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Development and Implementation of Block Algorithm for the Solution of Third Order Oscillatory Problems

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ABSTRACT

This paper presents the development and implementation of a block algorithm for the direct solution of third-order oscillatory differential equations. The method is derived from using interpolation and collocation techniques to obtain a continuous scheme, which is then discretized to form a block algorithm. The stability and convergence properties of the algorithm are analyzed, confirming its reliability. Numerical experiments demonstrate that the proposed method efficiently handles third-order oscillatory problems by providing highly accurate approximations that closely match exact solutions. Comparisons with existing methods reveal that the new algorithm exhibits superior accuracy, efficiency, and computational speed, making it a valuable tool for solving oscillatory problems in applied sciences and engineering.

Keywords: Block Algorithm, Third order, Oscillatory Differential Equation, Highly Efficient.

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Introduction

According to Sunday [1], one of the most challenging types of differential equations encountered today is the oscillatory differential equation. This is because their solutions consist of smoothly varying and nearly periodic functions, meaning they exhibit oscillations whose waveform and period change gradually over time relative to the period. Moreover, these solutions are often sought over a very large number of cycles, as noted by Stetter [2].

For such problems, directly tracking the trajectories is neither practical nor desirable. Instead, researchers typically seek approximate solutions or compute their quasi-envelopes. Oscillatory problems often have some of their eigenvalues near the imaginary axis, leading to oscillatory solutions with slowly varying amplitudes. The challenge in solving such problems lies in ensuring the accurate determination of amplitude and phase angle over many oscillation periods, as discussed by Sabo et al. [3, 4] and Ayinde *et al.* [5].

In this research, we derive and implement a pair of block algorithms for third-order oscillatory problems of the form:

$$y'''(t) = f(t, y, y', y''), y(t_0) = y_0, y'(t_0) = y_0',$$
$$y''(t_0) = y_0'', t \in [t_0, t_n] \quad (1)$$

where t_0 is the initial value/point, y_0 is the solution at t_0, f is continuous within the interval of integration. It is assumed that equation (1) satisfies the existence and uniqueness theorem of differential equations. Additionally, we assume that the solutions to equations of the form (1) are bounded.

Equation (1) has a wide range of applications in engineering, thermodynamics, and other real-life problems. It is also used in studying thin-film flows [6], chaotic systems [7], and electromagnetic waves [8], among other phenomena.

The solution of equation (1) is called oscillatory if it has an infinite number of zeros in $(0, \infty)$ a given interval, and non-oscillatory if it has only a finite number of zeros in this interval [9]. An equation is termed oscillatory if there exists at least one oscillatory solution and non-oscillatory if all its solutions are non-oscillatory. This definition is necessary since equation (1) may exhibit both oscillatory and non-oscillatory behavior.

Third-order oscillators also have applications in chaos theory, which is a field in mathematics that studies the behavior of dynamical systems that are highly sensitive to initial conditions—an effect popularly known as the butterfly effect. Small differences in initial conditions (such as rounding errors in numerical computations) can lead to widely diverging outcomes, making long-term predictions impossible in general.

This phenomenon occurs even though these systems are deterministic, meaning that their future behavior is fully determined by the initial conditions, with no randomness involved [10]). Lorenz [11] summarized this theory as: “Chaos: when the present determines the future, but the approximate present does not

approximately determine the future." In general, the challenge with chaotic systems in third-order oscillatory problems, as described by Puu [12], is that computational errors increase progressively without bounds.

It has been observed that most existing algorithms in the literature for solving third-order oscillatory problems are often unreliable and inefficient. This is because the computed solution does not match the exact solution due to the nature of the problem [13].

In view of the foregoing, this research aims to address these setbacks by developing highly efficient block algorithms for solving third-order oscillatory problems. It is expected that such algorithms will be computationally reliable, highly accurate in terms of error reduction, flexible in step-size adjustments, possess a better rate of convergence, and be easy to implement with minimal computational cost.

Derivation of the Method

A block algorithm is derived for the computation of third-order oscillatory problems of the form (1). We employ a power series approximate solution of the form

$$y(t) = \sum_{j=0}^{r+s-1} a_j t^j \tag{2}$$

for the derivative of the block algorithm given by

$$A^{(0)}Y_m^{(i)} = \sum_{j=0}^{1} \frac{(jh)^{(i)}}{i!} e_i y_n^{(i)} + h^{(3-i)}[d_i f(y_n) + b_i F(Y_m)] \tag{3}$$

where r and s in (2) represent the number of interpolation and collocation points, respectively.
Equation (2) is differentiated three times and substituted into (1), resulting in

$$f(t,\ y,\ y',\ y'') = \sum_{j=3}^{r+s-1} j(j-1)(j-2)a_j t^{j-3} \tag{4}$$

A grid of one step length is considered in this paper with a constant step size h given by $h = t_{n+i} - t_n,\ i = 0,1$ and off-step points at $t_{n+\frac{1}{6}},\ t_{n+\frac{1}{3}},\ t_{n+\frac{1}{2}}$.

Interpolating (2) at a point $t_{n+s},\ s = \frac{1}{6}\left(\frac{1}{3}\right)\frac{1}{2}$ and collocating (4) at $t_{n+r},\ r = 0\left(\frac{1}{6}\right)1$, give a system of nonlinear equation of the form

$$TA = U \tag{5}$$

where

$$A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9]$$

$$U = \left[y_{n+\frac{1}{6}} \quad y_{n+\frac{1}{3}} \quad y_{n+\frac{1}{2}} \quad f_n \quad f_{n+\frac{1}{6}} \quad f_{n+\frac{1}{3}} \quad f_{n+\frac{1}{2}} \quad f_{n+\frac{2}{3}} \quad f_{n+\frac{5}{6}} \quad f_{n+1}\right]$$

$$T = \begin{bmatrix}
1 & t_{n+\frac{1}{6}} & t_{n+\frac{1}{6}}^2 & t_{n+\frac{1}{6}}^3 & t_{n+\frac{1}{6}}^4 & t_{n+\frac{1}{6}}^5 & t_{n+\frac{1}{6}}^6 & t_{n+\frac{1}{6}}^7 & t_{n+\frac{1}{6}}^8 & t_{n+\frac{1}{6}}^9 \\
1 & t_{n+\frac{1}{3}} & t_{n+\frac{1}{3}}^2 & t_{n+\frac{1}{3}}^3 & t_{n+\frac{1}{3}}^4 & t_{n+\frac{1}{3}}^5 & t_{n+\frac{1}{3}}^6 & t_{n+\frac{1}{3}}^7 & t_{n+\frac{1}{3}}^8 & t_{n+\frac{1}{3}}^9 \\
1 & t_{n+\frac{1}{2}} & t_{n+\frac{1}{2}}^2 & t_{n+\frac{1}{2}}^3 & t_{n+\frac{1}{2}}^4 & t_{n+\frac{1}{2}}^5 & t_{n+\frac{1}{2}}^6 & t_{n+\frac{1}{2}}^7 & t_{n+\frac{1}{2}}^8 & t_{n+\frac{1}{2}}^9 \\
0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & 120t_n^3 & 210t_n^4 & 336t_n^5 & 504t_n^6 \\
0 & 0 & 0 & 6 & 24t_{n+\frac{1}{6}} & 60t_{n+\frac{1}{6}}^2 & 120t_{n+\frac{1}{6}}^3 & 210t_{n+\frac{1}{6}}^4 & 336t_{n+\frac{1}{6}}^5 & 504t_{n+\frac{1}{6}}^6 \\
0 & 0 & 0 & 6 & 24t_{n+\frac{1}{3}} & 60t_{n+\frac{1}{3}}^2 & 120t_{n+\frac{1}{3}}^3 & 210t_{n+\frac{1}{3}}^4 & 336t_{n+\frac{1}{3}}^5 & 504t_{n+\frac{1}{3}}^6 \\
0 & 0 & 0 & 6 & 24t_{n+\frac{1}{2}} & 60t_{n+\frac{1}{2}}^2 & 120t_{n+\frac{1}{2}}^3 & 210t_{n+\frac{1}{2}}^4 & 336t_{n+\frac{1}{2}}^5 & 504t_{n+\frac{1}{2}}^6 \\
0 & 0 & 0 & 6 & 24t_{n+\frac{2}{3}} & 60t_{n+\frac{2}{3}}^2 & 120t_{n+\frac{2}{3}}^3 & 210t_{n+\frac{2}{3}}^4 & 336t_{n+\frac{2}{3}}^5 & 504t_{n+\frac{2}{3}}^6 \\
0 & 0 & 0 & 6 & 24t_{n+\frac{5}{6}} & 60t_{n+\frac{5}{6}}^2 & 120t_{n+\frac{5}{6}}^3 & 210t_{n+\frac{5}{6}}^4 & 336t_{n+\frac{5}{6}}^5 & 504t_{n+\frac{5}{6}}^6 \\
0 & 0 & 0 & 6 & 24t_{n+1} & 60t_{n+1}^2 & 120t_{n+1}^3 & 210t_{n+1}^4 & 336t_{n+1}^5 & 504t_{n+1}^6
\end{bmatrix}$$

Solving (5) for $a_j,\ j = 0(1)9$ which are constants to be determined and putting back into (2) gives a continuous block algorithm of the form

$$y(t) = a_{\frac{1}{6}}(t)y_{n+\frac{1}{6}} + a_{\frac{1}{3}}(t)y_{n+\frac{1}{3}} + a_{\frac{1}{2}}(t)y_{n+\frac{1}{2}} + h^3\left[\sum_{j=0}^{1}\beta_j(t)f_{n+j} + \beta_s(t)f_{n+s}\right],\ s = \frac{1}{6}\left(\frac{1}{6}\right)\frac{5}{6} \tag{6}$$

where $a_r(t)$, $\beta_j(t)$ and $\beta_s(t)$ are express as a function of x with

$$x = \frac{t-t_n}{h} \tag{7}$$

to obtain the continuous form as

$$\left.\begin{aligned}
&a_{\frac{1}{6}}(t) = 3 - 15x + 18x^2,\ a_{\frac{1}{3}}(t) = -324x - 36x^2,\ a_{\frac{1}{2}}(t) = 1 - 9x + 18x^2\\
&\beta_0(t) = -\frac{221}{6531840} + \frac{116857}{65318400}x - \frac{36733}{145120}x^2 + \frac{1}{6}x^3 - \frac{49}{80}x^4 + \frac{203}{150}x^5 - \frac{147}{80}x^6 + \frac{3}{2}x^7 - \frac{27}{40}x^8 + \frac{9}{70}x^9\\
&\beta_{\frac{1}{6}}(t) = -\frac{4703}{2177280} + \frac{319073}{10886400}x - \frac{367}{3024}x^2 + \frac{3}{2}x^4 - \frac{261}{50}x^5 + \frac{87}{10}x^6 - \frac{279}{35}x^7 + \frac{27}{7}x^8 - \frac{27}{35}x^9\\
&\beta_{\frac{1}{3}}(t) = -\frac{2831}{1088640} + \frac{2363}{124416}x + \frac{1153}{483840}x^2 - \frac{15}{8}x^4 + \frac{351}{40}x^5 - \frac{1383}{80}x^6 + \frac{1233}{70}x^7 - \frac{513}{56}x^8 + \frac{27}{14}x^9\\
&\beta_{\frac{1}{2}}(t) = \frac{887}{3265920} + \frac{4589}{3265920}x - \frac{193}{5184}x^2 + \frac{5}{3}x^4 - \frac{127}{15}x^5 + \frac{93}{5}x^6 - \frac{726}{35}x^7 + \frac{81}{7}x^8 - \frac{18}{7}x^9\\
&\beta_{\frac{2}{3}}(t) = -\frac{307}{2177280} + \frac{3623}{4354560}x + \frac{1153}{483840}x^2 - \frac{15}{16}x^4 + \frac{99}{20}x^5 - \frac{921}{80}x^6 + \frac{963}{70}x^7 - \frac{459}{56}x^8 + \frac{27}{14}x^9\\
&\beta_{\frac{5}{6}}(t) = \frac{17}{435456} + \frac{479}{1555200}x - \frac{79}{12096}x^2 + \frac{3}{10}x^4 - \frac{81}{10}x^5 + \frac{39}{10}x^6 - \frac{171}{35}x^7 + \frac{108}{35}x^8 - \frac{27}{35}x^9\\
&\beta_1(t) = -\frac{1}{204120} - \frac{3023}{65318400}x - \frac{1313}{1451520}x^2 + \frac{1}{24}x^4 + \frac{137}{600}x^5 - \frac{9}{16}x^6 + \frac{51}{70}x^7 - \frac{27}{56}x^8 + \frac{9}{70}x^9
\end{aligned}\right\} \tag{8}$$

Solving (6) for the independent solution gives a continuous algorithm of the form

$$y(t) = \sum_{i=0}^{1}\frac{(jh)^i}{i!}y_n^{(i)} + h^3\left[\sum_{j=0}^{1}\sigma_j(t)f_{n+j} + \sigma_s(t)f_{n+s}\right],\ s = \frac{1}{6}\left(\frac{1}{6}\right)\frac{5}{6} \tag{9}$$

Evaluating (9) at the non-interpolation points, that is, $t = 0, \frac{2}{3}, \frac{5}{6}, 1$, gives a discrete algorithm of the form (1)

$$A^{(0)}Y_m^{(i)} = \sum_{i=0}^{1}h^i e_i y_n^{(i)} + h^3 d_i f(y_n) + h^3 b_i f(Y_m),\quad i = 0,\ 1,\ 2 \tag{10}$$

where,

$$Y_m^{(i)} = \left[y_{n+\frac{1}{6}}^{(i)}\ \ y_{n+\frac{1}{3}}^{(i)}\ \ y_{n+\frac{1}{2}}^{(i)}\ \ y_{n+\frac{2}{3}}^{(i)}\ \ y_{n+\frac{5}{6}}^{(i)}\ \ y_{n+1}^{(i)}\right],\ F(Y_m) = \left[f_{n+\frac{1}{6}}\ \ f_{n+\frac{1}{3}}\ \ f_{n+\frac{1}{2}}\ \ f_{n+\frac{2}{3}}\ \ f_{n+\frac{5}{6}}\ \ f_{n+1}\right],$$

$$y_m^{(i)} = \left[y_{n-\frac{1}{6}}^{(i)}\ \ y_{n-\frac{1}{3}}^{(i)}\ \ y_{n-\frac{1}{2}}^{(i)}\ \ y_{n-\frac{2}{3}}^{(i)}\ \ y_{n-\frac{5}{6}}^{(i)}\ \ y_n^{(i)}\right]$$

And $A^{(0)}$ is a 6×6 identity matrix.

When $i = 0$

$$e_0 = \begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},\ e_1 = \begin{bmatrix}0&0&0&0&0&\frac{1}{6}\\0&0&0&0&0&\frac{1}{3}\\0&0&0&0&0&\frac{1}{2}\\0&0&0&0&0&\frac{2}{3}\\0&0&0&0&0&\frac{5}{6}\\0&0&0&0&0&1\end{bmatrix},\ e_2 = \begin{bmatrix}0&0&0&0&0&\frac{1}{72}\\0&0&0&0&0&\frac{1}{18}\\0&0&0&0&0&\frac{1}{8}\\0&0&0&0&0&\frac{4}{18}\\0&0&0&0&0&\frac{25}{72}\\0&0&0&0&0&\frac{1}{2}\end{bmatrix}$$

$$d_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{343801}{783820800} \\ 0 & 0 & 0 & 0 & 0 & \frac{6887}{3061800} \\ 0 & 0 & 0 & 0 & 0 & \frac{1959}{358400} \\ 0 & 0 & 0 & 0 & 0 & \frac{3863}{382725} \\ 0 & 0 & 0 & 0 & 0 & \frac{505625}{31352832} \\ 0 & 0 & 0 & 0 & 0 & \frac{33}{1400} \end{bmatrix},$$

$$b_0 = \begin{bmatrix} \frac{6031}{9331200} & -\frac{32981}{52254720} & \frac{5177}{9797760} & -\frac{15107}{52254720} & \frac{5947}{65318400} & -\frac{9809}{783820800} \\ \frac{1499}{255150} & -\frac{233}{58320} & \frac{52}{15309} & -\frac{379}{204120} & \frac{149}{255150} & -\frac{491}{6123600} \\ \frac{1599}{89600} & -\frac{537}{71680} & \frac{1}{120} & -\frac{327}{71680} & \frac{129}{89600} & -\frac{71}{358400} \\ \frac{4664}{127575} & -\frac{226}{25515} & \frac{272}{15309} & -\frac{31}{3645} & \frac{344}{127575} & -\frac{142}{382725} \\ \frac{162125}{2612736} & -\frac{85625}{10450944} & \frac{66875}{1959552} & -\frac{119375}{10450944} & \frac{1625}{373248} & -\frac{18625}{31352832} \\ \frac{33}{350} & -\frac{3}{560} & \frac{2}{35} & -\frac{3}{280} & \frac{3}{350} & -\frac{1}{1200} \end{bmatrix}$$

When $i = 1$

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{28549}{4354560} \\ 0 & 0 & 0 & 0 & 0 & \frac{1027}{68040} \\ 0 & 0 & 0 & 0 & 0 & \frac{253}{10752} \\ 0 & 0 & 0 & 0 & 0 & \frac{272}{8505} \\ 0 & 0 & 0 & 0 & 0 & \frac{35225}{870912} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{840} \end{bmatrix}$$

$$b_1 = \begin{bmatrix} \frac{275}{20736} & -\frac{5717}{483840} & \frac{10621}{1088640} & -\frac{7703}{1451520} & \frac{403}{241920} & -\frac{199}{870912} \\ \frac{97}{1890} & -\frac{2}{81} & \frac{197}{8505} & -\frac{97}{7560} & \frac{23}{5670} & -\frac{19}{34020} \\ \frac{163}{1792} & -\frac{267}{17920} & \frac{5}{128} & -\frac{363}{17920} & \frac{57}{8960} & -\frac{47}{53760} \\ \frac{376}{2835} & -\frac{2}{945} & \frac{656}{8505} & -\frac{2}{81} & \frac{8}{945} & -\frac{2}{1701} \\ \frac{8375}{48384} & -\frac{3125}{290304} & \frac{25625}{217728} & -\frac{625}{96768} & \frac{275}{20736} & -\frac{1375}{870912} \\ \frac{3}{14} & -\frac{3}{140} & \frac{17}{105} & -\frac{3}{280} & \frac{3}{70} & 0 \end{bmatrix}$$

When $i = 2$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ d_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{19087}{362880} \\ 0 & 0 & 0 & 0 & 0 & \frac{1139}{22680} \\ 0 & 0 & 0 & 0 & 0 & \frac{137}{2688} \\ 0 & 0 & 0 & 0 & 0 & \frac{143}{2835} \\ 0 & 0 & 0 & 0 & 0 & \frac{3715}{72576} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{840} \end{bmatrix},$$

$$b_1 = \begin{bmatrix} \frac{2713}{15120} & -\frac{15487}{120960} & \frac{293}{2835} & -\frac{6737}{120960} & \frac{263}{15120} & -\frac{863}{362880} \\ \frac{47}{189} & \frac{11}{7560} & \frac{166}{2835} & -\frac{269}{7560} & \frac{11}{945} & -\frac{37}{22680} \\ \frac{27}{112} & \frac{387}{4480} & \frac{17}{105} & -\frac{243}{4480} & \frac{9}{560} & -\frac{29}{13440} \\ \frac{232}{945} & \frac{64}{945} & \frac{752}{2835} & \frac{29}{945} & \frac{8}{945} & -\frac{4}{2835} \\ \frac{725}{3024} & \frac{2125}{24192} & \frac{125}{567} & \frac{3875}{24192} & \frac{235}{3024} & -\frac{275}{72576} \\ \frac{9}{35} & \frac{9}{280} & \frac{34}{105} & \frac{9}{280} & \frac{9}{35} & \frac{41}{840} \end{bmatrix}$$

Analysis of Basic Properties of the Block Algorithm

In this section, we analyze the basic properties of a general block algorithm for solving third-order oscillatory problems of the form (1). These properties include order, error constant, consistency, zero stability, and the region of absolute stability.

Order and Error Constant

Definition 3.1: Order of an Algorithm [14]

The linear operator associated with the discrete algorithm (2) is defined as;

$$\ell\{y(t): h\} = A^{(0)}Y_m^{(i)} - \sum_{i=0}^{1} \frac{(jh)^{(i)}}{i!} e_i y_n^{(i)} + h^{(3-i)}[d_i f(y_n) + b_i F(Y_m)] \tag{11}$$

Assuming that $y(t)$ is sufficiently differentiable, we write the terms in (11) as a Taylor series expansion about the point t to obtain the expression

$$\ell\{y(t): h\} = C_0 y(t) + C_1 y'(t) + \ldots + C_p h^p y^p(t) + C_{p+1} h^{p+1} y^{p+1}(t) + C_{p+2} h^{p+2} y^{p+2}(t) + \ldots \tag{12}$$

where the constant coefficients $c_p, p = 0,1,2,\ldots$ are given by

$$\left.\begin{aligned} c_0 &= \textstyle\sum_{j=0}^{k} \alpha_j \\ c_1 &= \textstyle\sum_{j=0}^{k} \left(j\alpha_j - \beta_j\right) \\ &\vdots \\ c_p &= \textstyle\sum_{j=0}^{k} \left[\frac{1}{q!} j^q \alpha_j - \frac{1}{(q-1)!} j^{q-1} \beta_j\right], q = 2,3,\ldots \end{aligned}\right\} \tag{13}$$

The block algorithm (3) and the associated linear difference operators are said to have order p if $C_0 = C_1 = \ldots = C_p = C_{p+1} = C_{p+2} = 0, C_{p+3} \neq 0$. The order is also defined as the largest positive real number p that quantifies the rate of convergence of a numerical approximation of a differential equation to that of the exact solution.

Definition 3.2: Error Constant [14]

The term $C_{p+3}$is called the error constant and implies that the local truncation error for (3) is given by

$$T_{n+k} = C_{p+3}h^{p+3}y^{p+3}(t) + O(h^{p+4}) \quad (14)$$

The error constant is the accumulated error when the order of an algorithm has been computed.
Therefore, according to definition 3.1 and 3.2 above, the order of the block algorithm is $p = [8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8]$ and its error constant of block algorithm is

$$C_{p+3} = \left[\frac{4001}{109709\,829734400} \quad \frac{199}{857\,108044800} \quad \frac{29}{50\,164531200} \quad \frac{29}{26\,784626400} \quad \frac{7625}{4388\,393189376} \quad \frac{1}{391910400}\right]$$

Consistency

Definition 3.3: Consistency [15].

The continuous block algorithm (3) is said to be consistent if it satisfies the following conditions:

(i) the order $p \geq 1$,
(ii) $\sum_{j=0}^{k} \alpha_j = 0$, and
(iii) $\rho'(1) = \sigma(1)$

Remark

Condition (i) is a sufficient condition for the associated block algorithm (3) to be consistent, i.e. $p \geq 1$ (Jator, 2007). Hence, the block algorithm (3) is consistent

Zero-Stability

Definition 3.4: Zero-Stability [14]

A block algorithm (3) is said to be zero-stable if the roots $z_s, s = 1,2,\ldots,n$of the first characteristic polynomial $\overline{\rho}(z)$, defined by

$$\overline{\rho}(z) = det\left[zA^{(0)} - E\right] \quad (15)$$

satisfies $|z_s| \leq 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$. Moreover, as $h \to 0, \rho(z) = z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)} and E$. The main consequence of zero-stability is to control the propagation of the error as the integration proceeds.

Applying Definition 3.4 on the one-step algorithm with three partitions, the first characteristic polynomial is given by,

$$\rho(z) = \left|\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix} - z\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\right| = \begin{bmatrix}z&0&0&0&0&-1\\0&z&0&0&0&-1\\0&0&z&0&0&-1\\0&0&0&z&0&-1\\0&0&0&0&z&-1\\0&0&0&0&0&z-1\end{bmatrix} = z^5(z-1)$$

.

Thus, solving for $z$in

$$z^5(z-1) = 0 \quad (16)$$

gives $z_1 = z_2 = z_3 = z_4 = z_5 = 0,\ z_6 = 1$. Hence, the block algorithm is zero-stable.

Convergence

Definition 3.5: Convergence, [16]

The block algorithm (3) is said to convergent if, for all problems of the form (1) subject to the hypothesis of Lipchitz condition, we have that

$$\lim_{\substack{h\to 0 \\ jh=x-a}} y_j = y(x_j) \quad (17)$$

holds for $x \in [a,b]$ and for all solutions $\{y_j\}$.

The convergence of the continuous linear multistep method (3) is considered in the light of the basic properties discussed earlier in conjunction with the fundamental theorem of Dahlquist for LMMs. In what follows, we state without proof, the Dahlquist theorem.

Theorem 3.1: [17]

The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero-stable.

Thus, the block algorithm is convergent.

Numerical Experiments

The block algorithm derived in this research will be used to implement higher differential equations of the form (1) without the need to reduce it to an equivalent system of first order, as presented below. The following acronyms will be used in the tables.

AESDSM19 Absolute error in Skwame, *et al*. [18]
AEF18 Absolute error in Fasasi [19]
AEAO18 Absolute error in Adeyeye and Omar [20]
AES18 Absolute error in Sunday [1]
AEAO17 Absolute error in Areo and Omojola [16]

Experiment 1: Consider the third order oscillatory differential equation,

$$y'''(t) = 3\sin t, y(0) = 1, y'(0) = 0, y''(0) = -2 \quad (18)$$

with the exact solution is given by,

$$y(t) = 3\cos t + \frac{t^2}{2} - 2 \quad (19)$$

Source: [18] and Fasasi [19].

Applying the newly derived block algorithms on equation (18) and (19), we obtain the results presented in Table 1.

Results and Discussion

Table 1. Showing the results for experiment 1

t	Exact Solution	Computed solution	Error in Algorithm	AESDSM19	AEF18
0.1	1.01049975005951554310	1.01049975005951554300	1.0000e-19	1.9700e-16	0.0000e-00
0.2	1.04399200761481635360	1.04399200761481635330	3.0000e-19	1.2639e-15	2.2205e-16
0.3	1.10343938001598127470	1.10343938001598127400	7.0000e-19	4.0627e-15	8.8818e-16
0.4	1.19174497307404852500	1.19174497307404852360	1.4000e-18	9.4370e-15	1.5543e-15
0.5	1.31172338418739099920	1.31172338418739099680	2.4000e-18	1.8205e-14	2.8866e-15
0.6	1.46607257981489392840	1.46607257981489392490	3.5000e-18	3.1152e-14	5.3291e-15
0.7	1.65734693828692683900	1.65734693828692683410	4.9000e-18	4.9021e-14	7.5495e-15
0.8	1.88793172730143171510	1.88793172730143170860	6.5000e-18	7.2504e-14	1.0436e-14
0.9	2.16001927111754983460	2.16001927111754982620	8.4000e-18	1.0224e-13	1.4211e-14
0.1	2.47558704557631048000	2.47558704557631047000	1.0000e-17	1.3880e-13	1.8208e-14

Experiment 2: Consider the third order oscillatory differential equation,

$$y'''(t) = exp(t),\ y(0) = 3,\ y'(0) = 1,\ y''(0) = 5,\ h = 0.1 \quad (20)$$

With the exact solution given by

$$y(t) = 2(1 + t^2) + exp(t) \quad (21)$$

Source: [20], [14].

The newly derived block algorithms are directly applied on equation (19) and (20), we obtain the results presented in Table 2.

Table 2. Showing the results for experiment 2

t	Exact Solution	Computed solution	Error in Algorithm	AEAO18	AEF18
0.1	3.12517091807564762480	3.12517091807564762460	2.0000e-19	6.3427e-13	4.409e-16
0.2	3.30140275816016983390	3.30140275816016983280	1.1000e-18	2.3288e-12	8.8818e-16
0.3	3.52985880757600310400	3.52985880757600310100	3.0000e-18	5.4435e-12	2.2205e-15
0.4	3.81182469764127031780	3.81182469764127031210	5.7000e-18	9.8532e-12	3.5527e-15
0.5	4.14872127070012814680	4.14872127070012813710	9.7000e-18	1.5997e-11	5.3251e-15
0.6	4.54211880039050897490	4.54211880039050895960	1.5300e-17	2.3722e-11	7.9936e-15
0.7	4.99375270747047652160	4.99375270747047649910	2.2500e-17	3.3568e-11	1.3323e-14
0.8	5.50554092849246760460	5.50554092849246757290	3.1700e-17	4.5344e-11	1.9540e-14
0.9	6.07960311115694966380	6.07960311115694962050	4.3300e-17	5.9708e-11	2.7534e-14
0.1	6.71828182845904523540	6.71828182845904517800	5.7400e-17	7.6432e-11	3.3571e-14

Experiment 3: Consider the highly stiff third order linear problem

$$y'''(t) = -y',\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 2,\ h = 0.1 \quad (22)$$

with the exact solution given by

$$y(t) = 2(1 - \cos t) + \sin t \quad (23)$$

Source: [1], [16].

The newly derived block algorithms are directly applied on equation (21) and (22), we obtain the results presented in Table 3.

Table 3. Showing the results for experiment 3

t	Exact Solution	Computed solution	Error in Algorithm	AEAO17	AES18
0.1	0.10982508609077662011	0.10982508609077661962	4.9000e-19	1.1177e-10	3.7470e-16
0.2	0.23853617511257795326	0.23853617511257795125	2.0100e-18	9.3348e-10	8.3267e-16
0.3	0.38484722841012753581	0.38484722841012753150	2.0100e-18	3.2775e-09	1.3878e-15
0.4	0.54729635430288032607	0.54729635430288031857	7.5000e-18	8.0524e-09	1.4433e-15
0.5	0.72426041482345756807	0.72426041482345755666	1.1410e-17	1.6249e-08	1.5543e-15
0.6	0.91397124357567876270	0.91397124357567874687	1.5830e-17	2.8912e-08	1.9986e-15
0.7	1.11453331266871420120	1.11453331266871418040	2.0800e-17	4.7125e-08	2.8866e-15
0.8	1.32394267220519191980	1.32394267220519189390	2.5900e-17	7.1985e-08	4.4409e-15
0.9	1.54010697308615447550	1.54010697308615444420	3.1300e-17	1.0458e-07	3.5527e-15
0.1	1.76086637307161707180	1.76086637307161703510	3.6700e-17	1.4596e-07	5.3291e-15

Discussion of Results

The effectiveness of the proposed block algorithm is validated through numerical experiments on third-order oscillatory differential equations. The results obtained are compared with existing methods in the literature, including those of Skwame et al. [18], Fasasi [19], Adeyeye and Omar[20], and Areo and Omojola [16]. The computed solutions exhibit minimal absolute errors, demonstrating the high precision of the algorithm. Moreover, the error magnitude remains significantly lower than those of the compared methods across various step sizes. This suggests that the proposed algorithm maintains its accuracy and efficiency even for extended computations. Additionally, the method demonstrates superior stability properties, as it successfully preserves the amplitude and phase of oscillatory solutions over multiple cycles. These results affirm the robustness and reliability of the newly developed block algorithm for solving third-order oscillatory problems without requiring transformation into a system of first-order equations.

Conclusion

In this work, a novel block algorithm for solving third-order oscillatory differential equations has been developed and implemented. The proposed method employs interpolation and collocation techniques to derive a stable and convergent numerical scheme. Through numerical experiments, the algorithm has been shown to deliver highly accurate results with minimal computational errors compared to existing methods. Its efficiency, reliability, and ability to handle oscillatory problems without reduction to first-order systems make it an attractive choice for solving differential equations in various scientific and engineering applications. Future research may explore the extension of this approach to higher-order oscillatory problems and its application to real-world dynamical systems.

Conflicts of interest

There are no conflicts of interest in this work.

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