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Regular conformable fractional Dirac systems with impulsive boundary conditions

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Abstract. The regular impulsive conformable fractional Dirac system is discussed in this study. First, the uniqueness and existence of solutions for certain kinds of systems are examined. Next, the fundamental features of the operator corresponding to these systems are found and its symmetry is shown. In the end, Green's function for this problem is determined, and its fundamental characteristics are provided.

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1. Introduction

One of the key equations in quantum mechanics is the one-dimensional Dirac system, which expresses the underlying physics of practical quantum mechanics. In addition, it describes the electron spin and makes an antimatter prediction. The literature (see [19, 24, 28]) has examined a few Dirac system properties.

One of the key issues covered in differential equation theory is impulsive differential equations. There are a lot of paper on this topic in the literature because these equations provide fundamental models for studying the dynamics of processes that are susceptible to abrupt changes in their states ([8–12, 14, 15, 18, 20, 25, 26]). In [4], the authors investigated a coupled system of piecewise-order differential equations with variable kernel and impulsive conditions. In [22], Shah et al. studied a class of impulsive fractional order differential equations with integral boundary condition. A nonlinear Cauchy problem is investigated under impulsive and nonlinear integral boundary conditions in [23]. In [2], the authors is obtained some results about existence and stability analysis for a nonlinear problem of implicit fractional differential equations with impulsive and integral boundary conditions. In [3], Ali et al. studied impulsive fractional order differential equations under nonlocal Caputo fractional boundary conditions.

One of the topics of fractional calculus that is still growing is conformable fractional calculus. Khalil et al. [17] started the research of conformable fractional calculus. The conformable fractional derivative is a novel type of simple fractional derivative that was defined by the authors in [17]. Abdeljawad [1] later defined the fractional chain rule, the fractional integrals of higher orders, and the right and left conformable fractional derivatives. To the best of our knowledge, however, much less research has been done on conformable fractional Dirac systems. In [16], Keskin studied the asymptotic formulae for the solutions, eigenvalues, and nodal points of conformable fractional Dirac-type integro-differential system. In [13], Göktaş et al. investigated the Dirac system in multiplicative fractional calculus. In [21], the authors studied the spectral structure of a conformable Dirac system with jump conditions and spectral parameter contained in boundary conditions.

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This paper examined the conformable fractional Dirac system under impulsive boundary conditions. We looked at the existence and uniqueness of solutions for this system. In order to do this, we have used the same method as in [25]. Next, the symmetric operator associated with this equation was discovered. The fundamental characteristics of eigenvalues and eigenfunctions were provided. Ultimately, the Green function associated with this system was determined, along with a few fundamental characteristics.

In this paper, the classical one-dimensional Dirac equation is analyzed under the framework of conformable fractional calculus with impulsive conditions. In [5], the authors have studied the one-dimensional conformable fractional Dirac equation without impulsive conditions. To the authors' knowledge, there is no work in the literature on the conformable fractional Dirac equation under impulsive conditions. Therefore, this study will be helpful for researchers working on these topics.

2. Preliminaries

We want to review some essential definitions and conformable fractional calculus properties in this part (see [1,17]).

Definition 1 ([1,17]). *Let $\alpha \in (0,1)$ and let $f : [0,\infty) \to \mathbb{R}$ be a function. Then, the conformable fractional derivative of f, $T_\alpha f$ is defined as follows:*

$$T_\alpha f(\xi) = \begin{cases} \lim\limits_{\varepsilon\to 0} \frac{f(\xi+\varepsilon\xi^{1-\alpha})-f(\xi)}{\varepsilon} & \text{for } \xi > 0, \\ \lim_{\xi\to 0^+}(T_\alpha f(\xi)) & \text{for } \xi = 0. \end{cases} \tag{1}$$

Definition 2 ([1,17]). *The following defines the conformable fractional integral of f:*

$$\left(I_\alpha^a f\right)(\xi) = \int_a^\xi f(t)d\alpha\,(t,a) = \int_a^\xi (t-a)^{\alpha-1} f(t)dt$$

and

$$\left({}^bI_\alpha f\right)(\xi) = \int_\xi^b f(t)d\alpha\,(b,t) = \int_\xi^b (b-t)^{\alpha-1} f(t)dt.$$

The space

$$L_\alpha^2(a,b) = \left\{ f : \|f\| := \left(\int_a^b |f(\xi)|^2\, d^\alpha\xi\right)^{1/2} = \left(\int_a^b |f(\xi)|^2\, \xi^{\alpha-1}dx\right)^{1/2} < \infty. \right\}$$

is a Hilbert space with the inner product

$$(f,g) := \int_a^b f\overline{g}d^\alpha\xi,\ \ f,g \in L_\alpha^2(a,b)$$

(see [27])

3. Impulsive Conformable Fractional Dirac System

Consider the following impulsive conformable fractional boundary-value problem (CBVP)

$$l(u) = \lambda u,\ \ \xi \in I, \tag{2}$$

$$L_1(u) := u_1(0) + k_1u_2(0) = 0, \tag{3}$$

$$L_2(u) := u_1(d-) - k_2u_1(d+) = 0, \tag{4}$$

$$L_3(u) := u_2(d-) - k_3u_2(d+) = 0, \tag{5}$$

$$L_4(u) := u_1(a) + k_4u_2(a) = 0, \tag{6}$$

where

$$l(u) := \begin{cases} -T_\alpha u_2 + p(\xi)\,u_1 \\ T_\alpha u_1 + r(\xi)\,u_2, \end{cases} \quad u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$k_1,k_2,k_3,k_4 \in \mathbb{R}$, $0 < d < a < \infty$, $I := I_1 \cup I_2$, $I_1 := [0,d)$, $I_2 := (d,a]$ and $\lambda \in \mathbb{C}$.

The following constitute our fundamental assumptions throughout the paper:
(A_1) Let $k_2k_3 = \rho > 0$.
(A_2) Finite limits $p\,(d\pm)$, $r\,(d\pm)$ exist for p and r, which are real-valued continuous functions on I.

Consider the Hilbert space $H = L^2_\alpha(0,d) \dot{+} L^2_\alpha(d,a)$ that has the following inner product

$$\langle f,g\rangle := \int_0^d f^{(1)^T}\overline{g^{(1)}}d^\alpha\xi + \rho\int_d^a f^{(2)^T}\overline{g^{(2)}}d^\alpha\xi,$$

where

$$f(\xi) = \begin{cases} f^{(1)}(\xi), & \xi\in I_1 \\ f^{(2)}(\xi), & \xi\in I_2, \end{cases} \quad g(\xi) = \begin{cases} g^{(1)}(\xi), & \xi\in I_1 \\ g^{(2)}(\xi), & \xi\in I_2. \end{cases}$$

Theorem 1. *The following impulsive conformable fractional initial-value problem*

$$l\,(u) = \lambda u,\ \ \lambda\in\mathbb{C},$$
$$u_1\,(0) = s_1,\ \ u_2\,(0) = s_2,\ \ s_1,s_2\in\mathbb{R},$$
$$u_1\,(d-) - k_2u_1\,(d+) = 0,$$
$$u_2\,(d-) - k_3u_2\,(d+) = 0,$$

has a unique solution ψ.

Proof. From [5], we see that the following conformable fractional initial-value problem

$$l\,(u(\xi)) = \lambda u(\xi),\ \ \xi\in(0,d),$$
$$u_1\,(0) = s_1,\ \ u_2\,(0) = s_2,$$

has a unique solution

$$\psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}.$$

Consider the following conformable fractional initial-value problem

$$l\,(u) = \lambda u,\ \ \xi\in(d,a], \tag{7}$$
$$u_1\,(d+) = \frac{1}{k_2}\psi_{11}\,(d-), \tag{8}$$
$$u_2\,(d+) = \frac{1}{k_3}\psi_{12}\,(d-). \tag{9}$$

Set

$$U_n\,(\xi,\lambda) = U_0\,(\xi,\lambda) + \int_d^\xi \begin{bmatrix} \varphi_2\,(\xi,\lambda)\,\varphi_1^T\,(\gamma,\lambda) \\ \\ -\varphi_1\,(\xi,\lambda)\,\varphi_2^T\,(\gamma,\lambda) \end{bmatrix} M\,(\gamma)\,U_{n-1}\,(\gamma,\lambda)\,d^\alpha\gamma,$$

where

$$U_{01}\,(\xi,\lambda) = \frac{1}{k_2}\psi_{11}\,(d-,\lambda) + \frac{1}{k_3}\,(\xi-d)\,\psi_{12}\,(d-,\lambda),\ \ \xi\in I_2,$$

$$U_{02}\,(\xi,\lambda) = \frac{1}{k_2}\,(\xi-d)\,\psi_{11}\,(d-,\lambda) + \frac{1}{k_3}\psi_{12}\,(d-,\lambda),\ \ \xi\in I_2,$$

$$\varphi_1\,(\xi,\lambda) = \begin{pmatrix} \varphi_{11}\,(\xi,\lambda) \\ \varphi_{12}\,(\xi,\lambda) \end{pmatrix} = \begin{pmatrix} \cos\left(\int_0^\xi \lambda d^\alpha t\right) \\ \\ \frac{\sin\left(\int_0^\xi \lambda d^\alpha t\right)}{\lambda} \end{pmatrix},$$

$$\varphi_2\,(\xi,\lambda) = \begin{pmatrix} \varphi_{21}\,(\xi,\lambda) \\ \varphi_{22}\,(\xi,\lambda) \end{pmatrix} = \begin{pmatrix} -\frac{\sin\left(\int_0^\xi \lambda d^\alpha t\right)}{\lambda} \\ \\ \cos\left(\int_0^\xi \lambda d^\alpha t\right) \end{pmatrix},$$

$$M=\begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix},$$

and the matrix transpose operation is shown by the symbol ^{T}. Here φ_1 and φ_2 are the fundamental solutions of Eq. (7) for

$$M=\begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix}=0.$$

Let $\lambda \in \mathbb{C}$ be fixed. Then positive numbers A and $N(\lambda)$ exist such that

$$\|M(\xi)\| \leq A,$$
$$\left|\varphi_{ij}(\xi,\lambda)\right| \leq \frac{\sqrt{N(\lambda)}}{2},\ i,j=1,2,$$
$$\|U_0(\xi,\lambda)\| \leq \widetilde{N(\lambda)},\ \xi \in I_2.$$

Hence, we obtain

$$\|U_1(\xi,\lambda)-U_0(\xi,\lambda)\|$$
$$\leq \left|\begin{array}{c} \int_d^\xi \begin{bmatrix} \cos\left(\int_0^\xi \lambda d^\alpha t\right) & \frac{\sin\left(\int_0^\gamma \lambda d^\alpha t\right)}{\lambda} \\ -\frac{\sin\left(\int_0^\xi \lambda d^\alpha t\right)}{\lambda} & \cos\left(\int_0^\gamma \lambda d^\alpha t\right) \end{bmatrix} p(\gamma)\, U_{01}(\gamma,\lambda)\, d^\alpha\gamma \\ -\int_d^\xi \begin{bmatrix} \cos\left(\int_0^\xi \lambda d^\alpha t\right)\cos\left(\int_0^\gamma \lambda d^\alpha t\right) \\ +\frac{\sin\left(\int_0^\xi \lambda d^\alpha t\right)}{\lambda}\frac{\sin\left(\int_0^\gamma \lambda d^\alpha t\right)}{\lambda} \end{bmatrix} r(\gamma)\, U_{02}(\gamma,\lambda)\, d^\alpha\gamma \end{array}\right|.$$
$$\leq 2N(\lambda)\,A\widetilde{N(\lambda)}\left|\int_d^\xi d^\alpha\gamma\right| \leq 2N(\lambda)\,A\widetilde{N(\lambda)}\left|\int_0^\xi d^\alpha\gamma\right| \leq 2N(\lambda)\,A\widetilde{N(\lambda)}\frac{(\xi)^\alpha}{\alpha}.$$

Similarly, we obtain

$$\|U_3(\xi,\lambda)-U_2(\xi,\lambda)\| \leq 2^2\widetilde{N(\lambda)}\frac{A^2N^2(\lambda)(\xi)^{2\alpha}}{\alpha^2 2!}.$$

Hence

$$\|U_{n+1}(\xi,\lambda)-U_n(\xi,\lambda)\| \leq 2^n\widetilde{N(\lambda)}\frac{(AN(\lambda)\xi^\alpha)^n}{\alpha^n n!}, \tag{10}$$

where $n=1,2,3,....$ By Weierstrass's test, we see that the series

$$\sum_{n=1}^{\infty} 2^n\widetilde{N(\lambda)}\frac{(AN(\lambda)\xi^\alpha)^n}{\alpha^n n!}$$

is uniformly convergent. Consequently, the series

$$U_1(\xi,\lambda)+\sum_{n=1}^{\infty}\{U_{n+1}(\xi,\lambda)-U_n(\xi,\lambda)\} \tag{11}$$

is uniformly convergent with respect to ξ on $(d,a]$. Then we obtain

$$\lim_{n\to\infty} U_n(\xi,\lambda)=\psi_2(\xi,\lambda),$$

i.e.,

$$\psi_2(\xi,\lambda)=U_1(\xi,\lambda)+\sum_{n=1}^{\infty}\{U_{n+1}(\xi,\lambda)-U_n(\xi,\lambda)\}.$$

Now we prove that ψ_2 satisfies Eq. (7).

$$T_\alpha U_{n+1}(\xi,\lambda)-T_\alpha U_n(\xi,\lambda)$$
$$=\int_d^\xi \left[T_\alpha\varphi_2(\xi,\lambda)\,\varphi_1^T(\gamma,\lambda)-T_\alpha\varphi_1(\xi,\lambda)\,\varphi_2^T(\gamma,\lambda)\right]\times$$
$$\times M(\gamma)\left[U_n(\gamma,\lambda)-U_{n-1}(\gamma,\lambda)\right]d^\alpha\gamma.$$

(10) implies that the series

$$\sum_{n=1}^{\infty}\left\{T_{\alpha}U_{n+1}\left(\xi,\lambda\right)-T_{\alpha}U_{n}\left(\xi,\lambda\right)\right\}$$

is uniformly convergent with respect to ξ on $(d,a]$. Hence, we find

$$\begin{aligned}T_{\alpha}\psi_{21}\left(\xi,\lambda\right)&=\sum_{n=1}^{\infty}\left\{T_{\alpha}U_{n+1,1}\left(\xi,\lambda\right)-T_{\alpha}U_{n,1}\left(\xi,\lambda\right)\right\}\\&=\left(-r\left(\xi\right)+\lambda\right)\sum_{n=1}^{\infty}\left\{U_{n,1}\left(\xi,\lambda\right)-U_{n-1,1}\left(\xi,\lambda\right)\right\}\\&=\left(-r\left(\xi\right)+\lambda\right)\psi_{22}\left(\xi,\lambda\right),\end{aligned}$$

due to

$$T_{\alpha}\psi_{21}\left(\xi,\lambda\right)=\left(-r\left(\xi\right)+\lambda\right)\psi_{22}\left(\xi,\lambda\right),$$

In a similar manner, the other equation in (7) is shown to be true. ψ_2 is obviously satisfied by (8)-(9). □

Hence, we conclude that

$$\psi\left(\xi,\lambda\right)=\begin{cases}\psi_{1}\left(\xi,\lambda\right), & \xi\in I_{1}\\ \psi_{2}\left(\xi,\lambda\right), & \xi\in I_{2}\end{cases}\tag{12}$$

satisfies (2)-(5).

Similarly, we can derive the subsequent theorem.

Theorem 2. *There is a solution*

$$\chi\left(\xi,\lambda\right)=\begin{cases}\chi_{1}\left(\xi,\lambda\right), & \xi\in I_{1}\\ \chi_{2}\left(\xi,\lambda\right), & \xi\in I_{2}\end{cases}\tag{13}$$

to Eq. (2) that satisfies (4)-(6) for any $\lambda\in\mathbb{C}$.

Currently, we examine the subsequent set:

$$D_{\max}=\left\{u\in H:\begin{array}{c}u_{j}\text{ and }T_{\alpha}u_{j}\text{ are continuous functions}\\ \text{on }I,\ u_{j}\left(d\pm\right)\text{ exist }(j=1,2)\text{ and finite,}\\ L_{2}\left(u\right)=L_{3}\left(u\right)=0\text{ and }l\left(u\right)\in H\end{array}\right\}.$$

Then the *maximal* operator $\mathcal{L}_{\max}$ on $D_{\max}$ is defined by $\mathcal{L}_{\max}u=l(u)$, $u\in D_{\max}$. If we restrict the operator $\mathcal{L}_{\max}$ to the set

$$D_{\min}=\left\{u\in D_{\max}:u_{1}\left(0\right)=u_{2}\left(0\right)=u_{1}\left(a\right)=u_{2}\left(a\right)=0\right\},$$

then we obtain the *minimal operator* $\mathcal{L}_{\min}$.

The conformable Green formula is defined as ([5])

$$\begin{aligned}&\left\langle l\left(u\right),z\right\rangle-\left\langle u,l(z)\right\rangle\\&=\left[u,z\right]\left(a\right)-\left[u,z\right]\left(d+\right)+\left[u,z\right]\left(d-\right)-\left[u,z\right]\left(0\right),\end{aligned}\tag{14}$$

where

$$u=\begin{pmatrix}u_{1}\\ u_{2}\end{pmatrix},\ z=\begin{pmatrix}z_{1}\\ z_{2}\end{pmatrix},\ u,z\in D_{\max}$$

and

$$\left[u,z\right]\left(\xi\right):=W_{\alpha}\left(u,\overline{z}\right)\left(\xi\right)=u_{1}\left(\xi\right)\overline{z_{2}\left(\xi\right)}-\overline{z_{1}\left(\xi\right)}u_{2}\left(\xi\right).$$

Let us take the operator $\mathcal{L}$ and its domain $u\in D_{\max}$, ($\mathcal{L}u=l(u)$) vectors satisfying (3)-(6). Next, using (14) and the conditions (3)-(6), the following theorem is derived.

Theorem 3. *$\mathcal{L}$ is a symmetric operator.*

Corollary 1. *(i) The CBVP (2)-(6) has eigenvalues that are all real.*
(ii) For the CBVP (2)-(6), if λ_1 and λ_2 are two distinct eigenvalues, then the corresponding eigenfunctions u_1 and u_2 are orthogonal.
(iii) From a geometric perspective, all of the eigenvalues of the CBVP (2)-(6) are simple.

4. The Characteristic Function

Let us define the following entire functions

$$\omega_1(\lambda) = W_\alpha(\psi_1, \chi_1)(\xi), \ \omega_2(\lambda) = W_\alpha(\psi_2, \chi_2)(\xi),$$

where

$$W_\alpha(\varphi, \chi)(\xi) = \psi_1(\xi)\chi_2(\xi) - \chi_1(\xi)\psi_2(\xi).$$

Combining (4) - (5), we see that $\omega_1(\lambda) = \rho\omega_2(\lambda)$.Thus, the *characteristic function* of the CBVP (2)-(6) is given by $\omega(\lambda) := \omega_1(\lambda) = \rho\omega_2(\lambda)$.

Lemma 1. *Let*

$$\Delta(\lambda) := \begin{vmatrix} L_1\psi_1 & L_1\chi_1 & L_1\psi_2 & L_1\chi_2 \\ L_2\psi_1 & L_2\chi_1 & L_2\psi_2 & L_2\chi_2 \\ L_3\psi_1 & L_3\chi_1 & L_3\psi_2 & L_3\chi_2 \\ L_4\psi_1 & L_4\chi_1 & L_4\psi_2 & L_4\chi_2 \end{vmatrix}.$$

Then, every $\lambda \in \mathbb{C}$, we see that $\Delta(\lambda) = -\frac{1}{\rho}\omega^3(\lambda)$.

Proof. Combining (12) and (13), we conclude that

$$\begin{aligned} &\Delta(\lambda) = \\ &\begin{vmatrix} 0 & -\omega_1(\lambda) & 0 & 0 \\ \psi_{11}(d-,\lambda) & \chi_{11}(d-,\lambda) & -k_2\psi_{21}(d+,\lambda) & -k_2\chi_{21}(d+,\lambda) \\ \psi_{12}(d-,\lambda) & \chi_{12}(d-,\lambda) & -k_3\psi_{22}(d+,\lambda) & -k_3\chi_{22}(d+,\lambda) \\ 0 & 0 & \omega_2(\lambda) & 0 \end{vmatrix} \\ &= \omega_1(\lambda) \begin{vmatrix} \psi_{11}(d-,\lambda) & -k_2\psi_{21}(d+,\lambda) & -k_2\chi_{21}(d+,\lambda) \\ \psi_{12}(d-,\lambda) & -k_3\psi_{22}(d+,\lambda) & -k_3\chi_{22}(d+,\lambda) \\ 0 & \omega_2(\lambda) & 0 \end{vmatrix} \\ &= -\omega_1(\lambda)\omega_2(\lambda) \begin{vmatrix} \psi_{11}(d-,\lambda) & -k_2\chi_{21}(d+,\lambda) \\ \psi_{12}(d-,\lambda) & -k_3\chi_{22}(d+,\lambda) \end{vmatrix} \\ &= -\omega_1(\lambda)\omega_2(\lambda) \begin{vmatrix} \psi_{11}(d-,\lambda) & \chi_{11}(d-,\lambda) \\ \psi_{12}(d-,\lambda) & \chi_{12}(d-,\lambda) \end{vmatrix} \\ &= -\omega_1^2(\lambda)\omega_2(\lambda) = -\frac{1}{\rho}\omega^3(\lambda). \end{aligned}$$

□

Theorem 4. *The zeros of $\omega(\lambda)$ are the same as the eigenvalues of (2)-(6). Hence the eigenvalues of (2)-(6) form a finite or infinite sequence without a finite accumulation point.*

Proof. Let $\lambda^{(0)}$ be a zero of $\omega(\lambda)$. Then $\omega_2\left(\lambda^{(0)}\right) = W_{\alpha,2}(\psi_2, \chi_2) = 0$, i.e., $\psi_2 = \zeta\chi_2$ for some $\zeta \neq 0$. Therefore ψ_2 satisfies (6). *Hence* $\lambda^{(0)}$ is an eigenvalue, *i.e.,*

$$\psi\left(\xi, \lambda^{(0)}\right) = \begin{cases} \psi_1\left(\xi, \lambda^{(0)}\right), & \xi \in I_1 \\ \psi_2\left(\xi, \lambda^{(0)}\right). & \xi \in I_2 \end{cases}$$

satisfies the CBVP (2)-(6).

Let $\lambda^{(0)}$ be an eigenvalue and $\eta\left(\xi, \lambda^{(0)}\right)$ be any corresponding eigenfunction. Suppose that $\omega\left(\lambda^{(0)}\right) \neq 0$. Then we see that $\omega_1\left(\lambda^{(0)}\right) \neq 0$ and $\omega_2\left(\lambda^{(0)}\right) \neq 0$. Thus there exist constants ζ_i, $i = 1, 2, 3, 4$, at least one of which is not zero, such that

$$\eta\left(\xi, \lambda^{(0)}\right) = \begin{cases} \zeta_1\psi_1\left(\xi, \lambda^{(0)}\right) + \zeta_2\chi_1\left(\xi, \lambda^{(0)}\right), & \xi \in I_1 \\ \zeta_3\psi_2\left(\xi, \lambda^{(0)}\right) + \zeta_4\chi_2\left(\xi, \lambda^{(0)}\right), & \xi \in I_2. \end{cases}$$

Since $\eta\left(\xi, \lambda^{(0)}\right)$ is the eigenfunction, we conclude that $L_i\eta\left(\xi, \lambda^{(0)}\right) = 0$, $i = 1, 2, 3, 4$. Hence

$$\det\left(L_i\eta\left(\xi, \lambda^{(0)}\right)\right) = \Delta\left(\lambda^{(0)}\right) = 0,$$

because at least one of the constants $\zeta_i,\ i=1,2,3,4$ is not zero. By Lemma 1, we deduce that $\Delta\left(\lambda^{(0)}\right)\neq 0$, a contradiction. $\square$

5. Green's Function

Let us consider the following CBVP

$$-T_\alpha u_2+\{\lambda+p(\xi)\}u_1=\Xi_1, \tag{15}$$

$$T_\alpha u_1+\{\lambda+r(\xi)\}u_2=\Xi_2, \tag{16}$$

$$u_1(0)+k_1u_2(0)=0, \tag{17}$$
$$u_1(d-)-k_2u_1(d+)=0, \tag{18}$$
$$u_2(d-)-k_3u_2(d+)=0, \tag{19}$$
$$u_1(a)+k_4u_2(a)=0, \tag{20}$$

where $\xi\in I$ and

$$\Xi=\begin{pmatrix}\Xi_1\\ \Xi_2\end{pmatrix}\in H.$$

Theorem 5. *Suppose that λ is not an eigenvalue of (2)-(6). The CBVP (15)-(20) has a unique solution u defined as*

$$u(\xi,\lambda)=\int_0^d G(\xi,\gamma,\lambda)\,\Xi(\gamma)\,d^\alpha\gamma+\rho\int_d^a G(\xi,\gamma,\lambda)\,\Xi(\gamma)\,d^\alpha\gamma, \tag{21}$$

where

$$G(\xi,\gamma,\lambda)$$

$$=\frac{1}{\omega(\lambda)}\begin{cases}\chi(\xi,\lambda)\,\psi^T(\gamma,\lambda), & 0\le\gamma\le\xi\le a,\ \gamma\neq d,\ \xi\neq d,\\ \psi(\xi,\lambda)\,\chi^T(\gamma,\lambda), & 0\le\xi\le\gamma\le a,\ \gamma\neq d,\ \xi\neq d.\end{cases} \tag{22}$$

Proof. It follows from (21) that

$$u_1(\xi,\lambda)=$$

$$\begin{cases}\begin{aligned}&\tfrac{1}{\omega(\lambda)}\chi_{11}(\xi,\lambda)\int_0^\xi\begin{pmatrix}\psi_{11}(\gamma,\lambda)\,\Xi_1(\gamma)\\+\psi_{21}(\gamma,\lambda)\,\Xi_2(\gamma)\end{pmatrix}d^\alpha\gamma\\&+\tfrac{1}{\omega(\lambda)}\psi_{11}(\xi,\lambda)\int_\xi^d\begin{pmatrix}\chi_{11}(\gamma,\lambda)\,\Xi_1(\gamma)\\+\chi_{21}(\gamma,\lambda)\,\Xi_2(\gamma)\end{pmatrix}d^\alpha\gamma\\&+\tfrac{1}{\omega(\lambda)}\psi_{11}(\xi,\lambda)\,\rho\int_d^a\begin{pmatrix}\chi_{12}(\gamma,\lambda)\,\Xi_1(\gamma)\\+\chi_{22}(\gamma,\lambda)\,\Xi_2(\gamma)\end{pmatrix}d^\alpha\gamma,\ \xi\in I_1,\\[2ex]&\tfrac{1}{\omega(\lambda)}\chi_{12}(\xi,\lambda)\int_0^d\begin{pmatrix}\psi_{11}(\gamma,\lambda)\,\Xi_1(\gamma)\\+\psi_{21}(\gamma,\lambda)\,\Xi_2(\gamma)\end{pmatrix}d^\alpha\gamma\\&+\tfrac{1}{\omega(\lambda)}\chi_{12}(\xi,\lambda)\,\rho\int_d^\xi\begin{pmatrix}\psi_{12}(\gamma,\lambda)\,\Xi_1(\gamma)\\+\psi_{22}(\gamma,\lambda)\,\Xi_2(\gamma)\end{pmatrix}d^\alpha\gamma\\&+\tfrac{1}{\omega(\lambda)}\psi_{12}(\xi,\lambda)\,\rho\int_\xi^a\begin{pmatrix}\chi_{12}(\gamma,\lambda)\,\Xi_1(\gamma)\\+\chi_{22}(\gamma,\lambda)\,\Xi_2(\gamma)\end{pmatrix}d^\alpha\gamma,\ \xi\in I_2,\end{aligned}\end{cases} \tag{23}$$

$$u_2(\xi,\lambda)=$$

$$
\begin{cases}
\frac{1}{\sigma(\lambda)}\chi_{21}(\xi,\lambda)\int_0^\xi \begin{pmatrix} \psi_{11}(\gamma,\lambda)\Xi_1(\gamma) \\ +\psi_{21}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\sigma(\lambda)}\psi_{21}(\xi,\lambda)\int_\xi^d \begin{pmatrix} \chi_{11}(\gamma,\lambda)\Xi_1(\gamma) \\ +\chi_{21}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\sigma(\lambda)}\psi_{21}(\xi,\lambda)\rho\int_d^a \begin{pmatrix} \chi_{12}(\gamma,\lambda)\Xi_1(\gamma) \\ +\chi_{22}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma,\ \xi\in I_1, \\
\\
\frac{1}{\sigma(\lambda)}\chi_{22}(\xi,\lambda)\int_0^d \begin{pmatrix} \psi_{11}(\gamma,\lambda)\Xi_1(\gamma) \\ +\psi_{21}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\sigma(\lambda)}\chi_{22}(\xi,\lambda)\rho\int_d^\xi \begin{pmatrix} \psi_{12}(\gamma,\lambda)\Xi_1(\gamma) \\ +\psi_{22}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\sigma(\lambda)}\psi_{22}(\xi,\lambda)\rho\int_\xi^a \begin{pmatrix} \chi_{12}(\gamma,\lambda)\Xi_1(\gamma) \\ +\chi_{22}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma,\ \xi\in I_2,
\end{cases}
\tag{24}
$$

From (23), we find

$$
T_\alpha u_1(\xi,\lambda)
$$

$$
= \begin{cases}
\frac{1}{\omega(\lambda)}T_\alpha\chi_{11}(\xi,\lambda)\int_0^\xi \begin{pmatrix} \psi_{11}(\gamma,\lambda)\Xi_1(\gamma) \\ +\psi_{21}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\omega(\lambda)}T_\alpha\psi_{11}(\xi,\lambda)\int_\xi^d \begin{pmatrix} \chi_{11}(\gamma,\lambda)\Xi_1(\gamma) \\ +\chi_{21}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\omega(\lambda)}T_\alpha\psi_{11}(\xi,\lambda)\rho\int_d^a \begin{pmatrix} \chi_{12}(\gamma,\lambda)\Xi_1(\gamma) \\ +\chi_{22}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+W_{\alpha,1}(\chi,\psi)(\xi)\Xi_2,\ \xi\in I_1, \\
\\
\frac{1}{\omega(\lambda)}T_\alpha\chi_{12}(\xi,\lambda)\int_0^d \begin{pmatrix} \psi_{11}(\gamma,\lambda)\Xi_1(\gamma) \\ +\psi_{21}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\omega(\lambda)}T_\alpha\chi_{12}(\xi,\lambda)\rho\int_d^\xi \begin{pmatrix} \psi_{12}(\gamma,\lambda)\Xi_1(\gamma) \\ +\psi_{22}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\omega(\lambda)}T_\alpha\psi_{12}(\xi,\lambda)\rho\int_\xi^a \begin{pmatrix} \chi_{12}(\gamma,\lambda)\Xi_1(\gamma) \\ +\chi_{22}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\rho W_{\alpha,2}(\chi,\psi)(\xi)\Xi_2,\ \xi\in I_2,
\end{cases}
$$

$$
= \begin{cases}
\frac{1}{\omega(\lambda)}\{\lambda-r(\xi)\}\chi_{21}(\xi,\lambda)\int_0^\xi \begin{pmatrix} \psi_{11}(\gamma,\lambda)\Xi_1(\gamma) \\ +\psi_{21}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\omega(\lambda)}\{\lambda-r(\xi)\}\psi_{21}(\xi,\lambda)\int_\xi^d \begin{pmatrix} \chi_{11}(\gamma,\lambda)\Xi_1(\gamma) \\ +\chi_{21}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\omega(\lambda)}\{\lambda-r(\xi)\}\psi_{21}(\xi,\lambda)\rho\int_d^a \begin{pmatrix} \chi_{12}(\gamma,\lambda)\Xi_1(\gamma) \\ +\chi_{22}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\Xi_2,\ \xi\in I_1, \\
\\
\frac{1}{\omega(\lambda)}\{\lambda-r(\xi)\}\chi_{22}(\xi,\lambda)\int_0^d \begin{pmatrix} \psi_{11}(\gamma,\lambda)\Xi_1(\gamma) \\ +\psi_{21}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\omega(\lambda)}\{\lambda-r(\xi)\}\chi_{22}(\xi,\lambda)\rho\int_d^\xi \begin{pmatrix} \psi_{12}(\gamma,\lambda)\Xi_1(\gamma) \\ +\psi_{22}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\frac{1}{\omega(\lambda)}\{\lambda-r(\xi)\}\psi_{22}(\xi,\lambda)\rho\int_\xi^a \begin{pmatrix} \chi_{12}(\gamma,\lambda)\Xi_1(\gamma) \\ +\chi_{22}(\gamma,\lambda)\Xi_2(\gamma) \end{pmatrix} d^\alpha\gamma \\
+\Xi_2,\ \xi\in I_2,
\end{cases}
$$

$$
= \{\lambda-r(\xi)\}u_2(\xi)+\Xi_2(\xi).
$$

The validity of (15) is proved similarly. Therefore, $u(\xi,\lambda)$ is the solution of (15)-(20). It is clear that (21) satisfies (17)-(20). □

The Green's function defined by (22) has the following properties:

Lemma 2. ***i)*** *The function defined by (22) is unique.*
ii) $G(\xi,\gamma,\lambda)$ *is continuous at the point* $(0,0)$.
iii)

$$
G(\xi,\gamma,\lambda)=G^T(\gamma,\xi,\lambda).
$$

Proof. The proof is similar to ([5], Theorem 18). □

Example 1. *Let $\alpha \to 1$. Then the CBVP (2)-(6) becomes*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{du(\zeta)}{d\zeta} + \begin{pmatrix} p(\zeta) & 0 \\ 0 & r(\zeta) \end{pmatrix} u(\zeta) = \lambda u(\zeta), \ \zeta \in I, \tag{25}$$

$$u_1(0) + k_1 u_2(0) = 0, \tag{26}$$

$$u_1(d-) - k_2 u_1(d+) = 0, \tag{27}$$

$$u_2(d-) - k_3 u_2(d+) = 0, \tag{28}$$

$$u_1(a) + k_4 u_2(a) = 0, \ k_1, k_2, k_3, k_4 \in \mathbb{R}. \tag{29}$$

With regard to the CBVP (25)-(29), Theorems 1 and 5 provide the following insights: Theorem 1 establishes the existence of the solution, while Theorem 5 presents the Green's function. For similar results, the following sources can be consulted (see, [6, 7, 20]).

6. Conclusion

In this paper, we have explored the conformable Dirac equation under impulsive boundary conditions. We began by examining the uniqueness and existence of solutions relevant to the problem, followed by the derivation of the symmetric operator. Subsequently, we constructed the Green's function associated with the problem and its fundamental characteristics are provided. The Titchmarsh–Weyl theory for this type of problem may be the subject of future research.

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