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Research Article

Approximate controllability result for backward stochastic evolution inclusions in Hilbert spaces

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Abstract

In this paper, we study semilinear backward stochastic evolution inclusion systems in Hilbert spaces. First, we prove the existence of mild solution of the semilinear backward stochastic evolution inclusion systems using a multivalued fixed point theorem. Then, we obtain the approximate controllability result for semilinear backward stochastic evolution inclusion systems through the linear systems corresponding to these semilinear backward stochastic evolution inclusion systems under appropriate conditions. In particular, our study extends the results of the concept of approximate controllability to backward stochastic evolution inclusion systems.

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Keywords. backward stochastic evolution inclusions, approximate controllability, mild solutions, multivalued maps, Hausdorff measure of noncompactness

1. Introduction

Controllability theory, which has an important place in engineering and science, has developed considerably. Controllability works based on differential systems have been studied recently [21, 22, 28]. In [11, 20, 26, 27], the researchers discussed approximate controllability results for some differential systems using multivalued analysis and fixed point approaches. In addition, controllability problems for different stochastic and deterministic systems have been studied by many researchers [5, 14, 18, 25, 29]. Especially, stochastic systems attract a lot of attention because they provide better performance. Stochastic differential systems appear in the analysis of different engineering fields (electrical, mechanical). Many studies have been conducted on stochastic differential systems and the approximate controllability of these systems under appropriate conditions has been investigated [1, 2, 10, 15, 19, 24].

On the other hand, backward stochastic differential equations (BSDE), an important type of stochastic equations, attract the attention of researchers. These equations have important applications in stochastic control and mathematical finance. BSDE studies started with the publication of Pardoux and Peng [23], and there are many articles on this subject for finite and infinite spaces [3, 8, 23]. Dauer et. al. studied the approximate controllability of a semilinear backward stochastic evolution equation (BSEE) in Hilbert

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spaces [4]. Mahmudov and McKibben obtained optimal control and some existence results for the BSEE [4,17]. Govindan also studied mild solutions for such equations [9]. Recently, Lü and Neerven obtained some well-posedness results in Banach spaces for BSEE [16]. Then, Essaky et. al. extended these results to the inclusions and obtained the existence result of a mild L^p-solution of the backward stochastic evolution inclusion (BSEI) [7]. Nevertheless, so far very few studies have been reported for backward stochastic differential inclusion [13].

In this paper, we extend the BSEE studied by Dauer et al [4] to the semilinear BSEI in the following form:

$$\begin{cases} dx(t) \in -\left[Hx(t) + Gu(t,x,y) + K(t,x(t),y(t))\right]dt - y(t)dw(t) \\ x(T) = \zeta \in L^2\left(\Omega, \vartheta_T, X\right) \end{cases} \tag{1.1}$$

where $t \in [0,T]$, X, Y are separete Hilbert spaces, $H : D(H) \subset X \to X$ generates a strongly continuous semigroup $\{\phi(t) : t \in [0,T]\}$ such that ϕ is a bounded linear operator, $G : Y \to X$ is a bounded linear operator, w is a Q-Wiener proces on complete probalitiy space (Ω, ϑ_T, P), the control function u in $L^2_\vartheta\left([0,T], Y\right)$ and K is a multivalued operator.

Approximate controllability results for BSEI in the form (1.1) have not been presented in the literature. This information motivated us to do this study. We prove that the semilinear BSEI system (1.1) has a mild solution with a different approach than in [4], and then we focused on the approximate controllability result under appropriate conditions in Hilbert spaces.

Additionally, the progression of the paper continues as follows: Firstly, some definitions and preliminary information that will be used throughout the article are given, and the problem is also explained. Then, we prove the existence of mild solutions for system (1.1). We model our proof with a multivalued fixed point theorem (Theorem 2.1). Further, one of the important arguments in the results is concept of μ-condensing related to the Hausdorff measure of noncompactness. Finally, under suitable conditions, we obtain the approximate controllability result of system (1.1) through the linear system corresponding to (1.1).

2. Preliminaries

Let (Ω, ϑ_T, P) be a complete probalitiy space, where filtration ϑ_T $(0 \le t \le T)$ is a right-continuous increasing family and ϑ_0 contains all $P-$null sets. Then we take the separete Hilbert spaces X, Y, Z and w is a Q-Wiener proces on (Ω, ϑ_T, P) such that Q is a bounded linear convariance operator with $trQ < \infty$. We suppose that $\{e_n\}$ is a complete orthonormal system in Z and there exist a bounded sequence λ_n of non-negative real numbers such that $Qe_n = \lambda_n e_n$, $n \ge 1$, also $\{\varphi_n\}$ is a sequence of independent Brownian motions with

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \langle e_n, e\rangle \varphi_n(t), \quad e \in Z, t \in [0,T].$$

Let $L_2^0 = L_2\left(Q^{1/2}Z;\ X\right)$ be the space of all Hilbert-Schmidt operators with $\langle \psi, \beta\rangle_{L_2^0} = tr\left[\psi Q \beta^*\right]$ and $L_2\left(\Omega, \vartheta_T, X\right)$ be Hilbert space of all ϑ_T measurable square-integrable random variables in X. Let $L^2_\vartheta\left(\Omega, C[0,T], X\right)$ the Banach space of all X-valued ϑ_t- measurable for $t \in [0,T]$, continuous functions $x(t) : [0,T] \times \Omega \to X$ and for fixed $w \in \Omega$,

$$\|x\| = \left\{ E\left(\sup_{t\in[0,T]} \|x(t,w)\|^2 \right) \right\}^{1/2} < \infty$$

is satisfied. Also, for any $\alpha \in \mathbb{R}$, $t \in [0,T]$, let $N_\alpha[t,T]$ be a Banach space;

$$N_\alpha[t,T] = L^2_\vartheta\left(\Omega, C[t,T], X\right) \times L^2_\vartheta\left([t,T], L_2^0\right).$$

X is a Hilbert space and

$$P(X) := \{M \subset X : M \neq \varnothing\},$$
$$P_b(X) := \{M \subset X : M \text{ is nonempty and bounded}\},$$
$$P_{cv}(X) := \{M \subset X : M \text{ is nonempty and convex}\},$$
$$P_{cl,cv}(X) := \{M \subset X : M \text{ is nonempty, closed and convex}\}.$$

Let $A : M \subseteq X \to P(M)$ be a multivalued operator and $A(M)$ is defined by $A(M) = \bigcup_{a \in M} A(a)$. Assume that for every subset $N \subset X$, we put

$$A^{-1}(N) = \{a \in M : A(a) \cap N \neq \varnothing\}.$$

The graph $G(A)$ and the range $R(A)$ of A is defined by

$$G(A) = \{(a,b) \in M \times X : a \in M,\ b \in A(a)\}, \quad R(A) = \bigcup_{a \in M} A(a)$$

The multivalued operator $A : M \to P(M)$ is said to be upper semicontinuous (u.s.c.) if $A^{-1}(N)$ is closed in M, for any closed subset N of M. A is called compact if the set $A(N)$ is relatively compact in M, for any bounded subset N of M. Also, a multivalued mapping A has a closed graph (i.e., $a_n \to a$, $b_n \to b$, $b_n \in Aa_n \Longrightarrow b \in Aa$) if and only if A is u.s.c.

A Hausdorff measure of noncompactness μ is defined as follows

$$\mu(\Omega) = \inf\left\{r > 0 \mid \Omega \subset \bigcup_{i=1}^{n} B_r(x_i),\ \text{for some } x_i \in X\right\}$$

where $M \in P_b(X)$ and $B_r(x_i) = \{x \in X \mid d(x, x_i) < r\}$. Then the function μ satisfies the following conditions:

(1) $\mu(M) = 0$ if and only if Ω is relatively compact,
(2) $M_1 \subseteq M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$, for $M_1, M_2 \in P_b(X)$,
(3) $\mu(M \cup \{a\}) = \mu(M)$ for all $a \in X$, $M \in P_b(X)$,
(4) $\mu(\overline{co}(M)) = \mu(M)$; $\overline{co}$ is the closed convex hull of M,
(5) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$, for $M_1, M_2 \in P_b(X)$
(6) $\mu(\lambda M) = |\lambda|\mu(M)$, $\forall \lambda \in \mathbb{R}$.

More information about the measure of noncompactness can be found in Zeidler [30].

In addition, if $A : M \to P(X)$ is bounded and $\mu(A(M)) < \mu(M)$ is satisfied for all $M \in P_b(X)$ with $\mu(M) \neq 0$, the multivalued mapping A is called μ-condensing.

Theorem 2.1. [6] Let M be a bounded convex and closed subset of a Banach space X and let $A : M \to P_{cl,cv}(M)$ be μ-condensing and u.s.c. multivalued operator. Then A has a fixed point point.

Now let's consider the semilinear BSEI:

$$\begin{cases} dx(t) \in -\left[Hx(t) + Gu(t,x,y) + K(t,x(t),y(t))\right]dt - y(t)dw(t) \\ x(T) = \zeta \in L^2(\Omega, \vartheta_T, X) \end{cases} \tag{2.1}$$

where $H : D(H) \subset X \to X$ generates a compact and strongly continuous semigroup $\{\phi(t) : t \in [0,T]\}$ of bounded linear operator ϕ, i.e. for all $t \in [0,T]$ operator $\phi(t)$ is compact, $G : Y \to X$ is a bounded linear operator, $K : [0,T] \times X \times L_2^0 \to P(X)$ is a multivalued operator, w is a Q-Wiener process, the control u in $L_\vartheta^2([0,T], Y)$ and controlability operator $\Upsilon_0^t \in L(X)$ by

$$\Upsilon_0^t = \int_0^t \phi(s)GG^*\phi^*(s)ds, \quad 0 < t \leq T,$$

where G^* and $\phi^*(t)$ denote the adjoint of G and $\phi(t)$ respectively, also $\left\| a\left(aI + \Upsilon_0^t\right)^{-1} \right\| \leq 1$.

Definition 2.2. A pair of stochastic process $(x,y) \in N_\alpha[t,T]$ is called to be a mild solution of (2.1) if for all $t \in [0,T]$ there exists a function $k(t) \in K(t,x(t),y(t))$ such that

$$x(t) = \phi(T-t)\zeta + \int_t^T \phi(s-t)Gu(s)ds + \int_t^T \phi(s-t)k(s)ds + \int_t^T \phi(s-t)y(s)dw(s) \quad (2.2)$$

Definition 2.3. System (2.1) is called to be approximately controllable on $[0,T]$ if $\overline{F_0(\zeta)} = X,\ F_t(T) = \left\{x(t,\zeta,u) : u \in L^2_\vartheta([0,T],Y)\right\}.$

Now let's consider the linear system given as follows:

$$\begin{cases} x'(t) \in -[Hx(t) + Gu(t,x,y)]\,dt \\ x(0) = x_0 \in X. \end{cases} \quad (2.3)$$

Lemma 2.4. [18] The linear sysyem (2.3) is approximately controllable on $[0,T]$ if and only if $a\left(aI + \Upsilon_0^t\right) \to 0$ as $a \to 0^+$ in the strong operator topology.

3. Existence results

We will investigate the existence of the mild solution for the system (2.1). In particular, Theorem 2.1 plays a key role in achieving this goal. Now let's assume the following assumptions hold throughout this article:

(A$_1$) The multivalued mapping $K : [0,T] \times X \times L_2^0 \to P_{b,cv,cl}(X)$ is u.s.c., measurable and for each fixed $x \in X$, the set

$$K_x = \{k : k(t) \in K(t,x(t),y(t)),\ for\ t \in [0,T]\}$$

is nonempty. Also let $M : L_2^0([0,T],X) \to C([0,T],X)$ be a continuous linear mapping, then operator

$$M(K_x) : C([0,T],X) \to P_{b,cv,cl}(C([0,T],X))$$

is a closed graph operator in $C \times C$, where $M(K_x) = M \circ K_x$.

(A$_2$) For each $q > 0$, there exists a function $\widetilde{P_k}(q)$ which is positive and independent on q with

$$\sup_{E\|x\|^2 \le q}\ \sup_{k \in K(t,x,y)} E\|k\|^2 \le \widetilde{P_k}(q), \quad for\ t \in [0,T].$$

Also the operator $G : Y \to X$ and the function y are bounded such that

$$\widetilde{P}_G = \sup_{0\le t\le T}\|G(t)\|, \|G(t)\| \le \widetilde{P}_G \quad \text{and} \quad \widetilde{P}_y = \sup_{0\le t\le T}\|y(t)\|, \|y\| \le \widetilde{P}_y.$$

(A$_3$) $\phi : [0,T] \to [0,T]$ a linear bounded operator holds the following assumptions:

(i) For $t_1, t_2 \in [0,T]$, there exits a $\beta = \zeta L$ with $0 < \beta < 1$ such that

$$\|\phi(t_1) - \phi(t_2)\| \le \beta\|t_1 - t_2\|$$

is satisfied.

(ii) Let $\widetilde{P}_\phi = \sup_{0\le t\le T}\|\phi(t)\|$ and so $\|\phi(t)\| \le \widetilde{P}_\phi$.

Now we define the control function, as in study [4]:

$$\begin{aligned} u(s) &= -G^*\phi^*(s)\left(aI + \Upsilon_0^T\right)^{-1}(\phi(T)E\zeta - \rho) \\ &\quad -G^*\phi^*(s)\int_s^T \left(aI + \Upsilon_0^r\right)^{-1}\phi(r)Ek(r)dr. \end{aligned} \quad (3.1)$$

Theorem 3.1. *Assume that (A$_1$)-(A$_3$) are satisfied. Then, BSEI system (2.1) has a mild solution on* $[0,T]$.

Proof. Let set $M = \{x \in X : \|x\| \leq m\}$ and define the multivalued operator $\Psi : M \to P_{cl,cv}(M)$ by

$$\Psi(x) = \Bigg\{ z \in M : z(t) = \phi(T-t)\zeta + \int_t^T \phi(s-t)Gu(s)ds + \int_t^T \phi(s-t)k(s)ds + \int_t^T \phi(s-t)y(s)dw(s) \Bigg\}$$

where $k(t) \in K(t, x(t), y(t))$.

We must show that the operator Ψ satisfies the all requirements of Theorem 2.1.

Step 1 : $\Psi(x)$ is convex for each $x \in M$. Let $z_1, z_2 \in \Psi(x)$ and so $k_1, k_2 \in K_x$ such that for each $t \in [0,T]$, $\lambda \in [0,1]$, taking into account the (3.1), we can write

$$z_i(t) = \phi(T-t)\zeta + \int_t^T \phi(s-t)k_i(s)ds + \int_t^T \phi(s-t)y(s)dw(s) + \int_t^T \phi(s-t)G \times \Bigg\{ -G^*\phi^*(s)\left(aI + \Upsilon_0^T\right)^{-1}(\phi(T)E\zeta - \rho) - G^*\phi^*(s)\int_s^T (aI + \Upsilon_0^r)^{-1}\phi(r)Ek_i(r)dr \Bigg\} ds,\ i = 1,2.$$

$$\lambda z_1(t) + (1-\lambda)z_2(t)) = \phi(T-t)\zeta + \int_t^T \phi(s-t)\left[\lambda k_1(s) + (1-\lambda)k_2(s))\right]ds + \int_t^T \phi(s-t)y(s)dw(s) + \int_t^T \phi(s-t)G \times \Bigg\{ -G^*\phi^*(s)\left(aI + \Upsilon_0^T\right)^{-1}(\phi(T)E\zeta - \rho) - G^*\phi^*(s)\int_s^T (aI + \Upsilon_0^r)^{-1}\phi(r)E\left[\lambda k_1(r) + (1-\lambda)k_2(r))\right]dr \Bigg\} ds.$$

Since $k_i \in K_x$ and K_x is convex, $\lambda k_1(s) + (1-\lambda)k_2(s) \in K$. Thus, $\lambda z_1 + (1-\lambda)z_2 \in \Psi(x)$.

Step 2 : $\Psi(x)$ is closed for each $x \in M$. We see that Ψ has a closed graph. For this, let $x_n \to x_*$, $n \to \infty$, $z_n \in \Psi(x_n)$ and $z_n \to z_*$, $n \to \infty$. In this case there exists $k_n \in K_{x_n}$. We will prove that $z_* \in \Psi(x_*)$. We have,

$$z_n(t) = \phi(T-t)\zeta + \int_t^T \phi(s-t)k_n(s)ds + \int_t^T \phi(s-t)y(s)dw(s) + \int_t^T \phi(s-t)G \times \Bigg\{ -G^*\phi^*(s)\left(aI + \Upsilon_0^T\right)^{-1}(\phi(T)E\zeta - \rho) - G^*\phi^*(s)\int_s^T (aI + \Upsilon_0^r)^{-1}\phi(r)Ek_n(r)dr \Bigg\} ds. \tag{3.2}$$

Then, we must show that $k_* \in K_{x_*}$ with

$$z_*(t) = \phi(T-t)\zeta + \int_t^T \phi(s-t)k_*(s)ds + \int_t^T \phi(s-t)y(s)dw(s) + \int_t^T \phi(s-t)G \times \Bigg\{ -G^*\phi^*(s)\left(aI + \Upsilon_0^T\right)^{-1}(\phi(T)E\zeta - \rho) - G^*\phi^*(s)\int_s^T (aI + \Upsilon_0^r)^{-1}\phi(r)Ek_*(r)dr \Bigg\} ds.$$

By taking $n \to \infty$ in (3.2), we get

$$\Bigg\| z_n(t) - \phi(T-t)\zeta - \int_t^T \phi(s-t)k_n(s)ds - \int_t^T \phi(s-t)y(s)dw(s) - \int_t^T \phi(s-t)G\times$$

$$\left\{G^*\phi^*(s)\left(aI+\Upsilon_0^T\right)^{-1}(\phi(T)E\zeta-\rho)+G^*\phi^*(s)\int_s^T(aI+\Upsilon_0^r)^{-1}\phi(r)Ek_n(r)dr\right\}ds\Big\|^2$$
$\to 0.$

Give the linear and continouos operator $M : L^2([0,T],X) \to C([0,T],X)$ by

$$(M_k)(t)=\int_t^T\phi(s-t)k(s)ds+\int_t^T\phi(s-t)G\times\left\{-G^*\phi^*(s)\left(aI+\Upsilon_0^T\right)^{-1}(\phi(T)E\zeta-\rho)\right.$$
$$\left.-G^*\phi^*(s)\int_s^T(aI+\Upsilon_0^r)^{-1}\phi(r)Ek_i(r)dr\right\}ds.$$

From (A_1), it obtain that $M\circ K_x$ is a closed graph operator. Further,

$$\left(z_n(t)-\phi(T-t)\zeta+\int_t^T\phi(s-t)y(s)dw(s)-\int_t^T\phi(s-t)G\right.$$
$$\left.\times\left\{G^*\phi^*(s)\left(aI+\Upsilon_0^T\right)^{-1}(\phi(T)E\zeta-\rho)\right\}ds\right)\in M\left(K_{x_n}\right).$$

By (A_1) and $x_n\to x_*$, we reach

$$\left(z_*(t)-\phi(T-t)\zeta+\int_t^T\phi(s-t)y(s)dw(s)-\int_t^T\phi(s-t)G\right.$$
$$\left.\times\left\{G^*\phi^*(s)\left(aI+\Upsilon_0^T\right)^{-1}(\phi(T)E\zeta-\rho)\right\}ds\right)\in M\left(K_{x_*}\right).$$

Thus, $z_*\in\Psi(x_*)$. Consequently, Ψ has a closed graph and is therefore closed-valued. We also conclude that Ψ is u.s.c.

Step 3 : Ψ is a μ-condensing. Let's write the Ψ operator as the sum of Ψ_1 and Ψ_2 operators we define below:

$$\Psi=\Psi_1+\Psi_2,$$

$$\Psi_1=\phi(T-t)\zeta,$$
$$\Psi_2=\left\{z\in M : z(t)=\int_t^T\phi(s-t)Gu(s)ds+\int_t^T\phi(s-t)k(s)ds+\int_t^T\phi(s-t)y(s)dw(s)\right\}$$

For the proof we need to show that Ψ_1 is a contraction and Ψ_2 is compact. (see [12]). For each $t_i\in[0,T]$, $T-t_i=c_i$, $i=1,2$, considering assumption (A_3)-i, we have

$$\begin{aligned}\|\Psi_1x(t_1)-\Psi_2x(t_1)\| &= \zeta\|\phi(c_1)-\phi(c_2)\|\\ &\le \zeta L\|c_1-k_2\|\\ &= \beta\|t_1-t_2\|\end{aligned}$$

thus, Ψ_1 is a contraction. Then, let us show the compactness of Ψ_2, for each $t\in[0,T]$ and $z_2\in\Psi_2$ there exists $k\in K_x$ with

$$z_2(t)=\int_t^T\phi(s-t)Gu(s)ds+\int_t^T\phi(s-t)k(s)ds+\int_t^T\phi(s-t)y(s)dw(s)$$

if $t=T$, this case is trivial. Let $t\in[0,T)$ be fixed and $\eta\in(t,T)$, then we define

$$z_{2\eta}(t)=\phi(\eta)\int_{t-\eta}^T\phi(s-(t-\eta))Gu(s)ds+\phi(\eta)\int_{t-\eta}^T\phi(s-(t-\eta)k(s)ds$$
$$+\phi(\eta)\int_{t-\eta}^T\phi(s-(t-\eta)y(s)dw(s).$$

Since for each $t\in[0,T]$ $\phi(t)$ is compact , it follows that $\Psi_{2\eta}$ is compact in X.

For each $t\in[0,T]$, we have

$$E\|u(t)\|^2\le\frac{4\widetilde{P}_\phi^2\widetilde{P}_G^2}{a^2}\left(\widetilde{P}_\phi^2E\|\zeta\|^2+E\|h\|^2\right)+\frac{4E\widetilde{P}_\phi^4\widetilde{P}_G^2}{a^2}(T-t)\int_t^T\|Ek(s)\|^2ds$$
$$=P_u.$$

Thus, we have

$$\begin{aligned}E\left\Vert z_{2}(t)\right\Vert ^{2} &\leq 4E\left\Vert \int_{t}^{T}\phi(s-t)Gu(s)ds\right\Vert ^{2}+4E\left\Vert \int_{t}^{T}\phi(s-t)k(s)ds\right\Vert ^{2}\\ &\quad +4E\left\Vert \int_{t}^{T}\phi(s-t)y(s)dw(s)\right\Vert ^{2}\\ &\leq 4E\left(T-t\right)\int_{t}^{T}\left\Vert \phi(s-t)Gu(s)\right\Vert ^{2}ds+4E\left(T-t\right)\int_{t}^{T}\left\Vert \phi(s-t)k(s)\right\Vert ^{2}ds\\ &\quad +4Etr(Q)\int_{t}^{T}\left\Vert \phi(s-t)y(s)\right\Vert ^{2}ds\\ &\leq 4\left(T-t\right)\widetilde{P}_{\phi}^{2}\widetilde{P}_{G}^{2}P_{u}+4\left(T-t\right)\widetilde{P}_{\phi}^{2}\widetilde{P}_{k}+4tr(Q)\widetilde{P}_{\phi}^{2}\widetilde{P}_{y}^{2}.\end{aligned}$$

Then we can write

$$\begin{aligned}E\left\Vert z_{2}(t)-z_{2\eta}(t)\right\Vert ^{2} &\leq 3E\left\Vert \int_{t-\eta}^{T}\phi(s-t)Gu(s)ds\right\Vert ^{2}+3E\left\Vert \int_{t-\eta}^{T}\phi(s-t)k(s)ds\right\Vert ^{2}\\ &\quad +3E\left\Vert \int_{t-\eta}^{T}\phi(s-t)y(s)dw\left(s\right)\right\Vert ^{2}\\ &\leq 3\eta\widetilde{P}_{\phi}^{2}\widetilde{P}_{G}^{2}P_{u}+3\eta\widetilde{P}_{\phi}^{2}\widetilde{P}_{k}+3tr(Q)\widetilde{P}_{\phi}^{2}\widetilde{P}_{y}^{2}.\end{aligned}$$

When $\eta\to 0$ is taken, it can be seen that compact sets arbitrarily close to $\Psi_2(x(t))$, for each $t\in[0,T)$. Thus the set $\Psi_2(x(t))$ is compact, for each $t\in[0,T)$. Consequently we reach that Ψ is a μ-condensing.

As a reult, Ψ fulfill all requirements of Theorem 2.1. Thus $x\in\Psi x$ has a solution and so BSEI system (2.1) has a mild solution on $[0,T]$. □

4. Approximately Controllability

Theorem 4.1. *Suppose that F is uniformly bounded on respective domains and let the linear system (2.3) be approximately controllable. Then the BSEI system (2.1) is approximately controllable under assumptions (A_1)-(A_3).*

Proof. Let $x(t)$ be a fixed point on $\Psi x(t)$, also each fixed point of Ψ is a mild solution for system (2.1), for each $t\in[0,T]$. Then we have,

$$\begin{aligned}x\left(0\right) &= \phi(T)\zeta+\int_{0}^{T}\phi(s)k(s)ds+\int_{0}^{T}\phi(s)y(s)dw(s)\\ &\quad -\int_{0}^{T}\phi(s)GG^{*}\phi^{*}(s)\left(aI+\Upsilon_{0}^{T}\right)^{-1}\left(\phi(T)E\zeta-\rho\right)\\ &\quad -E\int_{0}^{T}\phi(s)GG^{*}\phi^{*}(s)\int_{s}^{T}\left(aI+\Upsilon_{0}^{r}\right)^{-1}\phi(r)k(r)drds\end{aligned}$$

$$= \phi(T)\zeta + \int_0^T \phi(s)k(s)ds + \int_0^T \phi(s)y(s)dw(s)$$

$$-\Upsilon_0^T \left(aI + \Upsilon_0^T\right)^{-1}(\phi(T)E\zeta - \rho) - E\int_0^T \Upsilon_0^r (aI + \Upsilon_0^r)^{-1}\phi(r)k(r)dr$$

$$= \phi(T)\zeta - a\left(aI + \Upsilon_0^T\right)^{-1}(\phi(T)E\zeta - \rho) + \int_0^T \phi(s)k(s)ds$$

$$+\int_0^T \phi(s)y(s)dw(s) - aE\int_0^T (aI + \Upsilon_0^s)^{-1}\phi(s)k(s)ds$$

$$= \widetilde{x} - a\left(aI + \Upsilon_0^T\right)^{-1}(\phi(T)E\zeta - \rho) - aE\int_0^T (aI + \Upsilon_0^s)^{-1}\phi(s)k(s)ds,$$

where $\widetilde{x} = \phi(T)\zeta + \int_0^T \phi(s)k(s)ds + \int_0^T \phi(s)y(s)dw(s)$, then we have

$$\begin{aligned} E\left\|x(0) - \widetilde{x}\right\|^2 &\le 4\left\|a(aI + \Upsilon_0^T)^{-1}\right\|^2\left(\widetilde{P}_\phi^2 E\left\|\zeta\right\|^2 + \left\|\rho\right\|^2\right) \\ &\quad +4\widetilde{P}_\phi^2 E\left(\int_0^T \left\|a(aI + \Upsilon_0^s)^{-1}\right\|\left\|k(s)\right\| ds\right)^2 \\ &\le 4\left\|a(aI + \Upsilon_0^T)^{-1}\right\|^2\left(\widetilde{P}_\phi^2 E\left\|\zeta\right\|^2 + \left\|\rho\right\|^2\right) \\ &\quad +4\widetilde{P}_\phi^2 E\widetilde{P}_k\left(\int_0^T \left\|a(aI + \Upsilon_0^r)^{-1}\right\|^2 ds\right). \end{aligned}$$

Letting $a \to 0^+$, we get $E\left\|x(0) - \widetilde{x}\right\|^2 \to 0$ and according to Lemma 2.4, this expresses the approximately controllablility of the system (2.1). □

5. Example

Consider the differantial inclusion

$$\begin{cases} dx_t(t,\lambda) \in -\left[x_{\lambda\lambda}(t,\lambda) + g(\lambda)u(t) + \widetilde{K}(t,(t,\lambda))\right]dt - y\left(t,\lambda\right)dw(t), \\ x(t,0) = x(t,\pi) = 0, \\ x(T) = p, \end{cases} \tag{5.1}$$

where $X = L_2\left[0,\pi\right]$, $p, g \in X$, $u \in L_2\left[0,T\right]$, $\widetilde{K} : \mathbb{R}\times\mathbb{R} \to P(\mathbb{R})$ a multivalued operator. Let $G : \mathbb{R} \to X$ a linear operator be defined as follow:

$$(Gu)(\lambda) = g(\lambda)u, \quad 0 \le \lambda \le \pi, \quad u \in \mathbb{R}, \ g(\lambda) \in X.$$

Let operator $H : X \to X$ defined by $Hy = y''$ and domain set

$$D(H) = \{y \in X \mid y, y' \text{ are absolutely continuous}, \ y'' \in X, \ y(0) = y(\pi) = 0\}$$

Then,

$$Hy = \sum_{n=1}^{\infty}\left(-n^2\right)(z, e_n)\, e_n, y \in D(H).$$

H generates a compact semigroup $\phi(t)$ given by

$$\phi(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}(y, \sqrt{2/\pi}\sin(n\pi))\sqrt{2/\pi}\sin(n\pi),\ n = 1, 2, ...$$

and define $K(t, x(.) = \widetilde{K}(t, x(.))$. Then, the linear system which corresponds to system (5.1) is approximately controllable. Hence, with the definitions of operators H, G and K as above, the system (5.1) can be written in the form (2.1) and thus by satisfying all the conditions of Theorem 4.1, the differential inclusion (5.1) is approximately controllable on $[0, T]$.

6. Conclusion

In this study, the controllability problem of BSEI systems in Hilbert spaces is discussed. We examine the approximate controllability of the semilinear BSEI by supposing that the semigroup generated in the linear part of the BSEI is strongly continuous and compact. To find the mild solutions of this inclusion system, we define an operator Ψ whose fixed points correspond to the solutions of the inclusion system. We then investigate the $\mu-$condensing and u.s.c. properties of the operator to obtain the existence of fixed points of Ψ. Finally, we achieve the approximate controllability of the corresponding linear system under appropriate conditions, which means that the semilinear BSEI system is approximately controllable. In addition, we provide an example for our main result. In the next study, we aim to study the controllability of fractional backward stochastic evolution inclusions, an analog of BSEI's.

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