}{2}}.$$

It is clear that $\mu^2(x,x)=0$ and $\mu^2(t,x)$ is bounded. Then, we get

$$\lim_{n\to\infty} S_n^{\alpha}\left(\mu^2(t,x);x\right) = \mu^2(x,x) = 0.$$

So, we obtain

$$\lim_{n\to\infty} nS_n^{\alpha}\left((t-x)^2\mu(t,x);x\right)=0. \tag{3.14}$$

Now, we can write the following equality from (3.13) and (3.14)

$$\lim_{n\to\infty} n\left[S_n^{\alpha}(f;x)-f(x)\right]=\frac{1}{2}\left[(4\alpha x^2+1)f'(x)+xf''(x)\right].$$

The proof is done.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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