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Research Article

Asymptotic formulae for modified Bernstein operators based on regular summability methods

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Abstract

In this paper, we get new Voronovskaja-type asymptotic formulae for modified Bernstein operators by using regular summability methods. We also display some significant special cases of our results including the methods of Cesàro summability, Riesz summability, Abel summability and Borel summability. At the end, we also discuss the similar results for the Kantorovich version of the operators.

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1. Introduction

While classical approximation operators generally approach the exact value of the function being approximated, they frequently struggle at points of discontinuity and converge to the average of the left and right limits instead of the function's value. However, there are notable exceptions, such as the Hermite-Fejér interpolation operator, which fails to converge at simple discontinuity points (see [5]). In such cases of irregular behavior, Cesàro-type matrix summability methods are powerful enough to overcome the loss of convergence. Moreover, even at points of continuity, the Cesàro summability method effectively corrects the Gibbs phenomenon encountered in certain approximation operators, such as partial sums of the Fourier series. More precisely, Fourier series of a continuous and periodic function may diverge; however its Cesàro mean is convergent to the function itself. This is the main idea of the Fejér approximation, which is well known in the literature (see [9]). On the other hand, some summability methods, such as subsequence matrix transformations, are used for the acceleration of convergence rate of a sequence (see [7, 13, 18, 22]). So, such methods may provide faster convergence. These facts clearly explain why we need summability methods in the approximation theory. Therefore, they have been actively used for many years (see, for instance, [1–3, 8, 10, 15, 17, 19, 20]).

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In this paper we use general summability methods to approximate continuous functions by means of the Bernstein operators. In this way we obtain some new Voronovskaja-type results for the classical Bernstein polynomials. In order to preserve the results in the classical approach, we will consider regular summability methods throughout the paper. In particular, we will use the well-known regular methods in summability theory such as Cesàro, Riesz, Abel and Borel. To the best of our knowledge, this will be the first study to use summability methods in the Voronovskaja-type results of Bernstein polynomials.

Let us now recall some of the concepts and results we will need in this article.

The classical Bernstein polynomials are defined by

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{k,n}(x), \ x \in [0,1], \ n \in \mathbb{N},$$

where $p_{k,n}(x)$ $(k = 0, 1, ..., n)$ are Bernstein basis functions given by

$$p_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Then, it is well-known that (see Bernstein [4]), for every $f \in C[0,1]$, the space of all real-valued continuous functions on $[0,1]$, the function f can be approximated uniformly by means of $B_n(f)$, which proves constructively the classical Weierstrass Approximation Theorem. Since Bernstein polynomials play a key role in approximation theory, they have been studied intensively until now and still continue to be investigated.

We know from Voronovskaja [21] that the following asymptotic formula for the Bernstein polynomials

$$\lim_{n\to\infty} n\left(B_n(f;x) - f(x)\right) = \frac{x(1-x)}{2} f''(x) \tag{1.1}$$

holds when f is a bounded function on $[0,1]$ and the second derivative f'' exists at a certain point x of $(0,1)$. This result of Voronovskaja was a first example of saturation of the Bernstein polynomials. So far, many significant modifications and generalizations of this asymptotic formula have been studied. The main goal of the present paper is to obtain new asymptotic formulae including Bernstein polynomials. To see that we consider two modifications of the Bernstein polynomials by using matrix summability methods and power series methods. Readers may find the following information and more from the summability theory in the books [6,11].

Let $A = [a_{jn}]$ $(j, n \in \mathbb{N})$ be an infinite matrix. For a given sequence (x_n), we say that (x_n) is A-convergent (or A-summable) to a number L provided that

$$\lim_{j\to\infty} \sum_{n=1}^{\infty} a_{jn} x_n = L, \tag{1.2}$$

where the series $\sum_{n=1}^{\infty} a_{jn} x_n$ is assumed to be convergent for each $j \in \mathbb{N}$. In this case, A is said to be a matrix summability method. Throughout the paper we consider nonnegative matrix summability methods, that is, $a_{jn} \geq 0$ for all $j, n \in \mathbb{N}$. Also, a matrix method A is said to be regular if it preserves the (usual) convergence and the corresponding limit value, that is, (1.2) holds whenever $\lim_{n\to\infty} x_n = L$. It is well-known that a matrix summability method is regular if and only if it satisfies the Silverman-Toeplitz conditions given as follows:

(i) $\lim_{j\to\infty} a_{jn} = 0$ for every $n \in \mathbb{N}$,

(ii) $\lim_{j\to\infty} \sum_{n=1}^{\infty} a_{jn} = 1$,

(iii) $\sup_{j\in\mathbb{N}} \sum_{n=1}^{\infty} |a_{jn}| < \infty$.

Now let $p_0 > 0$ and $p_n \geq 0$ for $n \in \mathbb{N}$. Assume that the power series

$$P(t) := \sum_{n=0}^{\infty} p_n t^n$$

has radius of convergence R with $0 < R \leq \infty$. Then, for a given sequence (x_n), we say that (x_n) is P_p-convergent to a number L if

$$\lim_{0<t\to R^-} \frac{1}{P(t)} \sum_{n=0}^{\infty} p_n x_n t^n = L, \tag{1.3}$$

where the power series $\sum_{n=0}^{\infty} p_n x_n t^n$ is assumed to have radius of convergence $\geq R$. In this case, the method P_p is said to be a power series method. It is known that a power series method is regular if and only if $\lim_{0<t\to R^-} \frac{p_n t^n}{P(t)} = 0$ for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We should note that, in (1.3), one can write x_{n+1} instead of x_n if the sequence is defined on $\mathbb{N}$.

With this terminology, in the next section we give the modifications of the Bernstein polynomials and obtain their new asymptotic formulae. Later, we display some special cases of our results. At the end of the paper, we also discuss similar results for the Kantorovich version of the operators. The last section is devoted to the concluding remarks.

2. Modifications of the operators and their asymptotic formulae

Now using the nonnegative regular matrix summability methods and power series methods, we consider the following modifications of Bernstein polynomials:

$$\mathcal{B}_j^{[A]}(f;x) := \sum_{n=1}^{\infty} a_{jn} B_n(f;x), \quad j \in \mathbb{N}, \tag{2.1}$$

and

$$\mathcal{B}_t^{[P]}(f;x) := \frac{1}{P(t)} \sum_{n=0}^{\infty} p_n t^n B_{n+1}(f;x), \quad 0 < t < R. \tag{2.2}$$

Observe that the operators $\mathcal{B}_t^{[P]}$ preserves linear functions while $\mathcal{B}_j^{[A]}$ do not. Hence, we need the next condition:

$$\sum_{n=1}^{\infty} a_{jn} = 1 \quad (j \in \mathbb{N}), \tag{2.3}$$

which is stronger than (ii). Under this assumption, the operators $\mathcal{B}_j^{[A]}$ preserve linear functions, too. From the regularity, one can immediately say that both operators (2.1) and (2.2) are uniformly convergent to f for every $f \in C[0,1]$, i.e.,

$$\lim_{j\to\infty} \mathcal{B}_j^{[A]}(f;x) = f(x) \text{ uniformly with respect to } x \in [0,1]$$

and

$$\lim_{0<t\to R^-} \mathcal{B}_t^{[P]}(f;x) = f(x) \text{ uniformly with respect to } x \in [0,1].$$

Now let

$$\tau_{j,r} := \sum_{n=1}^{\infty} \frac{a_{jn}}{n^r} \quad (j \in \mathbb{N} \text{ and } r > 0)$$

and

$$\beta_r(t) := \frac{1}{P(t)} \sum_{n=0}^{\infty} \frac{p_n t^n}{(n+1)^r} \quad (0 < t < R \text{ and } r > 0).$$

We first obtain the following result which gives the (uniform) approximation of continuous functions by our modified operators. This result also provides an error estimate for the rate of convergence.

Theorem 2.1. *Let $f \in C[0,1]$ and $x \in [0,1]$.*

(*a*) *For every* $j \in \mathbb{N}$, *we have*

$$\left|\mathcal{B}_j^{[A]}(f;x) - f(x)\right| = O\left(\sum_{n=1}^{\infty} a_{jn}\omega\left(f, \frac{1}{\sqrt{n}}\right)\right).$$

Furthermore, if $f \in Lip1$, *then*

$$\left|\mathcal{B}_j^{[A]}(f;x) - f(x)\right| = O\left(\tau_{j,1/2}\right).$$

(*b*) *For every* $t \in (0,R)$, *we have*

$$\left|\mathcal{B}_t^{[P]}(f;x) - f(x)\right| = O\left(\frac{1}{P(t)}\sum_{n=0}^{\infty} p_n t^n \omega\left(f, \frac{1}{\sqrt{n+1}}\right)\right).$$

Furthermore, if $f \in Lip1$, *then*

$$\left|\mathcal{B}_t^{[P]}(f;x) - f(x)\right| = O\left(\beta_{1/2}(t)\right).$$

Proof. For the classical Bernstein polynomials, the following facts are well-known (see [14]):

$$|B_n(f;x) - f(x)| \leq \frac{5}{4}\omega\left(f, \frac{1}{\sqrt{n}}\right) \text{ for } f \in C[0,1],$$

which gives

$$|B_n(f;x) - f(x)| \leq \frac{C}{\sqrt{n}}$$

for some $C > 0$ whenever $f \in Lip1$. From the similarity we just prove (a). Using defininitions (2.1), (2.2) and assumption (2.3) , we get

$$\begin{aligned}\left|\mathcal{B}_j^{[A]}(f;x) - f(x)\right| &\leq \sum_{n=1}^{\infty} a_{jn}\left|B_n(f;x) - f(x)\right| \\ &\leq \frac{5}{4}\sum_{n=1}^{\infty} a_{jn}\omega\left(f, \frac{1}{\sqrt{n}}\right),\end{aligned}$$

which gives the first part of (a). The remaining part of (a) is clear from the fact that $\omega(f,\delta) \leq C\delta$ for some $C > 0$ whenever $f \in Lip1$. □

In the next result, we see that the rate of convergence accelerates when the function has a first-order continuous derivative.

Theorem 2.2. *Let* $f \in C^1[0,1]$ *and* $x \in [0,1]$.

(*a*) *For every* $j \in \mathbb{N}$, *we have*

$$\left|\mathcal{B}_j^{[A]}(f;x) - f(x)\right| = O\left(\sum_{n=1}^{\infty} \frac{a_{jn}}{\sqrt{n}}\omega\left(f', \frac{1}{\sqrt{n}}\right)\right).$$

Furthermore, if $f' \in Lip1$, *then*

$$\left|\mathcal{B}_j^{[A]}(f;x) - f(x)\right| = O\left(\tau_{j,1}\right).$$

(*b*) *For every* $t \in (0,R)$, *we have*

$$\left|\mathcal{B}_t^{[P]}(f;x) - f(x)\right| = O\left(\frac{1}{P(t)}\sum_{n=0}^{\infty} \frac{p_n t^n}{\sqrt{n+1}}\omega\left(f', \frac{1}{\sqrt{n+1}}\right)\right).$$

Furthermore, if $f \in Lip1$, *then*

$$\left|\mathcal{B}_t^{[P]}(f;x) - f(x)\right| = O\left(\beta_1(t)\right).$$

Proof. The proofs are easily obtained from the following fact about the classical Bernstein polynomials (see [14]):

$$|B_n(f;x) - f(x)| \leq \frac{3}{4\sqrt{n}}\omega\left(f', \frac{1}{\sqrt{n}}\right) \text{ for } f \in C^1[0,1],$$

which gives

$$|B_n(f;x) - f(x)| \leq \frac{C}{n}$$

for some $C > 0$ whenever $f' \in Lip1$. □

Now we also need the following assumptions for these operators, respectively:

$$\lim_{j\to\infty} \frac{\tau_{j,2}}{\tau_{j,1}} = 0 \tag{2.4}$$

and

$$\lim_{0<t\to R^-} \frac{\beta_2(t)}{\beta_1(t)} = 0. \tag{2.5}$$

Then, here is our main theorem. This shows how Voronovskaja-type results can be improved with the help of regular summability methods.

Theorem 2.3. *Assume that f is a bounded function on $[0,1]$ and the second derivative f'' exists at a certain point x of $(0,1)$. Then, we get the following:*

(*a*) *If $A = [a_{jn}]$ $(j, n \in \mathbb{N})$ is a nonnegative regular matrix summability method satisfying* (2.3) *and* (2.4)*, then*

$$\lim_{j\to\infty} \tau_{j,1}^{-1}\left(\mathcal{B}_j^{[A]}(f;x) - f(x)\right) = \frac{x(1-x)}{2}f''(x)$$

holds. In particular, if $f''(x) \neq 0$, then the difference $\mathcal{B}_j^{[A]}(f;x) - f(x)$ is exactly of order $\tau_{j,1}$.

(*b*) *If P_p is a nonnegative regular power series method satisfying* (2.5)*, then we get*

$$\lim_{0<t\to R^-} (\beta_1(t))^{-1}\left(\mathcal{B}_t^{[P]}(f;x) - f(x)\right) = \frac{x(1-x)}{2}f''(x).$$

In particular, if $f''(x) \neq 0$, then the difference $\mathcal{B}_t^{[P]}(f;x) - f(x)$ is exactly of order $\beta_1(t)$.

Proof. (*a*) From Taylor's theorem, there exists a bounded function μ_x on $[0,1]$ satisfying $\lim_{y\to x} \mu_x(y) = 0$ such that

$$f(y) = f(x) + (y-x)\,f'(x) + \frac{(y-x)^2}{2}f''(x) + (y-x)^2\mu_x(y)$$

holds for all $y \in [0,1]$. Then, we obtain from (2.1) and (2.3) that

$$\begin{aligned}
\mathcal{B}_j^{[A]}(f;x) - f(x) &= \mathcal{B}_j^{[A]}\left(f(y) - f(x);x\right) \\
&= \frac{x(1-x)f''(x)}{2}\tau_{j,1} \\
&\quad + \sum_{n=1}^{\infty} a_{jn} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \mu_x\left(\frac{k}{n}\right) p_{k,n}(x).
\end{aligned}$$

Hence we get

$$\tau_{j,1}^{-1}\left(\mathcal{B}_j^{[A]}(f;x) - f(x)\right) = \frac{x(1-x)}{2}f''(x) + \Psi_j(x),$$

where

$$\Psi_j(x) = \tau_{j,1}^{-1} \sum_{n=1}^{\infty} a_{jn} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \mu_x\left(\frac{k}{n}\right) p_{k,n}(x).$$

For the proof, it is enough to show that

$$\lim_{j\to\infty} \Psi_j(x) = 0. \tag{2.6}$$

From the boundedness of μ_x on $[0,1]$, there exists $M>0$ such that $|\mu_x(y)| \le M$. Also, since $\lim_{y\to x} \mu_x(y) = 0$, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $|y-x|<\delta$ implies $|\mu_x(y)| < \varepsilon$. Then, we get

$$\begin{aligned}
|\Psi_j(x)| &\le \tau_{j,1}^{-1} \sum_{n=1}^{\infty} a_{jn} \sum_{k=0}^{n} \left(\frac{k}{n}-x\right)^2 \left|\mu_x\left(\frac{k}{n}\right)\right| p_{k,n}(x) \\
&\le \tau_{j,1}^{-1} \sum_{n=1}^{\infty} a_{jn} \left\{ \varepsilon \sum_{k:\left|\frac{k}{n}-x\right|<\delta} \left(\frac{k}{n}-x\right)^2 p_{k,n}(x) \right. \\
&\quad \left. + M \sum_{k:\left|\frac{k}{n}-x\right|\ge\delta} \left(\frac{k}{n}-x\right)^2 p_{k,n}(x) \right\} \\
&\le \tau_{j,1}^{-1} \sum_{n=1}^{\infty} a_{jn} \left\{ \varepsilon \sum_{k=0}^{n} \left(\frac{k}{n}-x\right)^2 p_{k,n}(x) \right. \\
&\quad \left. + \frac{M}{\delta^2} \sum_{k=0}^{n} \left(\frac{k}{n}-x\right)^4 p_{k,n}(x) \right\}.
\end{aligned}$$

We may also write from Theorem 1.5.1 in [14] that

$$\sum_{k=0}^{n} \left(\frac{k}{n}-x\right)^2 p_{k,n}(x) \le \frac{1}{n} \quad \text{and} \quad \sum_{k=0}^{n} \left(\frac{k}{n}-x\right)^4 p_{k,n}(x) \le \frac{C}{n^2}$$

for some $C>0$. Combining the above facts we observe that

$$|\Psi_j(x)| \le \varepsilon + \frac{MC}{\delta^2} \frac{\tau_{j,2}}{\tau_{j,1}}. \tag{2.7}$$

Therefore, (2.6) follows from (2.4) and (2.7), which completes the proof.

(b) Using the same idea as in (a), from the definition of operators (2.2), there exists a bounded function μ_x on $[0,1]$ satisfying $\lim_{y\to x} \mu_x(y) = 0$ such that

$$(\beta_1(t))^{-1} \left(\mathcal{B}_t^{[P]}(f;x) - f(x) \right) = \frac{x(1-x)}{2} f''(x) + \Theta(x,t),$$

holds, where

$$\Theta(x,t) := (\beta_1(t))^{-1} \sum_{n=0}^{\infty} p_n t^n \sum_{k=0}^{n+1} \left(\frac{k}{n+1}-x\right)^2 \mu_x\left(\frac{k}{n+1}\right) p_{k,n+1}(x).$$

Now it is enough to show that

$$\lim_{0<t\to R^-} \Theta(x,t) = 0. \tag{2.8}$$

Then, using the boundedness of μ_x and the limit condition $\lim_{y\to x}\mu_x(y)=0$, we may write that, for every $\varepsilon>0$,

$$\begin{aligned}|\Theta(x,t)| &\leq (\beta_1(t))^{-1}\sum_{n=0}^{\infty}p_nt^n\left\{\varepsilon\sum_{k=0}^{n+1}\left(\frac{k}{n+1}-x\right)^2p_{k,n+1}(x)\right.\\ &\quad\left.+\frac{M}{\delta^2}\sum_{k=0}^{n+1}\left(\frac{k}{n+1}-x\right)^4p_{k,n+1}(x)\right\}\\ &\leq \varepsilon+\frac{MC}{\delta^2}\frac{\beta_2(t)}{\beta_1(t)},\end{aligned}$$

where M,C and δ are the same as in (a). Now taking limit as $0<t\to R^-$ on the both sides of the last inequality and also using (2.5), we immediately get (2.8), which is the desired result. □

Remark 2.4. As can be seen, we needed condition (2.4) (or, (2.5) respectively) to prove Theorem 2.3. In the next section we will give examples of the existence of regular summability methods that do not satisfy condition (2.4). On the other hand, taking

$$\varepsilon_n(x):=n(B_n(f;x)-f(x))-\frac{x(1-x)}{2}f''(x), \tag{2.9}$$

one can easily see from (1.1) that

$$\lim_{n\to\infty}\varepsilon_n(x)=0.$$

Then, using (2.1), (2.3) and (2.9), we may write that

$$\begin{aligned}\mathcal{B}_j^{[A]}(f;x) &= \sum_{n=1}^{\infty}a_{jn}\left(f(x)+\frac{\varepsilon_n(x)+x(1-x)f''(x)/2}{n}\right)\\ &= f(x)+\tau_{j,1}\frac{x(1-x)}{2}f''(x)+\sum_{n=1}^{\infty}a_{jn}\frac{\varepsilon_n(x)}{n},\end{aligned}$$

which implies

$$\tau_{j,1}^{-1}\left(\mathcal{B}_j^{[A]}(f;x)-f(x)\right)=\frac{x(1-x)}{2}f''(x)+\tau_{j,1}^{-1}\sum_{n=1}^{\infty}a_{jn}\frac{\varepsilon_n(x)}{n}.$$

Then, in order to obtain an alternative proof of Theorem 2.3 we now need the following:

$$\lim_{j\to\infty}\tau_{j,1}^{-1}\sum_{n=1}^{\infty}a_{jn}\frac{\varepsilon_n(x)}{n}=0. \tag{2.10}$$

Although

$$\lim_{j\to\infty}\tau_{j,1}=0 \text{ and } \lim_{j\to\infty}\sum_{n=1}^{\infty}a_{jn}\frac{\varepsilon_n(x)}{n}=0$$

are known from regularity of the method $A=[a_{jn}]$, we cannot guarantee the truth of (2.10) for any regular method A. We therefore arrive at the following open problems.

Open Problems:

(a) For any nonnegative regular matrix summability method, does Theorem 2.3 hold true without condition (2.4) or (2.10)?

(b) For any nonnegative regular power series method, does Theorem 2.3 hold true without condition (2.5)?

Remark 2.5. If f is a bounded function on $[0,1]$ and the second derivative f'' exists at a certain point x of $(0,1)$, then (1.1) says that the rate of convergence for the classical Bernstein polynomials is exactly of order $1/n$ while, according to Theorem 2.3 (a), it is of order $\tau_{j,1}$ for the modified Bernstein operators. For example, if we consider the Cesàro matrix summability method $A = C_1 = [c_{jn}]$ defined by

$$c_{jn} := \begin{cases} \frac{1}{j}, & \text{if } n = 1, 2, ..., j \\ 0, & \text{otherwise,} \end{cases} \tag{2.11}$$

then we see that our rate of convergence for the modified Bernstein operators become

$$\tau_{j,1} = \frac{1 + (1/2) + \cdots + (1/j)}{j},$$

which is slower than the rate of $1/n$ for the classical Bernstein polynomials. But this is not always the case. Now, if we make a small change in the definition of Bernstein polynomials as follows

$$B_n^*(f;x) := \begin{cases} 2B_n(f;x), & \text{if } n \text{ even} \\ 0, & \text{if } n \text{ odd,} \end{cases}$$

then it will no longer be possible to approximate the function f by means of the sequence $\{B_n^*(f)\}$ (in the classical sense). However, it is possible to approximate f by the Cesàro transform of $\{B_n^*(f)\}$ given by

$$\mathcal{B}_j^{*[C_1]}(f;x) := \frac{1}{j} \sum_{n=1}^{j} B_n^*(f;x).$$

So, this points to the existence of an alternative method to compensate for the loss of convergence. On the other hand, in order to accelerate the convergence, one can consider the following regular subsequence matrix transformation $A = G = [g_{jn}]$ given by

$$g_{jn} := \begin{cases} 1, & \text{if } n = j^2 \\ 0, & \text{otherwise.} \end{cases}$$

In this case, the rate of convergence for the modified operators $\mathcal{B}_j^{[G]}(f;x)$ becomes $\tau_{j,1} = 1/j^2$, which is faster than $1/n$ for the classical operators $B_n(f;x)$.

3. Special cases

Now we give some important special cases of Theorem 2.3, which are also new in the literature.

First of all, if we take $A = I$, the identity matrix, Theorem 2.3 (a) reduces to the asymptotic formula (1.1).

3.1. Cesàro matrix summability

Now consider the Cesàro matrix summability method given by (2.11). Then it is easy to check that conditions (2.3) and (2.4) hold. Hence, we get from Theorem 2.3 (a) that if f is a bounded function on $[0,1]$ and the second derivative f'' exists at a certain point x of $[0,1]$, then the following new asymptotic formula is satisfied:

$$\lim_{j\to\infty} \left(\frac{1}{j} \sum_{n=1}^{j} \frac{1}{n} \right)^{-1} \left(\frac{1}{j} \sum_{n=1}^{j} B_n(f;x) - f(x) \right) = \frac{x(1-x)}{2} f''(x), \tag{3.1}$$

where $B_n(f;x)$ is the classical Bernstein polynomial. This asymptotic approximation is indicated in Figure 1 for the function $f(x) = 3x^2 - 2\sin(\pi x)$ and the values $j = 10, 30, 120$.

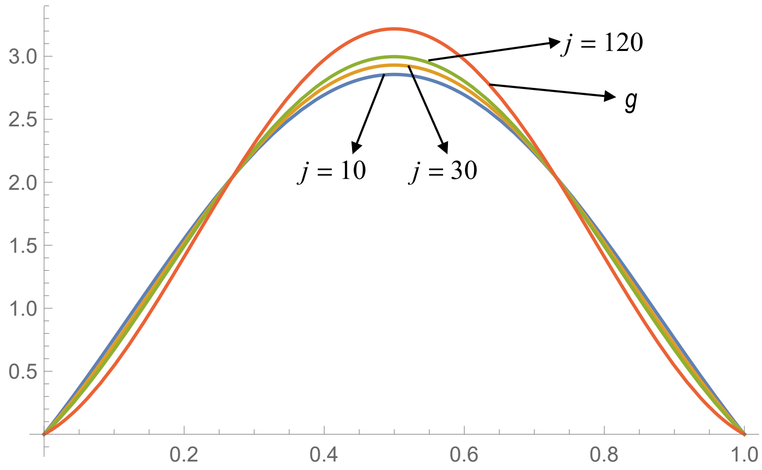

Figure 1. Asymptotic approximation in (3.1) for the function $f(x) = 3x^2 - 2\sin(\pi x)$ and the values $j = 10, 30, 120$. In this figure, $g(x) := \frac{x(1-x)}{2} f''(x)$.

3.2. Riesz matrix summability

For another special case of Theorem 2.3 (a), assume that (q_n) is a real sequence with $q_1 > 0$ and $q_n \geq 0$ $(n = 2, 3, ...)$, and also $Q_j := \sum_{n=1}^{j} q_n$. Now, take $A = R_q = [r_{jn}]$, the Riesz matrix summability method defined by

$$r_{jn} := \begin{cases} \frac{q_n}{Q_j}, & \text{if } n = 1, 2, ..., j \\ 0, & \text{otherwise.} \end{cases}$$

Then, observe that the matrix R_q satisfies (2.3). Also, (2.4) is equivalent to the next condition:

$$\lim_{j\to\infty} \frac{\sum_{n=1}^{j} \frac{q_n}{n^2}}{\sum_{n=1}^{j} \frac{q_n}{n}} = 0, \tag{3.2}$$

Hence, if (3.2) holds, then Theorem 2.3 (a) implies

$$\lim_{j\to\infty} \left(\frac{1}{Q_j} \sum_{n=1}^{j} \frac{q_n}{n} \right)^{-1} \left(\frac{1}{Q_j} \sum_{n=1}^{j} q_n B_n(f; x) - f(x) \right) = \frac{x(1-x)}{2} f''(x). \tag{3.3}$$

For example, if we take $q_n = 1$ for all $n \in \mathbb{N}$, observe that (2.4) holds and (3.3) reduces to (3.1). On the other hand, if we take $q_n = n$ $(n \in \mathbb{N})$, then it is easy to check that (3.2) holds, and we get

$$\lim_{j\to\infty} \frac{j+1}{2} \left(\frac{2}{j(j+1)} \sum_{n=1}^{j} n B_n(f; x) - f(x) \right) = \frac{x(1-x)}{2} f''(x).$$

We should remark that condition (2.4) or (3.2) does not hold for all nonnegative matrix summability methods. For example, consider the case of $q_n = \frac{1}{n}$ $(n \in \mathbb{N})$. Then, we observe that

$$\lim_{j\to\infty} \frac{\sum_{n=1}^{j} \frac{q_n}{n^2}}{\sum_{n=1}^{j} \frac{q_n}{n}} = \frac{\sum_{n=1}^{\infty} \frac{1}{n^3}}{\sum_{n=1}^{\infty} \frac{1}{n^2}} \geq \frac{6}{\pi^2}.$$

3.3. Abel power series method

Now we give some special cases of Theorem 2.3 (b). First, taking $p_n = 1$ for all $n \in \mathbb{N}_0$, we get the Abel method. In this case, observe that $P(t) = \frac{1}{1-t}$ for $-1 < t < 1$ and

$\beta_1(t) = \frac{(t-1)\ln(1-t)}{t}$ for $0 < t < 1$. We check that condition (2.5) is satisfied. Then, Theorem 2.3 (b) gives the following:

$$\lim_{0<t\to 1^-} \frac{t}{(t-1)\ln(1-t)} \left((1-t) \sum_{n=0}^{\infty} t^n B_{n+1}(f;x) - f(x) \right) = \frac{x(1-x)}{2} f''(x).$$

3.4. Borel power series method

For a final special case, take $p_n = \frac{1}{n!}$, $n \in \mathbb{N}_0$, which gives the Borel method. Then, we get $P(t) = e^t$ for $-\infty < t < \infty$ and $\beta_1(t) = \frac{t+e^t}{te^t}$ for $t \neq 0$. Since (2.5) is satisfied, Theorem 2.3 (b) implies

$$\lim_{0<t\to\infty} \left(\frac{te^t}{t+e^t} \right) \left(\frac{1}{e^t} \sum_{n=0}^{\infty} \frac{t^n B_{n+1}(f;x)}{n!} - f(x) \right) = \frac{x(1-x)}{2} f''(x).$$

4. Kantorovich version of the operators

The Kantorovich version (see [12]) of the classical Bernstein polynomials, which are known in the literature as Bernstein-Kantorovich polynomials, are defined by

$$K_n(f;x) := (n+1) \sum_{k=0}^{n} p_{k,n}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(u)du, \ x \in [0,1], \ n \in \mathbb{N}. \tag{4.1}$$

These operators enable us to approximate uniformly to not only continuous functions but also integrable functions on $[0,1]$. We also know that the asymptotic formula

$$\lim_{n\to\infty} n \left(K_n(f;x) - f(x) \right) = \left(-x + \frac{1}{2} \right) f'(x) + \frac{x(1-x)}{2} f''(x)$$

holds when f is a bounded function on $[0,1]$ and the derivatives f' and f'' exists at a certain point x of $(0,1)$.

Then our new operators based on nonnegative regular matrix summability and power series methods are given respectively by

$$\mathcal{K}_j^{[A]}(f;x) := \sum_{n=1}^{\infty} a_{jn} K_n(f;x), \quad j \in \mathbb{N}, \tag{4.2}$$

and

$$\mathcal{K}_t^{[P]}(f;x) := \frac{1}{P(t)} \sum_{n=0}^{\infty} p_n t^n K_{n+1}(f;x), \quad 0 < t < R. \tag{4.3}$$

We now obtain the next asymptotic formulae for (4.2) and (4.3).

Theorem 4.1. *Assume that f is a bounded function on $[0,1]$ and the first and second derivatives f' and f'' exist at a certain point x of $(0,1)$. Then, we get the following:*

(a) *If $A = [a_{jn}]$ $(j, n \in \mathbb{N})$ is a nonnegative regular matrix summability method satisfying* (2.3) *and* (2.4)*, then*

$$\lim_{j\to\infty} \tau_{j,1}^{-1} \left(\mathcal{K}_j^{[A]}(f;x) - f(x) \right) = \left(-x + \frac{1}{2} \right) f'(x) + \frac{x(1-x)}{2} f''(x)$$

holds.

(b) *If P_p is a nonnegative regular power series method satisfying* (2.5)*, then*

$$\lim_{0<t\to R^-} (\beta_1(t))^{-1} \left(\mathcal{K}_t^{[P]}(f;x) - f(x) \right) = \left(-x + \frac{1}{2} \right) f'(x) + \frac{x(1-x)}{2} f''(x)$$

holds.

Proof. (a) We may write from (4.2) and Taylor's theorem that

$$\begin{aligned} \mathcal{K}_j^{[A]}(f;x)-f(x) &= \mathcal{K}_j^{[A]}\left(f(y)-f(x);x\right) \\ &= f'(x)\mathcal{K}_j^{[A]}\left(y-x;x\right)+\frac{f''(x)}{2}\mathcal{K}_j^{[A]}\left((y-x)^2;x\right) \\ &\quad +\mathcal{K}_j^{[A]}\left((y-x)^2\mu_x(y);x\right). \end{aligned}$$

By a direct computation, we get

$$\begin{aligned} \mathcal{K}_j^{[A]}\left(y-x;x\right) &= \sum_{n=1}^{\infty} a_{jn}K_n(y-x;x) \\ &= \left(-x+\frac{1}{2}\right)\sum_{n=1}^{\infty}\frac{a_{jn}}{n+1} \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_j^{[A]}\left((y-x)^2;x\right) &= \sum_{j=1}^{\infty} a_{jn}K_n\left((y-x)^2;x\right) \\ &= -x^2\sum_{n=1}^{\infty}\frac{(n-1)\,a_{jn}}{(n+1)^2}+2x\sum_{n=1}^{\infty}\frac{na_{jn}}{(n+1)^2} \\ &\quad -x\sum_{n=1}^{\infty}\frac{a_{jn}}{n+1}+\frac{1}{3}\sum_{n=1}^{\infty}\frac{a_{jn}}{(n+1)^2}. \end{aligned}$$

Then, it follows from (2.4) that

$$\lim_{j\to\infty}\tau_{j,1}^{-1}\mathcal{K}_j^{[A]}\left(y-x;x\right)=-x+\frac{1}{2}$$

and

$$\lim_{j\to\infty}\tau_{j,1}^{-1}\mathcal{K}_j^{[A]}\left((y-x)^2;x\right)=x(1-x).$$

We also know from (4.1) that there exist positive constants C_1 and C_2 such that

$$K_n\left((y-x)^2;x\right)\le\frac{C_1}{n}\text{ and }K_n\left((y-x)^4;x\right)\le\frac{C_2}{n^2}.$$

Using this and also the fact that $\lim_{y\to x}\mu_x(y)=0$, we obtain that

$$\begin{aligned} \mathcal{K}_j^{[A]}\left((y-x)^2\mu_x(y);x\right) &= \sum_{n=1}^{\infty} a_{jn}K_n\left((y-x)^2\mu_x(y);x\right) \\ &\le \varepsilon\sum_{n=1}^{\infty} a_{jn}K_n\left((y-x)^2;x\right) \\ &\quad +\frac{M}{\delta^2}\sum_{n=1}^{\infty} a_{jn}K_n\left((y-x)^4;x\right) \\ &\le C_1\varepsilon\tau_{j,1}+\frac{MC_2}{\delta^2}\tau_{j,2}, \end{aligned}$$

where ε,δ and M are the same as in the proof of Theorem 2.3. Now, from (2.4), we immediately get

$$\lim_{j\to\infty}\tau_{j,1}^{-1}\mathcal{K}_j^{[A]}\left((y-x)^2\mu_x(y);x\right)=0.$$

Combining the above results, the proof is completed.

(b) We first observe that

$$\begin{aligned}\mathcal{K}_t^{[P_p]}(y-x;x) &= \frac{1}{P(t)}\sum_{n=0}^{\infty} p_n t^n K_{n+1}(y-x;x)\\ &= \left(-x+\frac{1}{2}\right)\frac{1}{P(t)}\sum_{n=0}^{\infty}\frac{p_n t^n}{n+2}\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_t^{[P_p]}\left((y-x)^2;x\right) &= \frac{1}{P(t)}\sum_{n=0}^{\infty} p_n t^n K_{n+1}\left((y-x)^2;x\right)\\ &= -\frac{x^2}{P(t)}\sum_{n=0}^{\infty}\frac{np_n t^n}{(n+2)^2}+\frac{2x}{P(t)}\sum_{n=0}^{\infty}\frac{(n+1)p_n t^n}{(n+2)^2}\\ &\quad -\frac{x}{P(t)}\sum_{n=0}^{\infty}\frac{p_n t^n}{n+2}+\frac{1}{3P(t)}\sum_{n=0}^{\infty}\frac{p_n t^n}{(n+2)^2}.\end{aligned}$$

Also from condition (2.5) one can obtain the following:

$$\begin{aligned}\lim_{0<t\to R^-}\frac{1}{\beta_1(t)P(t)}\sum_{n=0}^{\infty}\frac{p_n t^n}{n+2} &= 1,\\ \lim_{0<t\to R^-}\frac{1}{\beta_1(t)P(t)}\sum_{n=0}^{\infty}\frac{np_n t^n}{(n+2)^2} &= 1,\\ \lim_{0<t\to R^-}\frac{1}{\beta_1(t)P(t)}\sum_{n=0}^{\infty}\frac{(n+1)p_n t^n}{(n+2)^2} &= 1,\\ \lim_{0<t\to R^-}\frac{1}{\beta_1(t)P(t)}\sum_{n=0}^{\infty}\frac{p_n t^n}{(n+2)^2} &= 0.\end{aligned}$$

Therefore, the above results imply that

$$\lim_{0<t\to R^-}(\beta_1(t))^{-1}\mathcal{K}_t^{[P_p]}(y-x;x) = -x+\frac{1}{2}$$

and

$$\lim_{0<t\to R^-}(\beta_1(t))^{-1}\mathcal{K}_t^{[P_p]}\left((y-x)^2;x\right) = x(1-x).$$

Now, as in the proof of (a), it follows from (4.3) and Taylor's theorem that

$$\begin{aligned}\mathcal{K}_t^{[P_p]}(f;x)-f(x) &= \mathcal{K}_t^{[P_p]}(f(y)-f(x);x)\\ &= f'(x)\mathcal{K}_t^{[P_p]}(y-x;x)+\frac{f''(x)}{2}\mathcal{K}_t^{[P_p]}\left((y-x)^2;x\right)\\ &\quad +\mathcal{K}_t^{[P_p]}\left((y-x)^2\mu_x(y);x\right)\end{aligned}$$

holds. Now, if we first multiply both sides of the above equality by $(\beta_1(t))^{-1}$ and then take the limit as $0<t\to R^-$, we get the desired result. □

Finally, we should note that all the special cases given for Theorem 2.3 in Section 3 are also valid for Theorem 4.1; but we omit the details.

5. Concluding remarks

In this study, we have investigated the effects of regular summability methods on the approximation of functions by Bernstein polynomials. This has allowed us to obtain some new Voronovskaja results. In particular, we have for the first time used well-known regular methods such as Cesàro, Riesz, Abel and Borel in asymptotic formulae of a Voronovskaja type. When we compare our results with the classical ones, we have found that in cases where the classical convergence works, regular summability methods sometimes decelerate convergence and sometimes accelerate it. But more importantly, our approach has also responded to situations where the classical convergence fails.

In the future we plan to do similar work on the q-Bernstein polynomials, which were introduced by Phillips [16]. It is well known that to approximate continuous functions by means of q-Bernstein polynomials, instead of a fixed number $q \in (0,1)$, one needs a sequence (q_n) whose terms lie in the interval $(0,1)$ and satisfy the limit condition $\lim_{n\to\infty} q_n = 1$. Therefore, the work we plan to do may be more interesting because it is possible to weaken this limit condition with the help of regular summability methods.

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