http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 74, Number 1, Pages 68–78 (2025)
https://doi.org/10.31801/cfsuasmas.1485231
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: May 16, 2024; Accepted: December 13, 2024

Fischer-Marsden conjecture on K-paracontact manifolds and quasi-para-Sasakian manifolds

Mustafa ÖZKAN[1] and İrem KÜPELİ ERKEN[2]

[1,2]Department of Mathematics, Bursa Technical University, Bursa, TÜRKİYE

Abstract. The aim of this paper is to study of the non-trivial solutions of Fischer-Marsden conjecture on K-paracontact manifolds and 3-dimensional quasi-para-Sasakian manifolds. We prove that if a semi-Riemannian manifold of dimension $2n+1$ admits a non-trivial solution of Fischer-Marsden equation, then it has constant scalar curvature. We give a comprehensive classification for a $(2n+1)$-dimensional K-paracontact manifold which admits a non-trivial solution of Fischer-Marsden equation. We consider 3-dimensional quasi-para-Sasakian manifolds with β constant which admits Fischer-Marsden equation and prove that there are two possibilities. The first one is the scalar curvature $r=-6\beta^2$ and M^3 is Einstein. The second one is the manifold is paracosymplectic manifold and η-Einstein.

2020 Mathematics Subject Classification. 53B30, 53C25, 53D10.
Keywords. Fischer-Marsden equation, K-paracontact manifold, quasi-para-Sasakian manifold, gradient Ricci soliton.

1. Introduction

In modern physics, the general theory of relativity provides an interpretation of many cosmological events, from the expansion of the universe to black holes. A significant global solution of Einstein equation in general relativity is *static space-times.* A semi-Riemannian manifold (M^{2n+1},g) and positive function λ, we say that $(\bar{M}^{2n+2},\bar{g})=M^{2n+1}\times_\lambda \mathbb{R}$ endowed with the metric $\bar{g}=g-\lambda^2dt^2$ is a static space-time. In this case, the Einstein equation with perfect fluid as a matter field over $(\bar{M}^{2n+2},\bar{g})$ is given by

$$S_{\bar{g}}-\frac{r_{\bar{g}}}{2}\bar{g}=T, \tag{1}$$

where $T=\mu\lambda^2dt^2+\rho g$ is the stress-energy-momentum tensor of perfect fluid, $S_{\bar{g}}$ and $r_{\bar{g}}$ denotes the Ricci tensor and scalar curvature for the metric $\bar{g}$, resp. Moreover, the smooth functions μ and ρ are *energy density* and *pressure* of the perfect fluid, resp. *Static perfect fluid space-times* is a generalization of the static vacuum spaces and solution of (1). Also, it provides models for black holes, galaxies and stellars [7, 9]. Fischer-Marsden equation can be considered as a special case of the static perfect fluid space-times [5, Remark 1.3].

On the other hand, Fischer-Marsden conjecture is closely related the conjecture that known as *Cosmic no-hair conjecture.* We recall the Cosmic no-hair conjecture as *"the hemisphere $\mathbb{S}^n_+$ is the only possible n-dimensional positive static triple with single-horizon and positive scalar curveture"* [9].

Let (M^{2n+1},g) be a compact, orientable semi-Riemannian manifold. We denote the set of all unit volume semi-Riemannian metrics on (M^{2n+1},g) by $\mathcal{M}$. The linearization of the scalar curvature $\mathcal{L}_g(g^*)$ is given by

$$\mathcal{L}_g g^*=-\Delta_g(tr_g g^*)+div(div(g^*))-g(g^*,S),$$

[1] mustafa.ozkan@btu.edu.tr; 0000-0002-4483-2912
[2] irem.erken@btu.edu.tr-Corresponding author; 0000-0003-4471-3291.

where Δ_g, div, g^* and S denotes the negative Laplacian of the semi-Riemannian metric g, divergence operator, symmetric $(0,2)$ tensor field on M and the Ricci tensor, resp. The formal L^2-adjoint $\mathcal{L}_g g^*$ of the linearized scalar curvature operator $\mathcal{L}_g$ is defined by

$$\mathcal{L}_g^*(\lambda) = -(\Delta_g \lambda)g + Hess_g \lambda - \lambda S, \tag{2}$$

where $Hess_g \lambda(U,V) = \nabla_g^2 \lambda(U,V) = g(\nabla_U D\lambda, V)$ is the Hessian operator of the smooth function λ on M and D is the gradient operator of g. We refer the equation $\mathcal{L}_g^*(\lambda) = 0$ as Fischer-Marsden equation (FME). The pair (g,λ) that satisfying $\mathcal{L}_g^*(\lambda) = 0$ is called a solution of Fischer-Marsden equation. A solution with $\lambda = 0$ is called a *trivial solution.* We note that a complete Riemannian manifold that admits a non-trivial solution of Fischer-Marsden equation ($\lambda \neq 0$) has constant scalar curvature [1,10]. Moreover, Corvino [8] proved that a non-trivial solution of FME implies the warped product metric $g^* = g - \lambda^2 dt^2$ is Einstein. Further, we recall Fischer-Marsden conjecture [10] as "*a compact Riemannian manifold that admits a non-trivial solution of the equation $\mathcal{L}_g^*(\lambda) = 0$ is necessarily an Einstein manifold*". In the case of g is conformally flat, counter examples of this conjecture are given by Kobayashi [12] and Lafontaine [16]. This conjecture is investigated by various authors [2–4, 19, 20].

A *Ricci soliton* is a generalization of an Einstein metric [11]. A semi-Riemannian metric g on a semi-Riemannian manifold M^{2n+1} is said to be Ricci soliton if there exist a real number μ and a vector field V on M^{2n+1} satisfying

$$\mathcal{L}_V g + 2S + 2\mu g = 0, \tag{3}$$

where $\mathcal{L}_V g$ and S denote the Lie derivative along the vector field V and the Ricci tensor of g, resp. The vector field V is also called the potential vector field. If soliton constant μ is zero, negative or positive, then the Ricci soliton is said to be *steady, shrinking* or *expanding*, resp. Furthermore, if V is a gradient of a smooth function f, namely, $V = Df$, then the Ricci soliton is called a *gradient Ricci soliton* and the equation (3) becomes

$$Hess(f) + S = \mu g, \tag{4}$$

where $Hess(f)$ is the Hessian of f. In semi-Riemannian manifold M^{2n+1}, the metric g is said to be *gradient η-Ricci soliton* if it satisfies

$$Hess(f) + S = \mu_1 g + \mu_2 \eta \otimes \eta, \tag{5}$$

where f is a smooth function and μ_1, μ_2 are constants [6].

All of the mentioned works motivate us to study Fischer-Marsden conjecture on K-paracontact manifolds and 3-dimensional quasi-para-Sasakian manifolds. This paper is organized in the following way. In section 2, we recall some notations required for this paper. In section 3, first, we prove the counter-part of the theorem which was proved in 1975 [1, 10], namely, we show that in a semi-Riemannian manifold which admits non-trivial solution of Fischer-Marsden equation, the scalar curvature is constant. After that, we gave a comprehensive classification for a $(2n+1)$-dimensional K-paracontact manifold which admits a non-trivial solution of Fischer-Marsden equation. With this Theorem, we have shown one of the difference between contact geometry and paracontact geometry. Also, we prove that if the Ricci operator commutes for a K-paracontact manifold M^{2n+1} with a non-trivial solution of Fischer-Marsden equation, then M^{2n+1} is an Einstein manifold. We show that if a $2n+1$-dimensional para-Sasakian manifold admits a non-trivial solution of Fischer-Marsden equation, then it is Einstein. Moreover, for $n = 1$, the Ricci tensor is parallel and the manifold is Ricci-semisymmetric. We also investigate the relation between Fischer-Marsden conjecture and gradient Ricci solitons on K-paracontact manifolds. In Section 4, we consider 3-dimensional quasi-para-Sasakian manifolds with β constant which admits Fischer-Marsden equation and prove that there are two possibilities. The first one is the scalar curvature $r = -6\beta^2$ and M^3 is Einstein. The second one is the manifold is paracosymplectic manifold which is locally a product of the real line $\mathbb{R}$ and a 2-dimensional para-Kaehlerian manifold, and η-Einstein. Finally, we give the relation between Fischer-Marsden conjecture and gradient Ricci solitons and gradient η-Ricci solitons on quasi-para-Sasakian manifolds M^3.

2. Preliminaries

A $(2n+1)-$ dimensional manifold M is called *almost paracontact manifold* if it admits triple (F, ξ, η) satisfying the followings:

$$\eta(\xi) = 1, \quad F^2 = I - \eta \otimes \xi \tag{6}$$

and F induces on almost paracomplex structure on each fiber of $\mathcal{D} = ker(\eta)$, where F, ξ and η are $(1,1)-$tensor field, vector field and $1-$form, resp. As a natural consequence, the tensor field F has rank $2n$, $F\xi = 0$ and $\eta \circ F = 0$. Here, ξ denotes a certain vector field (referred to as the *Reeb* or *characteristic vector field*) which is dual to η and satisfying $d\eta(\xi, U) = 0$ for all $U \in \chi(M)$. If the structure (M, F, ξ, η) admits a pseudo-Riemannian metric such that

$$g(FU, FV) = -g(U, V) + \eta(U)\eta(V), \tag{7}$$

for all $U, V \in \chi(M)$, then we say that (M, F, ξ, η, g) is an *almost paracontact metric manifold*. It should be noted that a pseudo-Riemannian metric with a given almost paracontact metric manifold structure always have a signature of $(n+1, n)$. On an almost paracontant metric manifold, there always exists an orthogonal basis $\{U_1, \ldots, U_n, V_1, \ldots, V_n, \xi\}$, namely $F-$basis, such that $g(U_i, U_j) = -g(V_i, V_j) = \delta_{ij}$ and $V_i = FU_i$, for any $i, j \in \{1, \ldots, n\}$. Moreover, it is possible to establish the definition of a skew-symmetric tensor field (a 2-form), commonly referred to as the fundamental form, denoted as Φ, by using the equation

$$\Phi(U, V) = g(U, FV).$$

Within the framework of almost paracontact manifolds, the tensor $N^{(1)}$ of type $(1,2)$ can be introduced by

$$N^{(1)}(U, V) = [F, F](U, V) - 2d\eta(U, V)\xi$$

where

$$[F, F](U, V) = F^2[U, V] + [FU, FV] - F[FU, V] - F[U, FV]$$

is the Nijenhuis torsion of F. The almost paracontact manifold is designated as *normal*, when $N^{(1)} = 0$ [23].

Furthermore, an almost paracontact metric manifold is referred to as a *paracontact metric manifold* if the following condition is satisfied for all vector fields $U, V \in \chi(M)$:

$$d\eta(U, V) = g(U, FV) = \Phi(U, V).$$

In a paracontact metric manifold, a symmetric, trace-free operator h is defined as $h := \frac{1}{2}\mathcal{L}_\xi F$, where $\mathcal{L}$ represents the Lie derivative. It is important to note that h equals zero if and only if the vector field ξ is a killing vector. When ξ is a Killing vector, the paracontact metric manifold is referred to as a *K-paracontact manifold*. A normal almost paracontact metric manifold is said to be *para-Sasakian manifold* if $\Phi = d\eta$. Furthermore, a para-Sasakian manifold is also K-paracontact, with the reverse holding true solely in a three-dimensional [23]. An almost paracontact metric manifold is called *quasi-para-Sasakian* when both the structure is normal and its fundamental 2-form is closed.

Actually, three dimensional quasi-para-Sasakian and para-Sasakian manifolds are normal almost paracontact metric manifold in the type of (α, β) with $(0, \beta)$ and $(0, -1)$, resp. In the case of $\alpha = \beta = 0$, the manifold is paracosymplectic [21].

An almost paracontact metric manifold is said to be *η-Einstein* if its Ricci tensor S is of the form

$$S = \mu_1 g + \mu_2 \eta \otimes \eta \tag{8}$$

where μ_1 and μ_2 are smooth functions on the manifold. If M is para-Sasakian, then μ_1 and μ_2 are constants ([23, Proposition 4.7]). If $\mu_2 = 0$, then the manifold is said to be *Einstein*.

In a K-paracontact manifold, we have the following relations [23]:

$$\nabla_U \xi = -FU, \tag{9}$$

$$Q\xi = -2n\xi, \tag{10}$$

$$R(\xi, U)V = (\nabla_U F)V, \tag{11}$$

$$(\nabla_{FU} F)FV - (\nabla_U F)V = 2g(U, V)\xi - (U + \eta(U)\xi)\eta(V), \tag{12}$$

for all $U, V \in \chi(M)$. On K-paracontact manifold, from (8) and (10), we have $\mu_1 + \mu_2 = -2n$. So K-paracontact manifold is Einstein if and only if $S(U, V) = -2ng(U, V)$ for all $U, V \in \chi(M)$. Moreover, the following curvature identities holds for a three-dimensional quasi-para-Sasakian manifold with β constant [14, 15]:

$$
\begin{aligned}
\nabla_U \xi =& \beta F U, && (13)\\
R(U,V)W =& (2\beta^2+\frac{r}{2})(g(V,W)U-g(U,W)V)-(3\beta^2+\frac{r}{2})(g(V,W)\eta(U)\xi\\
&-g(U,W)\eta(V)\xi+\eta(V)\eta(W)U-\eta(U)\eta(W)V), && (14)\\
S(U,V) =& (\beta^2+\frac{r}{2})g(U,V)-(3\beta^2+\frac{r}{2})\eta(U)\eta(V), && (15)\\
QU =& (\beta^2+\frac{r}{2})U-(3\beta^2+\frac{r}{2})\eta(U)\xi, && (16)\\
Q\xi =& -2\beta^2\xi, && (17)
\end{aligned}
$$

where R, S and r are respectively Riemannian curvature, Ricci tensor and scalar curvature of M.

3. K-Paracontact Manifolds Satisfying Fischer-Marsden Equation

Theorem 1. *If a semi-Riemannian manifold (M^n, g) admits a non-trivial solution (g, λ) of Fischer-Marsden equation, then it has constant scalar curvature.*

Proof. Let (M^n, g) be a semi-Riemannian manifold and $\{e_i | 1 \le i \le n\}$ be a local frame on a normal coordinate system at any point $p \in M$. Therefore, from [18, Proposition 33, p. 73], we have

$$\nabla_{e_i} e_j = 0 \tag{18}$$

and

$$\nabla_U e_i = \sum_{i=1}^{n} x_j \nabla_{e_j} e_i = 0 \tag{19}$$

for vector field $U = \sum_{i=1}^{n} x_j e_j$ on a neighborhood of $p \in M$. We also know that

$$div(Hess\lambda)(U) = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} Hess\lambda)(U, e_i), \tag{20}$$

where $\varepsilon_i = g(e_i, e_i)$. Computing this covariant derivative, using (18), we have

$$
\begin{aligned}
(\nabla_{e_i} Hess\lambda)(U, e_i) =& \nabla_{e_i} Hess\lambda(U, e_i) - Hess\lambda(\nabla_{e_i} U, e_i) - Hess\lambda(U, \nabla_{e_i} e_i)\\
=& \nabla_{e_i} g(\nabla_U D\lambda, e_i) - g(\nabla_{\nabla_{e_i} U} D\lambda, e_i)\\
=& g(\nabla_{e_i} \nabla_U D\lambda, e_i) - g(\nabla_{\nabla_{e_i} U} D\lambda, e_i). && (21)
\end{aligned}
$$

On the other hand, using the Riemannian curvature tensor and (19), we obtain

$$
\begin{aligned}
g(R(e_i, U)D\lambda, e_i) =& g(\nabla_{e_i} \nabla_U D\lambda, e_i) - g(\nabla_U \nabla_{e_i} D\lambda, e_i) - g(\nabla_{[e_i, U]} D\lambda, e_i)\\
=& g(\nabla_{e_i} \nabla_U D\lambda, e_i) - g(\nabla_U \nabla_{e_i} D\lambda, e_i) - g(\nabla_{\nabla_{e_i} U} D\lambda, e_i). && (22)
\end{aligned}
$$

Using (21) and (22), one can get

$$(\nabla_{e_i} Hess\lambda)(U, e_i) = g(R(e_i, U)D\lambda, e_i) + g(\nabla_U \nabla_{e_i} D\lambda, e_i). \tag{23}$$

By the help of (19) and writing (23) in (20), we derive

$$
\begin{aligned}
div(Hess\lambda)(U) =& \sum_{i=1}^{n} \varepsilon_i g(R(e_i, U)D\lambda, e_i) + \sum_{i=1}^{n} \varepsilon_i g(\nabla_U \nabla_{e_i} D\lambda, e_i)\\
=& \sum_{i=1}^{n} \varepsilon_i g(R(e_i, U)D\lambda, e_i) + \sum_{i=1}^{n} \varepsilon_i U(g(\nabla_{e_i} D\lambda, e_i))\\
=& S(U, D\lambda) + U(\Delta\lambda), && (24)
\end{aligned}
$$

for all vector field U. From (24), we have

$$div(Hess\lambda) = Q(D\lambda) + d(\Delta\lambda). \tag{25}$$

Again, computing the divergence of λS, we obtain

$$div(\lambda S)(U) = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} \lambda S)(U, e_i)$$

$$=\sum_{i=1}^{n}\varepsilon_i[e_i(\lambda)S(U,e_i)+\lambda(\nabla_{e_i}S)(U,e_i)],$$

which gives

$$div(\lambda S)=Q(D\lambda)+\frac{\lambda}{2}dr. \tag{26}$$

At the end, by the parallelity of the semi-Riemannian metric g, we get

$$\begin{aligned}
div(\Delta\lambda.g)(U)=&\sum_{i=1}^{n}\varepsilon_i(\nabla_{e_i}\Delta\lambda.g)(U,e_i)\\
=&\sum_{i=1}^{n}\varepsilon_i[e_i(\Delta\lambda.g)(U,e_i)-\Delta\lambda.g(\nabla_{e_i}U,e_i)-\Delta\lambda.g(U,\nabla_{e_i}e_i)]\\
=&\sum_{i=1}^{n}\varepsilon[e_i(\Delta\lambda)g(U,e_i)+\Delta\lambda\{e_ig(U,e_i)-g(\nabla_{e_i}U,e_i)-g(U,\nabla_{e_i}e_i)\}]\\
=&\sum_{i=1}^{n}\varepsilon_ie_i(\Delta\lambda)g(U,e_i)\\
=&\sum_{i=1}^{n}\varepsilon_ig(U,e_i(\Delta\lambda)e_i)\\
=&g(U,d(\Delta\lambda)),
\end{aligned}$$

which implies

$$div(\Delta\lambda.g)=d(\Delta\lambda). \tag{27}$$

If (g,λ) is a non-trivial solution of the Fischer-Marsden equation, i.e. $\lambda\neq 0$, then from (2), we have

$$-(\Delta_g\lambda)g+Hess_g\lambda-\lambda S=0. \tag{28}$$

Taking the divergence in (28), and using (25), (26) and (27), we have

$$\frac{\lambda}{2}dr=0. \tag{29}$$

Since $\lambda\neq 0$, from (29), the scalar curvature r is constant. □

Proposition 1. *[4] If (g,λ) is a non-trivial solution of the Fischer-Marsden equation on a $(2n+1)$-dimensional paracontact metric manifold M, then the Riemanian curvature tensor and Fischer-Marsden equation can be expressed as*

$$R(U,V)D\lambda=U(\lambda)QV-V(\lambda)QU+\lambda\{(\nabla_UQ)V-(\nabla_VQ)U\}+U(f)V-V(f)U, \tag{30}$$

and

$$\nabla_UD\lambda=\lambda QU+fU, \tag{31}$$

where $f=-\frac{\lambda r}{2n}$, λ is a function of Fischer-Marsden equation and $U,V\in\chi(M)$.

On a K-paracontact manifold, we have $L_\xi Q=0$ [22]. Then using $L_\xi Q=0$ and (9), we have the following result.

Lemma 1. *On a $(2n+1)$-dimensional K-paracontact manifold, we have*

$$\nabla_\xi Q=QF-FQ. \tag{32}$$

Theorem 2. *Let (g,λ) be a non-trivial solution of Fischer-Marsden equation on a K-paracontact manifold M of dimension $(2n+1)$. Then either*

(1) *$\xi(\lambda)=\pm\lambda$, or*
(2) *the manifold is an Einstein manifold, or*

(3) *the $C \neq 0$ tensor defined by $C = Q + 2nI$ and $1 \leq rank(C_p) \leq n$ for all $p \in M$, where $C_p \neq 0$. Further, there exists a basis $\{U_1, V_1, \ldots, U_n, V_n, \xi\}$ of T_pM such that*

$$g_p(\xi, \xi) = 1, g_p(U_i, V_i) = \pm 1$$

and

$$C_{|<U_i,V_i>} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad or \quad C_{|<U_i,V_i>} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where there are exactly $rank(C_p)$ submatrices of the first type.

If $n = 1$, such a basis $\{\xi, U_1, V_1\}$ satisfies that $FU_1 = \pm U_1, FV_1 = \mp V_1$, and the tensor C can be written as

$$C_{|<U_i,V_i>} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. From (9) and (10), we derive

$$(\nabla_U Q)\xi = 2nFU + QFU. \tag{33}$$

Letting $U = \xi$ in (30), we get

$$\begin{aligned} R(\xi, V)D\lambda =& \xi(\lambda)QV - V(\lambda)Q\xi + \lambda\{(\nabla_\xi Q)V - (\nabla_V Q)\xi\} \\ &+ \xi(f)V - V(f)\xi. \end{aligned}$$

In above equation, using (10), (32) and (33), we obtain

$$R(\xi, V)D\lambda = \xi(\lambda)QV + 2nV(\lambda)\xi - \lambda FQV - 2n\lambda FV + \xi(f)V - V(f)\xi. \tag{34}$$

Taking the inner product of (34) with the vector field U, we get

$$\begin{aligned} -g(R(V, \xi)D\lambda, U) = &\xi(\lambda)S(V, U) + 2nV(\lambda)\eta(U) + \lambda S(FU, V) \\ &- 2n\lambda g(FV, U) + \xi(f)g(V, U) - V(f)\eta(U). \end{aligned} \tag{35}$$

From (11) and (35), we have

$$\begin{aligned} &g((\nabla_V F)U, D\lambda) + \xi(\lambda)S(V, U) + [2nV(\lambda) - V(f)]\eta(U) \\ &- 2n\lambda g(FV, U) + \xi(f)g(V, U) + \lambda S(FU, V) = 0. \end{aligned} \tag{36}$$

Letting $U = FU$ and $V = FV$ in (36), we obtain

$$g((\nabla_{FV} F)FU, D\lambda) + \xi(\lambda)S(FV, FU) + \xi(f)g(FV, FU) - 2n\lambda g(F^2V, FU) + \lambda S(F^2U, FV) = 0. \tag{37}$$

By substracting (37) from (36) and using the equations (6), (7), (10) and (12), we get

$$\begin{aligned} &2\xi(\lambda - f)g(U, V) - V((2n + 1)\lambda - f)\eta(U) - \xi(\lambda - f)\eta(U)\eta(V) - \xi(\lambda)S(V, U) \\ &+ 4n\lambda g(FV, U) + \lambda g(U, QFV + FQV) + \xi(\lambda)g(QFV, FU) = 0. \end{aligned} \tag{38}$$

Since S is a symmetric tensor, we also have

$$\begin{aligned} &2\xi(\lambda - f)g(U, V) - U((2n + 1)\lambda - f)\eta(V) - \xi(\lambda - f)\eta(U)\eta(V) - \xi(\lambda)S(V, U) \\ &+ 4n\lambda g(FU, V) + \lambda g(V, QFU + FQU) + \xi(\lambda)g(QFU, FV) = 0. \end{aligned} \tag{39}$$

The equations (38) and (39) implies

$$0 = U((2n + 1)\lambda - f)\eta(V) - V((2n + 1)\lambda - f)\eta(U) + 8n\lambda g(FV, U) + 2\lambda g(U, QFV + FQV). \tag{40}$$

Putting $U = FU$ and $V = FV$ in (40), we obtain

$$4n\lambda g(FV, U) = -\lambda[g(U, QFV) + g(U, FQV)].$$

Since $\lambda \neq 0$ on M, we derive

$$-4nFV = (QF + FQ)V, \tag{41}$$

for all $V \in \chi(M)$. Let $\{e_i, Fe_i, \xi\}$, $(i = 1, 2, \ldots, n)$ be a local orthonormal F-basis. Using (7), we get

$$g(FQe_i, Fe_i) = -g(Qe_i, e_i). \tag{42}$$

By the definition of the scalar curvature, (41) and (42), we have

$$\begin{aligned} r &= S(\xi,\xi) + \sum_{i=1}^{n} \varepsilon_i \{S(e_i,e_i) + S(Fe_i,Fe_i)\} \\ &= g(Q\xi,\xi) + \sum_{i=1}^{n} \varepsilon_i \{g(QFe_i + FQe_i, Fe_i)\} \\ &= -2n(2n+1). \end{aligned} \tag{43}$$

Therefore, from the Proposition 1 the following equation is valid

$$f = (2n+1)\lambda. \tag{44}$$

Taking the inner product of (34) with $D\lambda$ and using in (44), we obtain

$$\xi(\lambda)[QD\lambda + 2nD\lambda] + \lambda[QFD\lambda + 2nFD\lambda] = 0. \tag{45}$$

Letting $D\lambda = V$ in (41) implies $QFD\lambda = -4nFD\lambda - FQD\lambda$. Hence, using the last equation, (45) becomes

$$\xi(\lambda)[QD\lambda + 2nD\lambda] + \lambda[-2nFD\lambda - FQD\lambda] = 0. \tag{46}$$

Finally, applying F to (46) and using (6), we have

$$\xi(\lambda)[FQD\lambda + 2nFD\lambda] + \lambda[-2nD\lambda - QD\lambda] = 0.$$

After some calculations, the last two equations imply

$$[(\xi(\lambda))^2 - \lambda^2][QD\lambda + 2nD\lambda] = 0.$$

Then, either $\xi(\lambda) = \pm\lambda$ or $QD\lambda + 2nD\lambda = 0$. Assume that $\xi(\lambda) \neq \pm\lambda$. Hence, $QD\lambda + 2nD\lambda = 0$. Taking the covariant derivative of $QD\lambda + 2nD\lambda = 0$ along the vector field U and using (31), we get

$$(\nabla_U Q)D\lambda + \lambda Q^2 U + (2n\lambda + f)QU + 2nfU = 0.$$

Contracting above equation over U with respect to a local orthonormal F-basis, we obtain

$$\sum_{i=1}^{n} \varepsilon_i [g((\nabla_{e_i} Q)D\lambda, e_i) + g((\nabla_{Fe_i} Q)D\lambda, Fe_i)] + g((\nabla_\xi Q)D\lambda, \xi) + \lambda|Q|^2 + (2n\lambda + f)r + 2n(2n+1)f = 0. \tag{47}$$

Using the well-known formula $divQ = \frac{1}{2}dr$ and (43), since $\lambda \neq 0$, from (47) we derive $|Q|^2 = 4n^2(2n+1)$. Finally, using the last equation and (43), we compute

$$|Q - \frac{r}{2n+1}I|^2 = |Q|^2 - \frac{2r^2}{2n+1} + \frac{r^2}{2n+1} = 0. \tag{48}$$

From (43) and (48), we have $|C|^2 = 0$, where the tensor $C = Q + 2nI$. Then, there are two possibilities. If $C = 0$, then $Q = -2nI$. In the case of $C \neq 0$, since C is self-adjoint and $Ker(\eta)$ is C-invariant we have from [18, p.260] that, at each point $p \in M$, $Ker(\eta_p) = W_1 \oplus \cdots \oplus W_l$ for some $(1 \leq l \leq 2n)$, where V_k are mutually orthogonal subspaces that are C-invariant and on $C_{|W_k}$ has matrix of either type:

$$\begin{pmatrix} \bar{\gamma} & & & & \\ 1 & \bar{\gamma} & & 0 & \\ & 1 & \bar{\gamma} & & \\ & & \ddots & \ddots & \\ & 0 & & 1 & \bar{\gamma} \end{pmatrix}$$

relative to a basis $U_1, \ldots, U_r$ of $W_k, r \geq 1$, such that the only non-zero products are $g_p(U_i, U_j) = \pm 1$ if $i + j = r + 1$, or of type

$$\begin{pmatrix} a & b & & & & & & & & \\ -b & a & & & & & 0 & & & \\ 1 & 0 & a & b & & & & & & \\ 0 & 1 & -b & a & & & & & & \\ & & 1 & 0 & a & b & & & & \\ & & 0 & 1 & -b & a & & & & \\ & & & & \ddots & & \ddots & & & \\ & & 0 & & & & & & & \\ & & & & & & 1 & 0 & a & b \\ & & & & & & 0 & 1 & -b & a \end{pmatrix} (b \neq 0)$$

relative to a basis $U_1, V_1, \ldots, U_m, V_m$ of W_k, such that the only non-zero products are $g_p(U_i, U_j) = 1 = -g_p(V_i, V_j)$ if $i + j = m + 1$. For $n = 1$, the rest of the proof is similar to the proof of Theorem 3.2 in [17]. This completes the proof. □

Theorem 3. *Let (g, λ) be a non-trivial solution of Fischer-Marsden equation on a K-paracontact manifold of dimension $(2n+1)$ that Ricci operator commutes, i.e. $QF = FQ$. Then the manifold is an Einstein manifold.*

Proof. From the assumption, (41) returns

$$-4nFV = 2FQV. \tag{49}$$

Applying F to the (49) and using (6) and (10), we obtain $QV = -2nV$. Hence, the manifold is an Einstein manifold. □

Remark 1. *In a $(2n+1)$-dimensional para-Sasakian manifold M satisfies the relation $S(FU, FV) = -S(U, V) - 2n\eta(U)\eta(V)$ [23, Lemma 3.15]. Letting $V = FV$ in the last equation, one can observe that the Ricci tensor commutes.*

With the help of the Theorem 4.1 in [13], Theorem 3 and Remark 1, we can state the following corollary.

Corollary 1. *If a $(2n+1)$-dimensional para-Sasakian manifold admits a non-trivial solution of Fischer-Marsden equation, then it is an Einstein manifold. Moreover, for $n = 1$, the Ricci tensor is parallel and the manifold is Ricci-semisymmetric.*

Corollary 2. *If (M^{2n+1}, g) is a K-paracontact manifold admitting a non-trivial solution of the Fischer-Marsden equation with $QF = FQ$, then g is a gradient Ricci soliton.*

Proof. Since the Ricci operator commutes with F, we have $Q = -2nI$ from Theorem 3. Then using this and (44), the equation (31) becomes

$$\nabla_U D\lambda = \lambda U,$$

which gives

$$Hess(\lambda)(U, V) = \lambda g(U, V). \tag{50}$$

In the view of (50) and $Q = -2nI$, we have

$$Hess\lambda + S - (\lambda - 2n)g = 0. \tag{51}$$

It follows from (4) and (51), g is a gradient Ricci soliton.

□

4. 3-Dimensional Quasi-para-Sasakian Manifolds Admitting Fischer-Marsden Equation

In this section, we will consider 3-dimensional quasi-para-Sasakian manifolds with β constant which admits Fischer-Marsden equation. The general form of the following proposition is given in [14].

Proposition 2. *For a 3-dimensional quasi-para-Sasakian manifold M^3, the following equation holds*

$$(\nabla_V Q)\xi - (\nabla_\xi Q)V = -\beta(3\beta^2 + \frac{r}{2})FV,$$

for any vector field V.

Proof. Taking the covariant derivative of (17) along the vector field V and using the equations (13), (16) and (17), we get

$$(\nabla_V Q)\xi = -\beta(3\beta^2 + \frac{r}{2})FV. \tag{52}$$

Let $\{e, Fe, \xi\}$ be a local orthonormal F-basis. Using the well-known formula $divQ = \frac{dr}{2}$ and contracting (52) over V with respect to a local orthonormal F-basis, we obtain

$$\xi(r) = 0.$$

Similarly, taking the covariant derivative of (16) along ξ, we have

$$(\nabla_\xi Q)V = 0, \tag{53}$$

which completes the proof. □

Theorem 4. *Let (g, λ) be a non-trivial solution of Fischer-Marsden equation on a 3-dimensional quasi-para-Sasakian manifold M^3 with β constant. Then either*

(1) *the scalar curvature is $-6\beta^2$ and M^3 is Einstein, or*
(2) *M^3 is a paracosymplectic manifold which is locally a product of the real line $\mathbb{R}$ and a 2-dimensional para-Kaehlerian manifold, and η-Einstein.*

Proof. Letting $U = \xi$ in (30) and taking the inner product with E, we have

$$\begin{aligned} g(R(\xi, V)D\lambda, E) =& \xi(\lambda)S(V, E) - V(\lambda)S(\xi, E) + \lambda\{g((\nabla_\xi Q)V - (\nabla_V Q)\xi, E)\} \\ &+ \xi(f)g(V, E) - V(f)g(\xi, E). \end{aligned} \tag{54}$$

After some calculations, using the equations (16), (17) and (53) in (54), we get

$$g(R(\xi, V)D\lambda, E) = \xi(\lambda)S(V, E) + 2\beta^2 V(\lambda)\eta(E) + \lambda\beta(3\beta^2 + \frac{r}{2})g(FV, E) + \xi(f)g(V, E) - V(f)\eta(E). \tag{55}$$

We recall that the scalar curvature r is constant from Theorem 1. Putting $E = \xi$ in (55) and using the equation (15) and $f = -\frac{\lambda r}{2}$, we obtain

$$g(R(\xi, V)D\lambda, \xi) = (\frac{r}{2} + 2\beta^2)(V(\lambda) - \eta(V)\xi(\lambda)). \tag{56}$$

From (14), one can get

$$g(R(\xi, V)D\lambda, E) = -\beta^2(V(\lambda)\eta(E) - \xi(\lambda)g(V, E)). \tag{57}$$

For $E = \xi$ in (57), we obtain

$$g(R(\xi, V)D\lambda, \xi) = -\beta^2(V(\lambda) - \xi(\lambda)\eta(V)). \tag{58}$$

Therefore, the equations (56) and (58) imply

$$(\frac{r}{2} + 3\beta^2)(V(\lambda) - \eta(V)\xi(\lambda)) = 0. \tag{59}$$

From the above equation, two cases occur. We now check, case by case, whether (59) give rise to a local classification.

Case I: If $\frac{r}{2} + 3\beta^2 = 0$, then scalar curvature r is $-6\beta^2$.

Case II: If

$$V(\lambda) - \eta(V)\xi(\lambda) = 0, \tag{60}$$

then the gradient of λ is colinear with ξ, i.e. $D\lambda = \xi(\lambda)\xi$. Taking the covariant derivative of the last equation along the vector field U implies

$$\nabla_U D\lambda = \nabla_U(\xi(\lambda))\xi + \xi(\lambda)\nabla_U \xi. \tag{61}$$

Taking the inner product of (61) with V and using (13), we obtain

$$g(\nabla_U D\lambda, V) = U(\xi(\lambda))\eta(V) + \beta\xi(\lambda)g(FU, V). \tag{62}$$

Interchancing U and V in (62), we get

$$g(\nabla_V D\lambda, U) = V(\xi(\lambda))\eta(U) + \beta\xi(\lambda)g(FV, U). \tag{63}$$

Since the Hessian operator is symmetric, the equations (62) and (63) imply

$$U(\xi(\lambda))\eta(V) - V(\xi(\lambda))\eta(U) = 2\beta\xi(\lambda)g(U, FV). \tag{64}$$

Putting $U = FU$ and $V = FV$ in (64), we have

$$2\beta\xi(\lambda)g(FU, V) = 0.$$

If $\beta = 0$, then the manifold is paracosymplectic which is locally a product of the real line $\mathbb{R}$ and a 2-dimensional para-Kaehlerian manifold and η-Einstein from (15). Let $\xi(\lambda) = 0$ and $\beta \neq 0$. Then, from (60), λ is constant. Therefore, from (2), the Ricci operator S is zero. Hence, the manifold is Ricci flat. Using (15), we get $\beta = 0$, which is a contradiction of our assumption. So, this case does not occur. □

Corollary 3. *Let (M^3, g) is a quasi-para-Sasakian manifold that admitting non-trivial solution of Fischer-Marsden equation. Then either g is a gradient Ricci soliton or is a gradient η-Ricci soliton.*

Proof. From the assumption and Theorem 4, there are two possibilities.

Case I: If $r = -6\beta^2$, then the equations (16), (31) and $f = -\frac{\lambda r}{2}$ implies

$$\nabla_U D\lambda = \lambda\beta^2 U.$$

With similar idea in the proof of Corollary 2, we have

$$Hess\lambda + S - \beta^2(\lambda - 2)g = 0.$$

It means that g is a gradient Ricci soliton.

Case II: If $\beta = 0$, then we have $S(U, V) = \frac{r}{2}[g(U, V) - \eta(U)\eta(V)]$. On the other hand, using (16) and (31), we get $Hess\lambda = -\lambda\frac{r}{2}\eta \otimes \eta$. In the view of the last two equations, we obtain

$$Hess\lambda + S - \frac{r}{2}g + \frac{r}{2}(1 + \lambda)\eta \otimes \eta = 0,$$

which shows that from (5), g is a gradient η-Ricci soliton.

□

Author Contribution Statements All authors contributed equally to this work. All authors read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

Acknowledgements We would like to thank to Prof. Dr. Cengizhan Murathan for his valuable comments for the improvement of this paper.

References

[1] Bourguignon, J. P., Une stratifcation de l'espace des structures Riemanniennes, *Compos. Math.*, 30 (1975), 1—41.

[2] Chaubey, S. K., De, U. C., Suh Y. J., Kenmotsu manifolds satisfying the Fischer-Marsden equation, *J. Korean Math. Soc.*, 58(3) (2021), 597–607. https://doi.org/10.4134/JKMS.j190602

[3] Chaubey, S.K., Vîlcu, GE., Gradient Ricci solitons and Fischer–Marsden equation on cosymplectic manifolds, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.*, 116 (2022), 186. https://doi.org/10.1007/s13398-022-01325-2

[4] Chaubey, S. K., Khan, M.A., Al Kaabi, A.S.R., $N(\kappa)$-paracontact metric manifolds admitting the Fischer-Marsden conjecture, *AIMS Mathematics*, 9(1) (2023), 2232–2243. https://doi.org/10.3934/math.2024111

[5] Chen, X., Yang, Yifan, Static perfect fluid space-time on contact metric manifold, *Period. Math. Hung.*, 86 (2023), 160–171. https://doi.org/10.1007/s10998-022-00466-6

[6] Cho, J. T., Kimura M., Ricci solitons and real hypersurfaces in a complex space form, *Tohoku Math. J.*, 61(2) (2009), 205—212. https://doi.org/10.2748/tmj/1245849443

[7] Coutinho, F., Diógenes, R., Leandro, B., Ribeiro, E., Static perfect fluid space-time on compact manifolds, *Class. Quant. Grav.*, 37(1) (2019), 015003. https://doi.org/10.1088/1361-6382/ab5402

[8] Corvino, J., Scalar curvature deformations and a gluing construction for the Einstein constraint equations, *Commun. Math. Phys.*, 214 (2000), 137–189. https://doi.org/10.1007/PL00005533

[9] Costa, J., Diógenes, R., Pinheiro, N., Ribeiro, E., Geometry of static perfect fluid space-time, *Class. Quant. Grav.*, 40(20) (2023), 205012. https://doi.org/10.1088/1361-6382/acf8a7

[10] Fischer, A. E., Marsden, J., Manifolds of Riemannian metrics with prescribed scalar curvature, *Bull. Am. Math. Soc.*, 80 (1974), 479–484.

[11] Hamilton, R.S., The Ricci-flow on surfaces, *Contemporary Mathematics, Santa Cruz, CA,*, 71 (1986), 237–262. https://doi.org/10.1090/conm/071/954419

[12] Kobayashi, O., A differential equation arising from scalar curvature function, *J. Math. Soc. Jpn.*, 34 (1982), 665—675. https://doi.org/10.2969/jmsj/03440665
[13] Küpeli Erken, I., On normal almost paracontact metric manifolds of dimension 3, *Facta Univ. Ser. Math. Inform.*, 30(5) (2015), 777-788.
[14] Küpeli Erken, I., Classification of three-dimensional conformally flat quasi-para-Sasakian manifolds, *Honam Mathematical J.*, 41(3) (2019), 489-503. http://doi.org/10.5831/HMJ.2019.41.3.489
[15] Küpeli Erken, I., Curvature properties of quasi-para-Sasakian manifolds, *International Electronic Journal of Geometry*, 12(2) (2019), 210-217. https://doi.org/10.36890/iejg.628085
[16] Lafontaine, J., Sur la geometrie d'une generalisation de l'equation differentielle d'Obata, *J. Math. Pures Appl.*, 62 (1983), 63-72.
[17] Martin-Molina, V., Paracontact metric manifolds without a contact metric counterpart, *Taiwanese J. Math.*, 19(1) (2015), 175-191. https://doi.org/10.11650/tjm.19.2015.4447
[18] O'Neill, B., Semi-Riemann Geometry, Academic Press, New York, 1983.
[19] Patra, D.S., Ghosh, A., The Fischer–Marsden conjecture and contact geometry, *Period. Math. Hung.*, 76 (2018), 207-216. https://doi.org/10.1007/s10998-017-0220-1
[20] Sarkar, A., Biswas, G. G., Critical point equation on K-paracontact manifolds, *Balkan Journal of Geometry and Its Applications*, 25(1) (2020), 117-126.
[21] Welyczko, J., On Legendre curves in 3-dimensional normal almost paracontact metric manifolds, *Result Math.*, 54 (2009), 377-387. https://doi.org/10.1007/s00025-009-0364-2
[22] Yano, K., Integral Formulas in Riemannian Geometry, Marcel Dekker, New York, 1970.
[23] Zamkovoy, S., Canonical connections on paracontact manifolds, *Ann. Glob. Anal. Geom.*, 36 (2009), 37-60. https://doi.org/10.1007/s10455-008-9147-3