http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 4, Pages 969–981 (2024)
DOI:10.31801/cfsuasmas.1485446
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: May 17, 2024; Accepted: October 21, 2024

ALMOST INNER DERIVATIONS OF LEIBNIZ ALGEBRAS

Nil MANSUROĞLU[1] and Mücahit ÖZKAYA[2]

[1,2]Department of Mathematics, Kırşehir Ahi Evran University, Kırşehir, TÜRKİYE

Abstract. This work is presented the study on almost inner derivations of Leibniz algebras. In this note, we demonstrate the natural extensions of some general properties on derivations given for Lie algebras to Leibniz algebras with finite dimension, and also we investigate which statements a mapping have to hold to be an almost inner derivation.

1. Introduction

Leibniz algebras which were first initiated by Loday [10] are as a generalization of Lie algebras. Loday and Pirashvili in [11] investigated such algebras by using homological algebras. In literature, many papers have consisted of the results which show the similarities and the differences between Lie and Leibniz algebras. In the paper [9] M. Ladra and et al. studied the derivations of Leibniz algebras and they extended several common properties of derivations and automorphisms given for Lie algebras to Leibniz algebras with finite dimensions over $\mathbb{C}$. The paper [16] of C. Zargeh is proved that if Leibniz algebra $\mathcal{L}$ has a derivation $\delta : \mathcal{L} \to \mathcal{L}$ satisfying $\mathcal{L}^m \subset \delta(\mathcal{L})$ for some $m > 1$ where $\mathcal{L}^m$ is the m-th terms of lower central series of $\mathcal{L}$, then $\mathcal{L}$ is solvable. The derivations of Leibniz algebras are studied in many papers including [4, 5]. There exist still several open natural questions. One of those questions is on the almost inner derivations which were not considered for Leibniz algebras.

The principal goal of this note is to demonstrate the important consequences on almost inner derivations of Leibniz algebras which are analogs to the consequences in Lie algebras. Our fundamental starting point is presented by the papers [3, 7, 14, 15] which studied on almost inner derivations of Lie algebras.

This paper is planned as follows. Several definitions and notations are introduced in Section 2. Section 3 is presented to the notion of almost inner derivation. First,

2020 *Mathematics Subject Classification.* 17A32, 17A60.
Keywords. Leibniz algebra, derivation, almost inner derivation.

[1] nil.mansuroglu@ahievran.edu.tr-Corresponding author; 0000-0002-6400-2115
[2] ogr.m.ozkaya@ahievran.edu.tr; 0000-0002-6436-8360.

©2024 Ankara University
Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics

we examine some special types of derivations, this concerns the almost inner ones which form a generalization of the inner derivations. Then we derive a procedure to figure out the set of all almost inner derivations, and we also give an example for this method. In Section 4, we investigate which statements a general mapping have to hold to be an almost inner derivation by using the structure constants. In the concluding Section 5, we focus on fixed basis vectors for an arbitrary derivation. In particular, we prove that if any basis vector for all almost inner derivations is fixed, then the set of all almost inner derivations is equal to the set of all inner derivations.

2. Preliminaries

This section introduces the concepts of Lie algebra and Leibniz algebra which will be used in later sections. The material in this section is based on [1,2,8,10,13]. Given a field K with characteristic zero. Recall that an algebra $\mathcal{L}$ over K is Lie algebra if the algebra satisfies the following properties

$$\begin{aligned} &\text{(i)} \quad pp = 0, \ \text{(anti-commutativity)} \\ &\text{(ii)} \quad (pq)r + (qr)p + (rp)q = 0 \ \text{(Jacobi identity)} \end{aligned}$$

for all $p, q, r \in \mathcal{L}$. Let $\mathcal{L}$ be a Lie algebra and $\mathcal{I}$ be a subspace of $\mathcal{L}$. If $xy \in \mathcal{I}$ for all $x \in \mathcal{I}$ and $y \in \mathcal{L}$, $\mathcal{I}$ is said to be a Lie ideal of $\mathcal{L}$. The set of all linear maps on $\mathcal{L}$, $gl(\mathcal{L})$, becomes a Lie algebra with Lie product given by $[h_1, h_2] = h_1h_2 - h_2h_1$ for every $h_1, h_2 \in gl(\mathcal{L})$.
An algebra $\mathcal{L}$ over K with an operation $[,] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is said to be a left Leibniz algebra if $\mathcal{L}$ holds Leibniz identity

$$[[p, q], r] = [p, [q, r]] - [q, [p, r]]$$

for every p, q, r in $\mathcal{L}$. Similarly, we say a right Leibniz algebra if $\mathcal{L}$ holds Leibniz identity

$$[p, [q, r]] = [[p, q], r] - [[p, r], q].$$

We use left Leibniz algebra the rest of this paper. We give the left normed convention for Leibniz brackets, that is,

$$[p_1, p_2, p_3, \ldots, p_s] = [[\ldots[[p_1, p_2], p_3], \ldots], p_s]$$

for all $p_1, p_2, \ldots, p_s \in \mathcal{L}$.

It is clear that Leibniz algebra is obvious a generalization of Lie algebra. Given a subspace $\mathcal{I}$ of a Leibniz algebra $\mathcal{L}$, $\mathcal{I}$ is a subalgebra if $[p, q] \in \mathcal{I}$ for every $p, q \in \mathcal{I}$. If $[p, q] \in \mathcal{I}$ and $[q, p] \in \mathcal{I}$ for every $p \in \mathcal{L}$ and $q \in \mathcal{I}$, then we say $\mathcal{I}$ an ideal of $\mathcal{L}$ and we denote by $\mathcal{I} \trianglelefteq \mathcal{L}$. The left centre of $\mathcal{L}$ is denoted by $C^l(\mathcal{L}) = \{p \in \mathcal{L} | [p, q] = 0 \text{ for every } q \in \mathcal{L}\}$ and the right centre of $\mathcal{L}$ is represented by $C^r(\mathcal{L}) = \{p \in \mathcal{L} | [q, p] = 0 \text{ for every } q \in \mathcal{L}\}$. The centre of $\mathcal{L}$ is represented by $C(\mathcal{L}) = C^l(\mathcal{L}) \cap C^r(\mathcal{L})$. Given two Leibniz algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ over K, a linear mapping $\theta : \mathcal{L}_1 \to \mathcal{L}_2$ is said to be a homomorphism if it satisfies that $\theta([p,q]) = [\theta(p), \theta(q)]$ for every $p, q \in \mathcal{L}_1$. The series of ideals

$$\mathcal{L} = \mathcal{L}^1 \supseteq \mathcal{L}^2 \supseteq \ldots \supseteq \mathcal{L}^k \supseteq \mathcal{L}^{k+1} \supseteq \ldots$$

where for positive integer m, $\mathcal{L}^{m+1} = [\mathcal{L}, \mathcal{L}^m]$ is called the lower central series of $\mathcal{L}$. We say nilpotent of class c if a Leibniz algebra holds that $\mathcal{L}^{c+1} = 0$ but $\mathcal{L}^c \neq 0$. Hence, if $\mathcal{L}$ is nilpotent of class c, we have $\mathcal{L}^c \subseteq C^r(\mathcal{L})$. We also have $\mathcal{L}^c \subseteq C^l(\mathcal{L})$. Therefore, $\mathcal{L}^c \subseteq C^l(\mathcal{L}) \cap C^r(\mathcal{L}) = C(\mathcal{L})$ and $C(\mathcal{L}) \neq 0$.

3. Derivations of Leibniz Algebras

Definition 1. *Given a Leibniz algebra $\mathcal{L}$ over a field K. A derivation of $\mathcal{L}$ is a K-linear mapping $\delta : \mathcal{L} \to \mathcal{L}$ given by $\delta([p,q]) = [\delta(p), q] + [p, \delta(q)]$ for every $p, q \in \mathcal{L}$.*

By $der\mathcal{L}$, we represent the set of all derivations in $\mathcal{L}$. This set with the following multiplication

$$[,] : der\mathcal{L} \times der\mathcal{L} \to der\mathcal{L}$$

by $[\delta_1, \delta_2] = t(\delta_1)\delta_2 - \delta_2 t(\delta_1)$ where t is a linear operator with $t^2 = t$ is an algebra, it is called derivation algebra. Indeed, for any $\delta_1, \delta_2 \in der\mathcal{L}$ and $p, q \in \mathcal{L}$ we obtain

$$\begin{aligned}
[\delta_1, \delta_2]([p,q]) &= (t(\delta_1)\delta_2 - \delta_2 t(\delta_1))([p,q]) \\
&= f(\delta_1)([\delta_2(p), q] + [p, \delta_2(q)]) - \delta_2([t(\delta_1)(p), q] + [p, t(\delta_1)(q)]) \\
&= [t(\delta_1)\delta_2(p), q] - [\delta_2 t(\delta_1)(p), q] + [p, t(\delta_1)\delta_2(q)] - [p, \delta_2 t(\delta_1)(q)] \\
&= [[\delta_1, \delta_2](p), q] + [p, [\delta_1, \delta_2](q)].
\end{aligned}$$

It means that $[\delta_1, \delta_2]$ is a derivation of $\mathcal{L}$. In addition, $der\mathcal{L}$ is a Leibniz algebra. Clearly, $der\mathcal{L}$ is a Lie algebra if t is the identity map.

For any element a in $\mathcal{L}$, the left multiplication operator $\mathcal{L}_a : \mathcal{L} \to \mathcal{L}$ given by $\mathcal{L}_a(p) = [a, p]$ for $p \in \mathcal{L}$. Given a left multiplication $\mathcal{L}_a$, by Leibniz identity we obtain

$$\begin{aligned}
\mathcal{L}_a([p,q]) &= [a, [p,q]] \\
&= [[a,p], q] + [p, [a,q]] \\
&= [\mathcal{L}_a(p), q] + [p, \mathcal{L}_a(q)]
\end{aligned}$$

for all $p, q \in \mathcal{L}$. This shows that $\mathcal{L}_a$ is a derivation of $\mathcal{L}$ and it is said to be inner derivation. The set of all such derivations is represented by $id(L)$.

Lemma 1. *Given a Leibniz algebra $\mathcal{L}$ over K. Then $id(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$ with Lie product. Also $id(\mathcal{L})$ is a Lie ideal of $der\mathcal{L}$.*

Proof. Let $\mathcal{L}_a$ and $\mathcal{L}_b$ two inner derivations of $\mathcal{L}$. For all $p, q \in \mathcal{L}$ we obtain

$$\begin{aligned}
[\mathcal{L}_a, \mathcal{L}_b]([p,q]) &= (\mathcal{L}_a\mathcal{L}_b - \mathcal{L}_b\mathcal{L}_a)([p,q]) \\
&= \mathcal{L}_a([\mathcal{L}_b(p), q] + [p, \mathcal{L}_b(q)]) - \mathcal{L}_b([\mathcal{L}_a(p), q] + [p, \mathcal{L}_a(q)])
\end{aligned}$$

$$\begin{aligned} &= [\mathcal{L}_{[a,b]}(p), q] + [p, \mathcal{L}_{[a,b]}(q)] \\ &= [[[a,b],p],q] + [p,[[a,b],q]] \\ &= [[a,b],[p,q]] \\ &= \mathcal{L}_{[a,b]}([p,q]). \end{aligned}$$

Hence $id(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$. Moreover, for each element $\mathcal{L}_a \in id(\mathcal{L})$ and $\delta \in der\mathcal{L}$, we obtain

$$\begin{aligned} [\mathcal{L}_a, \delta]([p,q]) &= (\mathcal{L}_a\delta - \delta\mathcal{L}_a)([p,q]) \\ &= \mathcal{L}_a([\delta(p),q] + [p,\delta(q)]) - \delta([a,[p,q]] \\ &= -[\delta(a),[p,q]] \\ &= \mathcal{L}_{-\delta(a)}[p,q], \end{aligned}$$

as required. □

Definition 2. *A derivation $\delta \in der\mathcal{L}$ of a Leibniz algebra $\mathcal{L}$ is called an almost inner derivation if $\delta(p) \in [\mathcal{L}, p]$ for all $p \in \mathcal{L}$.*

By $aid(\mathcal{L})$, we represent the set of all almost inner derivations of $\mathcal{L}$. Since $[\mathcal{L}, p] = \{[q,p] | q \in \mathcal{L}\}$, it is obvious that the set of all inner derivations, $id(\mathcal{L})$, is a subset of $aid(\mathcal{L})$.

Lemma 2. *Given a Leibniz algebra $\mathcal{L}$ over K. Then $aid(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$ with Lie product. Also $aid(\mathcal{L})$ is a Lie ideal of $der\mathcal{L}$.*

Proof. Let $\delta_1, \delta_2 \in aid(\mathcal{L})$ and $p \in \mathcal{L}$. Then there are $q_1, q_2 \in \mathcal{L}$ with $\delta_1(p) = [q_1, p]$ and $\delta_2(p) = [q_2, p]$. By applying the derivation condition and Leibniz identity, we have

$$\begin{aligned} [\delta_1, \delta_2](p) &= (\delta_1\delta_2 - \delta_2\delta_1)(p) \\ &= \delta_1([q_2,p]) - \delta_2([q_1,p]) \\ &= [\delta_1(q_2),p] + [q_2,\delta_1(p)] - [\delta_2(q_1),p] - [q_1,\delta_2(p)] \\ &= [\delta_1(q_2) - \delta_2(q_1) + [q_2,q_1],p] \in [\mathcal{L},p]. \end{aligned}$$

Hence we obtain $[\delta_1, \delta_2] \in aid(\mathcal{L})$. So $aid(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$. Moreover, given $\delta \in der\mathcal{L}$ and $h \in aid(\mathcal{L})$. Since $h \in aid(\mathcal{L})$, there is an element $q \in \mathcal{L}$ satisfying $h(p) = [q,p]$. Then

$$\begin{aligned} [h,\delta](p) &= (h\delta - \delta h)(p) \\ &= [q,\delta(p)] - \delta([q,p]) \\ &= -[\delta(q),p]. \end{aligned}$$

Therefore we obtain that $aid(\mathcal{L})$ is a Lie ideal of $der\mathcal{L}$. □

Definition 3. *We say an almost inner derivation δ a central almost inner derivation if there is an element $p \in \mathcal{L}$ with $\delta - \mathcal{L}_p$ maps $\mathcal{L}$ to $C(\mathcal{L})$.*

The set of all central almost inner derivations of $\mathcal{L}$ is denoted by $caid(\mathcal{L})$. We have the following inclusions of Lie subalgebras

$$id(\mathcal{L}) \subseteq caid(\mathcal{L}) \subseteq aid(\mathcal{L}) \subseteq der\mathcal{L}.$$

Clearly, $caid(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$. To see that this subalgebra is a Lie ideal of $aid(\mathcal{L})$ we give the next lemma.

Lemma 3. *Given a Leibniz algebra $\mathcal{L}$ over a field K. Then $caid(\mathcal{L})$ is a Lie ideal of $aid(\mathcal{L})$.*

Proof. Let $\delta_1 \in caid(\mathcal{L})$ and $\delta_2 \in aid(\mathcal{L})$. Then there is an element $p \in \mathcal{L}$ satisfying $\delta_1 - \mathcal{L}_p = \delta_3$ maps $\mathcal{L}$ to $C(\mathcal{L})$ and there is an element $q \in \mathcal{L}$ with $\delta_2(p) = [q, p] \in [\mathcal{L}, p]$. To prove that $caid(\mathcal{L})$ is a Lie ideal of $aid(\mathcal{L})$, we need to show that $[\delta_2, \delta_1] \in caid(\mathcal{L})$. Since $aid(\mathcal{L})$ is an ideal of $der\mathcal{L}$ for every derivations of $\mathcal{L}$, it is clear that $[\delta_2, \delta_1] \in aid(\mathcal{L})$. Suppose that $\delta_4 = [\delta_2, \delta_1] - \mathcal{L}_{\delta_2(p)}$. For any element $r \in \mathcal{L}$,

$$\mathcal{L}_{\delta_2(p)}(r) = \mathcal{L}_{[q,p]}(r) = [[q, p], r] = [\delta_2, \mathcal{L}_p](r).$$

Then we get

$$\mathcal{L}_{\delta_2(p)} = [\delta_2, \mathcal{L}_p]. \tag{1}$$

By (1), we have

$$\delta_4 = [\delta_2, \delta_1] - [\delta_2, \mathcal{L}_p] = [\delta_2, \delta_1 - \mathcal{L}_p] = [\delta_2, \delta_3].$$

It follows that δ_3 maps $\mathcal{L}$ to $C(\mathcal{L})$ and δ_2 maps $C(\mathcal{L})$ to $C(\mathcal{L})$. Hence δ_4 maps $\mathcal{L}$ to $C(\mathcal{L})$. Since there is $\delta_2(p) \in \mathcal{L}$ such that $[\delta_2, \delta_1] - \mathcal{L}_{\delta_2(p)} = \delta_4$ maps $\mathcal{L}$ to $C(\mathcal{L})$, $[\delta_2, \delta_1] \in caid(\mathcal{L})$. □

The results obtained for the derivations of Leibniz algebras are given in the next theorem.

Theorem 1. *Given a Leibniz algebra $\mathcal{L}$. Then the following statements satisfy*

(i) *Let $\delta \in aid(\mathcal{L})$. Then $\delta(\mathcal{L}) \subseteq [\mathcal{L}, \mathcal{L}]$, $\delta(C(\mathcal{L})) = 0$ and $\delta(\mathcal{I}) \subseteq \mathcal{I}$ for any ideal of $\mathcal{L}$.*

(ii) *Let $\delta \in caid(\mathcal{L})$. Then there is an element $p \in \mathcal{L}$ such that $\delta|_{[\mathcal{L},\mathcal{L}]} = \mathcal{L}_p|_{[\mathcal{L},\mathcal{L}]}$.*

(iii) *If $\mathcal{L}$ is a Leibniz algebra with the nilpotency class 2, then $caid(\mathcal{L}) = aid(\mathcal{L})$.*

(iv) *If the centre of $\mathcal{L}$ is zero, then $caid(\mathcal{L}) = id(\mathcal{L})$.*

(v) *If $\mathcal{L}$ is a nilpotent Leibniz algebra, then $aid(\mathcal{L})$ is also nilpotent.*

Proof. (i) If $\delta \in aid(\mathcal{L})$, then for every element $p \in \mathcal{L}$ we have

$$\delta(p) \in [\mathcal{L}, p] \subseteq [\mathcal{L}, \mathcal{L}]. \tag{2}$$

Therefore, for each ideal $\mathcal{I}$ of $\mathcal{L}$ and $p \in \mathcal{I}$ we have

$$\delta(p) \in [\mathcal{L}, \mathcal{I}] \subseteq \mathcal{I} \text{ and } \delta(p) \in [\mathcal{I}, \mathcal{L}] \subseteq \mathcal{I}.$$

Thus $\delta(\mathcal{I}) \subseteq \mathcal{I}$. By (2), we obtain that for all $p \in C(\mathcal{L})$, $\delta(p) = 0$, that is, $\delta(C(\mathcal{L})) = 0$.

(ii) If $\delta \in caid(\mathcal{L})$, then there is an element $p \in \mathcal{L}$ such that $\delta_1 = \delta - \mathcal{L}_p$ maps $\mathcal{L}$ to the centre of $\mathcal{L}$. Namely, $\delta_1(\mathcal{L}) \subseteq C(\mathcal{L})$. Since δ_1 is a derivation of $\mathcal{L}$ and for every $a, b \in \mathcal{L}$,

$$\delta_1([a,b]) = [\delta_1(a), b] + [a, \delta_1(b)] = 0.$$

(iii) We know that from the inclusions of Lie subalgebras $caid(\mathcal{L}) \subseteq aid(\mathcal{L})$. Now we must only show that $aid(\mathcal{L}) \subseteq caid(\mathcal{L})$. Suppose that $\delta \in aid(\mathcal{L})$. Then there is an element $p \in \mathcal{L}$ such that $\delta(p) \in [\mathcal{L}, p] \subseteq [\mathcal{L}, \mathcal{L}]$. Moreover, if $\mathcal{L}$ is a nilpotent Leibniz algebra of class m, then $\mathcal{L}^m \subseteq C^l(\mathcal{L})$. By Proposition 4.2 in [6], $\mathcal{L}^m \subseteq C^r(\mathcal{L})$. Hence we obtain $\mathcal{L}^m \subseteq C^l(\mathcal{L}) \cap C^r(\mathcal{L}) = C(\mathcal{L}) \neq 0$. Since $\mathcal{L}$ is a nilpotent of class 2, we can write from (2)

$$\delta(\mathcal{L}) \subseteq [\mathcal{L}, \mathcal{L}] = \mathcal{L}^2 \subseteq C(\mathcal{L}).$$

This means that δ maps $\mathcal{L}$ to the centre of $\mathcal{L}$, that is, $\delta \in caid(\mathcal{L})$ and $aid(\mathcal{L}) \subseteq caid(\mathcal{L})$. Therefore $aid(\mathcal{L}) = caid(\mathcal{L})$.
(iv) We know that from the inclusions of Lie subalgebras $id(\mathcal{L}) \subseteq caid(\mathcal{L})$. Now we need only to show that $caid(\mathcal{L}) \subseteq id(\mathcal{L})$. Suppose that $\delta \in caid(\mathcal{L})$ and $C(\mathcal{L}) = 0$. Then there is an element $p \in \mathcal{L}$ satisfying $\delta - \mathcal{L}_p$ maps $\mathcal{L}$ to the centre of $\mathcal{L}$. Since $C(\mathcal{L}) = 0$, we have $(\delta - \mathcal{L}_p)(q) = 0$ for all $q \in \mathcal{L}$. Namely $\delta - \mathcal{L}_p = 0$. Thus $\delta = \mathcal{L}_p$. This shows that $\delta \in id(\mathcal{L})$ and $caid(\mathcal{L}) \subseteq id(\mathcal{L})$. Therefore, we obtain $caid(\mathcal{L}) = id(\mathcal{L})$.
(v) Suppose that $\mathcal{L}$ is a Leibniz algebra with the nilpotency class m ($\mathcal{L}^{m+1} = 0, \mathcal{L}^m \neq 0$). For $\delta \in aid(\mathcal{L})$ and $p \in \mathcal{L}$, from (2) we can define nilpotent operator,

$$\begin{aligned}
\delta^1(p) &\in [\mathcal{L}, p] \subseteq [\mathcal{L}, \mathcal{L}] = \mathcal{L}^2 \\
\delta^2(p) &\in [\mathcal{L}, [\mathcal{L}, p]] \subseteq [\mathcal{L}, [\mathcal{L}, \mathcal{L}]] = [\mathcal{L}, \mathcal{L}^2] = \mathcal{L}^3 \\
&\vdots \\
\delta^m(p) &\in [\mathcal{L}, [\ldots, [\mathcal{L}, p] \ldots]] \subseteq [\mathcal{L}, [\ldots, [\mathcal{L}, \mathcal{L}] \ldots]] = [\mathcal{L}, \mathcal{L}^m] = \mathcal{L}^{m+1}.
\end{aligned}$$

Since $\mathcal{L}$ is nilpotent of class m, then $\mathcal{L}^{m+1} = 0$, so $\delta^m = 0$. Therefore δ is a nilpotent. By Engel theorem [[6], Theorem 4.5], $aid(\mathcal{L})$ is nilpotent. □

Example 1. *Given a Leibniz algebra $\mathcal{L}$ over a field K with the basis $\{e_1, e_2, e_3, e_4, e_5\}$ by the following multiplication*

$$\begin{array}{lll}
[e_1, e_2] = e_2, & [e_2, e_1] = -e_2, & [e_4, e_1] = e_5, \\
[e_1, e_4] = e_4, & [e_2, e_3] = e_4, & [e_5, e_1] = -e_5, \\
[e_1, e_5] = e_5, & [e_3, e_2] = e_5, & [e_i, e_j] = 0
\end{array}$$

for other multiplications. We will compute $id(\mathcal{L})$, $aid(\mathcal{L})$ and $caid(\mathcal{L})$. Since every derivation δ of $\mathcal{L}$ is of the following form $\delta(e_1) = \alpha_1 e_2 + \alpha_2 e_5$, $\delta(e_2) = \beta_1 e_2 + \beta_2 e_5$, $\delta(e_3) = \gamma_1 e_3 + \gamma_2 e_4$, $\delta(e_4) = \sigma_1 e_4$, $\delta(e_5) = \tau_1 e_5$, we obtain

$$\begin{aligned}
der\mathcal{L} = \; & \{\delta | \delta(e_1) \in Span\{e_2, e_5\}, \delta(e_2) \in Span\{e_2, e_5\}, \delta(e_3) \in Span\{e_3, e_4\}, \\
& \delta(e_4) \in Span\{e_4\}, \delta(e_5) \in Span\{e_5\}\}.
\end{aligned}$$

By using the definition of inner derivation of $\mathcal{L}$. We obtain the following results

$$\begin{array}{lllll}
\mathcal{L}_{e_1}(e_1)=0, & \mathcal{L}_{e_2}(e_1)=-e_2, & \mathcal{L}_{e_3}(e_1)=0, & \mathcal{L}_{e_4}(e_1)=e_5, & \mathcal{L}_{e_5}(e_1)=-e_5,\\
\mathcal{L}_{e_1}(e_2)=e_2, & \mathcal{L}_{e_2}(e_2)=0, & \mathcal{L}_{e_3}(e_2)=e_5, & \mathcal{L}_{e_4}(e_2)=0, & \mathcal{L}_{e_5}(e_2)=0,\\
\mathcal{L}_{e_1}(e_3)=0, & \mathcal{L}_{e_2}(e_3)=e_4, & \mathcal{L}_{e_3}(e_3)=0, & \mathcal{L}_{e_4}(e_3)=0, & \mathcal{L}_{e_5}(e_3)=0,\\
\mathcal{L}_{e_1}(e_4)=e_4, & \mathcal{L}_{e_2}(e_4)=0, & \mathcal{L}_{e_3}(e_4)=0, & \mathcal{L}_{e_4}(e_4)=0, & \mathcal{L}_{e_5}(e_4)=0,\\
\mathcal{L}_{e_1}(e_5)=e_5, & \mathcal{L}_{e_2}(e_5)=0, & \mathcal{L}_{e_3}(e_5)=0, & \mathcal{L}_{e_4}(e_5)=0, & \mathcal{L}_{e_5}(e_5)=0.
\end{array}$$

It is clear to see that $\mathcal{L}_{e_5}=-\mathcal{L}_{e_4}=\mathcal{L}_{-e_4}$. *Hence we have*

$$id(\mathcal{L})=Span\{\mathcal{L}_{e_1},\mathcal{L}_{e_2},\mathcal{L}_{e_3},\mathcal{L}_{e_4}\}.$$

To obtain $aid(\mathcal{L})$ *we must calculate* $[\mathcal{L},e_i]$ *for all* $1\le i\le 5$,

$$\begin{array}{l}[\mathcal{L},e_1]=Span\{e_2,e_5\},[\mathcal{L},e_2]=Span\{e_2,e_5\},\\ [\mathcal{L},e_3]=Span\{e_4\}=[\mathcal{L},e_4],[\mathcal{L},e_5]=Span\{e_5\}.\end{array}$$

Hence we obtain

$$\begin{array}{rcl} aid(\mathcal{L}) & = & \{\delta|\delta(e_1)\in Span\{e_2,e_5\},\delta(e_2)\in Span\{e_2,e_5\},\delta(e_3)\in Span\{e_4\},\\ & & \delta(e_4)\in Span\{e_4\},\delta(e_5)\in Span\{e_5\}\}.\end{array}$$

To determine the set of all of the central almost inner derivation we need the centre of $\mathcal{L}$, $C(\mathcal{L})=0$. *Take* $\delta\in aid(\mathcal{L})$ *such that* $\delta(e_1)=0,\delta(e_2)=e_2,\delta(e_3)=0,\delta(e_4)=e_4$ *and* $\delta(e_5)=e_5$. *Now we need to show that there exists an element* p *in* $\mathcal{L}$ *such that* $\delta-\mathcal{L}_p$ *maps* $\mathcal{L}$ *to the centre of* $\mathcal{L}$. *Then we have for* $e_1\in\mathcal{L}$

$$\begin{array}{rcl}
(\delta-\mathcal{L}_{e_1})(e_1) & = & \delta(e_1)-\mathcal{L}_{e_1}(e_1)=0-[e_1,e_1]=0-0=0,\\
(\delta-\mathcal{L}_{e_1})(e_2) & = & \delta(e_2)-\mathcal{L}_{e_1}(e_2)=e_2-[e_1,e_2]=e_2-e_2=0,\\
(\delta-\mathcal{L}_{e_1})(e_3) & = & \delta(e_3)-\mathcal{L}_{e_1}(e_3)=0-[e_1,e_3]=0-0=0,\\
(\delta-\mathcal{L}_{e_1})(e_4) & = & \delta(e_4)-\mathcal{L}_{e_1}(e_4)=e_4-[e_1,e_4]=e_4-e_4=0,\\
(\delta-\mathcal{L}_{e_1})(e_5) & = & \delta(e_5)-\mathcal{L}_{e_1}(e_5)=e_5-[e_1,e_5]=e_5-e_5=0.
\end{array}$$

It follows that $\delta-\mathcal{L}_{e_1}$ *maps* $\mathcal{L}$ *to the centre of* $\mathcal{L}$. *For* e_2,e_3,e_4 *we have a similar result, that is why,* $aid(\mathcal{L})$ *consists of only inner derivations. Therefore,* $caid(\mathcal{L})=id(\mathcal{L})$. *As a result,* $\delta\in caid(\mathcal{L})$.

Definition 4. *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be two Leibniz algebras over* K. *The direct sum of the Leibniz algebras* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *which is denoted by* $\mathcal{L}_1\oplus\mathcal{L}_2$ *is the vector space direct sum with* $[\mathcal{L}_1,\mathcal{L}_2]=0$ *and* $[\mathcal{L}_2,\mathcal{L}_1]=0$.

Theorem 2. *Let* $\mathcal{G}$ *and* T *be two Leibniz algebras over* K. *Then* $aid(\mathcal{G}\oplus T)=aid(\mathcal{G})\oplus aid(T)$.

Proof. Let $\delta\in aid(\mathcal{G}\oplus T)$ and $p\in\mathcal{G}\oplus T$. Then $p=p_1+p_2$, where $p_1\in\mathcal{G},p_2\in T$. By the definition of almost inner derivation, $\delta(p)\in[\mathcal{G}\oplus T,p]$ and there is an element $q=q_1+q_2\in\mathcal{G}\oplus T$, where $q_1\in\mathcal{G},q_2\in T$ satisfying $\delta(p)=[q,p]\in[\mathcal{G}\oplus T,p]$. Doing some calculations we get

$$\begin{array}{rcl}\delta(p)=[q,p] & = & [q_1+q_2,p_1+p_2]\\ & = & [q_1,p_1]+[q_1,p_2]+[q_2,p_1]+[q_2,p_2]\end{array}$$

$$= \quad [q_1, p_1] + [q_2, p_2].$$

So $\delta(p) \in \mathcal{G} \oplus T$. We say $\delta_1 = \delta|_{\mathcal{G}} \in aid(\mathcal{G})$, similarly $\delta_2 = \delta|_T \in aid(T)$. Hence δ can be written as $\delta = \delta_1 + \delta_2$, which is defined as

$$\begin{aligned} \delta : \mathcal{G} \oplus T &\rightarrow \mathcal{G} \oplus T \\ p_1 + p_2 &\mapsto (\delta_1 + \delta_2)(p_1 + p_2) = \delta_1(p_1) + \delta_2(p_2). \end{aligned}$$

Furthermore, let $p = p_1 + p_2 \in \mathcal{G} \oplus T, \delta_1 \in aid(\mathcal{G}) \oplus 0$ and $\delta_2 \in 0 \oplus aid(T)$. Then

$$\begin{aligned} [\delta_1, \delta_2](p) &= [\delta_1, \delta_2](p_1 + p_2) \\ &= (\delta_1\delta_2 - \delta_2\delta_1)(p_1) + (\delta_1\delta_2 - \delta_2\delta_1)(p_2) \\ &= (\delta_1\delta_2)(p_1) - (\delta_2\delta_1)(p_1) + (\delta_1\delta_2)(p_2) - (\delta_2\delta_1)(p_2) \\ &= 0. \end{aligned}$$

This means that $[aid(\mathcal{G}), aid(T)] = 0$ and $[aid(T), aid(\mathcal{G})] = 0$. Thus, $aid(\mathcal{G} \oplus T) \subseteq aid(\mathcal{G}) \oplus aid(T)$. Conversely, let $\delta_1 + \delta_2 \in aid(\mathcal{G}) \oplus aid(T)$, where $\delta_1 \in aid(\mathcal{G}) \oplus 0$, $\delta_2 \in 0 \oplus aid(T)$ and

$$\begin{aligned} \delta_1 + \delta_2 : \mathcal{G} \oplus T &\rightarrow \mathcal{G} \oplus T \\ p_1 + p_2 &\mapsto (\delta_1 + \delta_2)(p_1 + p_2) = \delta_1(p_1) + \delta_2(p_2). \end{aligned}$$

By the definition of $aid(\mathcal{G} \oplus T)$, there are $q_1 \in \mathcal{G}$ and $q_2 \in T$ such that $\delta_1(p_1) = [q_1, p_1]$ and $\delta_2(p_2) = [q_2, p_2]$. Therefore, by the definition, there exists $p_1 + p_2 \in \mathcal{G} \oplus T$, where $p_1 \in \mathcal{G}$ and $p_2 \in T$. Since $[q_1, p_2] = [p_1, q_2] = 0$, we have

$$\begin{aligned} (\delta_1 + \delta_2)(p_1 + p_2) &= \delta_1(p_1) + \delta_2(p_2) \\ &= [q_1, p_1] + [q_2, p_2] \\ &= [q_1 + q_2, p_1 + p_2]. \end{aligned}$$

Then $[q_1 + q_2, p_1 + p_2] \in \mathcal{G} \oplus T$ and we obtain $q_1 + q_2 \in \mathcal{G} \oplus T$. This shows that $\delta_1 + \delta_2 \in \mathcal{G} \oplus T$, that is, $aid(\mathcal{G}) \oplus aid(T) \subseteq aid(\mathcal{G} \oplus T)$. As a result, we obtain $aid(\mathcal{G}) \oplus aid(T) = aid(\mathcal{G} \oplus T)$, as required. □

Theorem 3. *Let $\mathcal{L}$ be a Leibniz algebra and $id(\mathcal{L})$ be an ideal of $der\mathcal{L}$ in which each element is nilpotent. Then $aid(\mathcal{L})$ is nilpotent.*

Proof. If each element of $id(\mathcal{L})$ is nilpotent, then there is a positive integer m such that $\mathcal{L}_p^m \neq 0$ and $\mathcal{L}_p^{m+1} = 0$. We have $\mathcal{L}_p^{m+1}(q) \in [\mathcal{L}^{m+1}, q]$. By Corollary 4.8 in [6], $\mathcal{L}$ is nilpotent and so by Theorem 1 (v), $aid(\mathcal{L})$ is nilpotent. □

4. Structure Constants

In this section firstly we derive which conditions a general map have to satisfy to be an almost inner derivation. Recall that if $\mathcal{L}$ is a Leibniz algebra over K with basis $P = \{p_1, p_2, \ldots, p_m\}$, then all elements in $\mathcal{L}$ can be determined by the products

$[p_i, p_j]$. Moreover, each product $[p_i, p_j]$ is expressed by a linear combination of the elements of basis as the following

$$[p_i, p_j] = \sum_{l=1}^{m} c_{ij}^l p_l, \tag{3}$$

where for $1 \leq i, j, l \leq m$, c_{ij}^l are scalars in K. We say that the c_{ij}^l are the structure constants of $\mathcal{L}$ with respect to this basis. The structure constants of $\mathcal{L}$ depend on the choice of basis of $\mathcal{L}$, that is, for different bases, we have different structure constants (more details in [13]).

Since a derivation δ of $\mathcal{L}$ is linear, we have

$$\delta(p_i) = \sum_{j=1}^{m} \alpha_{ij} p_j,$$

where $A = [\alpha_{ij}]_{m\times m}$ is the corresponding matrix of derivation δ. Let p_i and p_j be arbitrary two basis vectors in P. Then

$$\delta([p_i, p_j]) = \sum_{l=1}^{m} c_{ij}^l \delta(p_l) = \sum_{k=1}^{m} (\sum_{l=1}^{m} \alpha_{lk} c_{ij}^l) p_k \tag{4}$$

and

$$\begin{aligned} [\delta(p_i), p_j] + [p_i, \delta(p_j)] &= \sum_{l=1}^{m} \alpha_{il} [p_l, p_j] + \sum_{l=1}^{m} \alpha_{jl} [p_i, p_l] \\ &= \sum_{k=1}^{m} (\sum_{l=1}^{m} (\alpha_{il} c_{lj}^k + \alpha_{jl} c_{il}^k)) p_k. \end{aligned} \tag{5}$$

Hence by (4) and (5), we obtain

$$\sum_{l=1}^{m} \alpha_{lk} c_{ij}^l = \sum_{l=1}^{m} (\alpha_{il} c_{lj}^k + \alpha_{jl} c_{il}^k)$$

for every $1 \leq i, j, k \leq m$. As every derivation, an inner derivation can also be represented by a matrix. Let $\mathcal{L}_{p_i}$ be an inner derivation. Then we have

$$\mathcal{L}_{p_i}(p_j) = [p_i, p_j] = \sum_{k=1}^{m} \beta_{jk} p_k.$$

Hence by the equation (3), we obtain $\beta_{jk} = c_{ij}^k$ for all $1 \leq i, j, k \leq m$. Given an arbitrary $p = \sum_{i=1}^{m} t_i p_i \in \mathcal{L}$, where $t_i \in K$ and let $B = [\beta_{ji}]_{m\times m}$ be the matrix representation of $\mathcal{L}_p$. By using bilinearity of Leibniz bracket, the entries of B are given by

$$\beta_{jk} = \sum_{i=1}^{m} t_i c_{ij}^k.$$

Moreover, there are other conditions imposed by the definition of an almost inner derivation. Indeed, take $\delta \in aid(\mathcal{L})$, there exists a_{ij} with $1 \leq i, j \leq m$, so that

$$\delta(p_i) \quad = \quad \sum_{j=1}^{m} a_{ij}[p_j, p_i] = \sum_{k=1}^{m}\sum_{j=1}^{m} a_{ij}c_{ji}^{k}p_k. \tag{6}$$

These values a_{ij} with $1 \leq i, j \leq m$ are referred to as the parameters of δ with respect to the basis P. By using the bilinearity of a derivation and the equation (6) for any $p = \sum_{i=1}^{m} \beta_i p_i \in \mathcal{L}$ where $\beta_i \in K$ for all $1 \leq i \leq m$, we have

$$\delta(p) = \sum_{i=1}^{m} \beta_i \delta(p_i) = \sum_{k=1}^{m}(\sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i a_{ij} c_{ji}^{k})p_k. \tag{7}$$

Besides, there exist $\gamma_j \in K$ for $1 \leq j \leq m$, so that

$$\delta(p) = [\sum_{j=1}^{m} \gamma_j p_j, p] = \sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i \gamma_j [p_j, p_i] = \sum_{k=1}^{m}(\sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i \gamma_j c_{ji}^{k})p_k. \tag{8}$$

Therefore, we have two ways to write $\delta(p)$. The equations (7) and (8) give a system of linear equations

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i a_{ij} c_{ji}^{k} = \sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i \gamma_j c_{ji}^{k}$$

for all $1 \leq i, j \leq m$. Equivalently,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i (a_{ij} - \gamma_j) c_{ji}^{k} = 0. \tag{9}$$

The aim is to obtain the conditions on the parameters a_{ij} with $1 \leq i, j \leq m$ such that there exist γ_j for which the system of equations (9) has a solution for all possible values of β_i. An arbitrary almost inner derivation $\delta : \mathcal{L} \to \mathcal{L}$ can be expressed as $p \mapsto A.p$ where $A = [\alpha_{ij}]$ is the matrix representation of δ and . is matrix multiplication. By the equation (6), the entries of A are given by

$$\alpha_{ij} = \sum_{k=1}^{m} a_{ij} c_{ji}^{k}.$$

5. Fixed Basis Vectors

Let $\mathcal{L}$ be an m-dimensional Leibniz algebra over K with the basis $P = \{p_1, p_2, \dots, p_m\}$. We denote by $C_{\mathcal{L}}(p)$ the centralizer of p which is defined by

$$C_{\mathcal{L}}(p) = \{q \in \mathcal{L} | [p, q] = [q, p] = 0\}.$$

Let $\delta \in aid(\mathcal{L})$, then there is a mapping $\varphi_\delta : \mathcal{L} \to \mathcal{L}$ satisfying $\delta(p) = [\varphi_\delta(p), p] \in [\mathcal{L}, p]$ for all $p \in \mathcal{L}$. This is not unique because for any $q \in C_{\mathcal{L}}(p)$, we can take

$\varphi_\delta(p) + q$ instead of $\varphi_\delta(q)$. Namely,

$$\delta(p) = [\varphi_\delta(p) + q, p] = [\varphi_\delta(p), q] + [q, p] = [\varphi_\delta(p), p].$$

In general, this map need not be linear. Let $p \in \mathcal{L}$, then p can be written as a linear combination of the basis P such that $p = \sum_{j=1}^{m} \alpha_j p_j$, where $\alpha_j \in K$. We represent by $t_i(p) = \alpha_i$ the i-th projection mapping of p with respect to the given basis.

Definition 5. *A basis vector p_i is called a fixed vector for δ with $\alpha \in K$ iff $t_i(\varphi_\delta(p_j)) = \alpha$ where $p_j \notin C_{\mathcal{L}}(p_i)$ for every $1 \leq j \leq m$.*

Example 2. *Let $\mathcal{L}$ be a 3-dimensional Leibniz algebra with the basis $\{p, q, r\}$ by the following rule $[p, p] = q$ and $[p, q] = r$. Then the centralizers for $p, q, r \in \mathcal{L}$, $C_{\mathcal{L}}(p) = Span\{r\}$, $C_{\mathcal{L}}(q) = Span\{q, r\}$, $C_{\mathcal{L}}(r) = Span\{p, q, r\}$. Let $\delta \in aid(\mathcal{L})$ and φ_δ be a mapping with*

$$\delta(p) = [\varphi_\delta(p), p] = q, \delta(q) = [\varphi_\delta(q), q] = r, \delta(r) = [\varphi_\delta(r), r] = 0.$$

Hence we obtain $\varphi_\delta(p) = p, \varphi_\delta(q) = p$ and $\varphi_\delta(r) \in Span\{p, q, r\}$. In particular, we take a map φ_δ with the following rule

$$\varphi_\delta(p) = p, \varphi_\delta(q) = p, \text{ and } \varphi_\delta(r) = q.$$

Thus, for $p \in \mathcal{L}$ we have

$$\begin{aligned} &p \notin C_{\mathcal{L}}(p), t_1(\varphi_\delta(p)) = t_1(p) = t_1(1.p + 0.q + 0.r) = 1, \\ &v \notin C_{\mathcal{L}}(p), t_1(\varphi_\delta(q)) = t_1(p) = t_1(1.p + 0.q + 0.r) = 1, \\ &r \in C_{\mathcal{L}}(p). \end{aligned}$$

p is fixed basis vector for δ with fixed value $\alpha = 1$. For $q \in \mathcal{L}$

$$\begin{aligned} &p \notin C_{\mathcal{L}}(q), t_2(\varphi_\delta(q)) = t_2(p) = t_2(1.p + 0.q + 0.r) = 0, \\ &q \in C_{\mathcal{L}}(q), \\ &r \in C_{\mathcal{L}}(q). \end{aligned}$$

q is fixed basis vector for δ with fixed value $\beta = 0$. Finally, for $r \in \mathcal{L}$ we obtain $p \in C_{\mathcal{L}}(r)$, $q \in C_{\mathcal{L}}(r)$, $w \in C_{\mathcal{L}}(r)$, this means that r is also fixed basis vector for δ.

Lemma 4. *Let $\mathcal{L}$ be a Leibniz algebra and $\delta \in aid(\mathcal{L})$ which is defined by a mapping $\varphi_\delta : \mathcal{L} \to \mathcal{L}$. If p_i is a fixed basis vector with fixed value α, then $\delta^{'} = \delta + \mathcal{L}_{\alpha p_i} \in aid(\mathcal{L})$ which is determined by a mapping $\varphi_{\delta'} : \mathcal{L} \to \mathcal{L}$ holding*

$$t_i(\varphi_{\delta'}(p_k)) = 2t_i(\varphi_\delta(p_k)), \quad t_j(\varphi_{\delta'}(p_k)) = 0$$

for every $1 \leq i, j, k \leq m$ and $i \neq j$.

Proof. Firstly, we will show that $\delta^{'} \in aid(\mathcal{L})$. For any $p \in \mathcal{L}$,

$$\begin{aligned} \delta^{'}(p) &= (\delta + \mathcal{L}_{\alpha p_i})(p) \\ &= [\varphi_\delta(p), p] + [\alpha p_i, p] \\ &= [\varphi_\delta(p) + \alpha p_i, p]. \end{aligned}$$

This implies that $[\varphi_\delta(p) + \alpha p_i, p] \in [\mathcal{L}, p]$. So $\delta^{'} \in aid(\mathcal{L})$ and $\delta^{'}$ is defined by the mapping

$$\begin{array}{rcl} {\varphi_{\delta'}}^* : \mathcal{L} & \to & \mathcal{L} \\ p & \mapsto & \varphi_\delta(p) + \alpha p_i. \end{array}$$

Now we define the mapping $\varphi_{\delta'} : \mathcal{L} \to \mathcal{L}$ such that

$$p \mapsto \begin{cases} \varphi_\delta(p) + \alpha p_i, & \text{if } p \notin \{p_1, p_2, \dots, p_m\} \\ \varphi_\delta(p) + t_i(\varphi_\delta(p))p_i, & \text{if } p \in \{p_1, p_2, \dots, p_m\}. \end{cases}$$

We need to prove that $\delta^{'}$ is determined by this map. Indeed for all $p \notin \{p_1, p_2, \dots, p_m\}$ we have $\varphi_{\delta'}(p) = {\varphi_{\delta'}}^*(p)$ and for all $p \in \{p_1, p_2, \dots, p_m\}$ there are two cases:
Case 1. If $p_j \notin C_{\mathcal{L}}(p_i)$, then we have $f_i(\varphi_\delta(p_j)) = \alpha$. Thus, ${\varphi_{\delta'}}^* = \varphi_{\delta'}$.
Case 2. If $p_j \in C_{\mathcal{L}}(p_i)$, then we have

$$\begin{aligned} \delta^{'}(p_j) &= (\delta + \mathcal{L}_{\alpha p_i})(p_j) \\ &= \delta(p_j) + [\alpha p_i, p_j] \\ &= [\varphi_\delta(p_j), p_j] \\ &= [\varphi_\delta(e_j) + t_i(\varphi_\delta(e_j))p_i, p_j] \\ &= [\varphi_{\delta'}(p_j), p_j]. \end{aligned}$$

Hence $\delta^{'}$ is given by $\varphi_{\delta'}$. By the definition of $\varphi_{\delta'}$, it is clear to show that $t_i(\varphi_{\delta'}(p_k)) = 2t_i(\varphi_\delta(p_k)), \quad t_j(\varphi_{\delta'}(p_k)) = 0$ for every $1 \le i, j, k \le m$ and $i \neq j$. □

Corollary 1. *Given a Leibniz algebra $\mathcal{L}$ and $\delta \in aid(\mathcal{L})$ which is defined by a mapping φ_δ. If each basis vector is fixed, then $\delta \in id(\mathcal{L})$.*

Proof. Let α_i be the fixed value of p_i. By Lemma 4, we obtain that

$$\delta^{'} = \delta + \mathcal{L}_p$$

where $p = \sum_{i=1}^m \alpha_i p_i$ is an almost inner derivation which is given by a mapping $\varphi_{\delta'}$ with $\varphi_{\delta'}(p_i) = 0$ for every $1 \le i \le m$. It follows that

$$\delta^{'}(p_i) = [\varphi_{\delta'}(p_i), p_i] = 0$$

for all p_i basis vectors. Hence we obtain $\delta^{'} = 0$ and $\delta = -\mathcal{L}_p$. This shows that $\delta \in id(\mathcal{L})$. □

Corollary 2. *If any basis vector for all almost inner derivation is fixed, then $aid(\mathcal{L}) = id(\mathcal{L})$.*

Proof. We know from the inclusions of Lie subalgebras $id(\mathcal{L}) \subseteq aid(\mathcal{L})$. By Corollary 1, we obtain that $aid(\mathcal{L}) \subseteq id(\mathcal{L})$. Therefore, $aid(\mathcal{L}) = id(\mathcal{L})$. □

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

Declaration of Competing Interests The authors declare that they have no competing interest.

Acknowledgements The authors would like to thank the anonymous referees for reading the manuscript carefully, correcting misprints and giving valuable comments and suggestions to improve the presentation of this paper.

References

[1] Bloh, A. M., A generalization of the concept of Lie algebras, *Doklady Akademii Nauk*, 165 (1965), 471-473.
[2] Bloh, A. M., A certain generalization of the concept of Lie algebras, *Algebra and Number Theory Moskow. Gos. Ped. Inst Ućen*, 375 (1971), 9-20.
[3] Burde, D., Dekimpe, K., Verbeke, B., Almost inner derivations of Lie algebras, *Journal of Algebra and Its Applications*, 17(11) (2018). https://doi.org/10.1142/S0219498818502146
[4] Centrone, L., Yasumura, F., Actions of Taft's algebras on finite dimensional algebras, *Journal of Algebra*, 560 (2020), 725-744. https://doi.org/10.1016/j.jalgebra.2020.06.007
[5] Centrone, L., Zargeh, C., Varieties of Null-Filiform Leibniz algebras under the action of Hopf algebras, *Algebra and Representation Theory*, 26 (2023), 631-648. https://doi.org/10.1007/s10468-021-10105-2
[6] Demir, I., Misra, K. C., Stitzinger, E., On some structures of Leibniz algebras, *Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics Contemporary Mathematics*, 623 (2014), 41-54. https://doi.org/10.1090/conm/623/12456
[7] Jacobson, N., A note on automorphisms and derivations of Lie algebras, *Proc. Amer. Math. Soc.*, 6 (1955), 281-283. https://doi.org/10.1090/S0002-9939-1955-0068532-9
[8] Jacobson, N., Lie Algebras, Dover Publications Interscience, 1979.
[9] Ladra, M., Rikhsiboev, I. M., Turdiboev, R. M., Automorphisms and derivations of Leibniz algebras, *Ukrains'kyi Matematychnyi Zhurnal*, 68(7) (2016), 933-944. https://umj.imath.kiev.ua/index.php/umj/article/view/1891
[10] Loday, J. L., Une version non commutative des algebres de Lie: les algebres de Leibniz, *Enseing. Math.*, 39 (1993), 269-293. https://doi.org/10.5169/seals-60428
[11] Loday, J.L., Pirashvili, T., Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Ann.*, 269(1) (1993), 139-158. https://doi.org/10.1007/BF01445099
[12] Mansuroğlu, N., Fundamentals of Lie Algebras, Gece Kitaplığı, 2022.
[13] Mansuroğlu, N., Özkaya, M., On the structure constants of Leibniz algebras, *International Advanced Research Journal in Science, Engineering and Technology*, 5(5) (2018), 67-69. Doi:10.17148/IARJSET.2018.5511
[14] Sato, T., The derivations of the Lie algebras, *Tôhoku Math. Jour.*, 23, (1971), 21-36.
[15] Shahryari, M., A note on derivations of Lie algebras, *Rev. Mat. Bull. Aust. Math. Soc.*, 84 (2011), 444-446. https://doi.org/10.1017/S0004972711002516
[16] Zargeh, C., Some remarks on derivations of Leibniz algebras, *International Journal of Algebra*, 6(30) (2012), 1471-1474. https://api.semanticscholar.org/CorpusID:131761265