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Conformal semi-invariant Riemannian maps to Sasakian manifolds

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Abstract. The idea of conformal semi-invariant Riemannian maps to almost Hermitian manifolds was first put forward by Şahin and Akyol in [3]. In this paper, we expand this idea to Sasakian manifolds which are almost contact metric manifolds. Hereby, we present conformal semi-invariant Riemannian maps from Riemannian manifolds to Sasakian manifolds. Then, we prepare a illustrative example and investigate the geometry of the leaves of $D_1, D_2, \bar{D}_1$ and $\bar{D}_2$. We find necessary and sufficient conditions for conformal semi-invariant Riemannian maps to be totally geodesic. Also, we investigate the harmonicity of such maps.

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1. Introduction

In [6], Fischer introduced Riemannian map between Riemannian manifolds as a generalization of an isometric immersion and Riemannian submersion that satisfies the well known generalized eikonal equation $\|\vartheta_*\|^2 = rank\vartheta$, which is a bridge between geometric optics and physical optics. Where ϑ is a Riemannian map and ϑ_* is its derivative map. Let $\vartheta : (S_1, g_{S_1}) \to (S_2, g_{S_2})$ be a smooth map between Riemannian manifolds such that $0 < rank\vartheta < \min\{\dim(S_1), \dim(S_2)\}$. We state the kernel space of ϑ_* by $V_q = \ker\vartheta_{*_q}$ at $q \in S_1$ and consider the orthogonal complementary space $H_q = (\ker\vartheta_{*_q})^\perp$ to $\ker\vartheta_{*_q}$ in T_qS_1. Then the tangent space T_qS_1 of S_1 at q has the decomposition $T_qS_1 = (\ker\vartheta_{*_q}) \oplus (\ker\vartheta_{*_q})^\perp = V_q \oplus H_q$. We state the range of ϑ_* by $range\vartheta_*$ at $q \in S_1$ and consider the orthogonal complementary space $(range\vartheta_{*_q})^\perp$ to $range\vartheta_{*_q}$ in the tangent space $T_{\vartheta(q)}S_2$ of S_2 at $\vartheta(q) \in S_2$. Since $rank\vartheta < \min\{\dim(S_1), \dim(S_2)\}$, we have $(\ker\vartheta_{*_q})^\perp \neq \{0\}$. Therefore the tangent space $T_{\vartheta(q)}S_2$ of S_2 at $\vartheta(q) \in S_2$ has the decomposition $T_{\vartheta(q)}S_2 = (range\vartheta_{*_q}) \oplus (range\vartheta_{*_q})^\perp$. Then ϑ is called Riemannian map at $q \in S_1$ if the horizontal restriction $\vartheta^h_{*_q} : (\ker\vartheta_{*_q})^\perp \to (range\vartheta_{*_q})$ is a linear isometry between the spaces $((\ker\vartheta_{*_q})^\perp, g_{S_1}|_{(\ker\vartheta_{*_q})^\perp})$ and $(range\vartheta_{*_q}, g_{S_2\vartheta(q)}|_{range\vartheta_{*_q}})$. In other words, ϑ satisfies

$$g_{S_2}(\vartheta_* A_1, \vartheta_* A_2) = g_{S_1}(A_1, A_2), \tag{1}$$

for all A_1, A_2 vector field tangent to $\Gamma(\ker\vartheta_{*_q})^\perp$.

Different features of Riemannian maps have been investigated extensively by many authors in [1,7,8,10,15–17,20,24,25,27–29]. Detailed development in the theory of Riemannian map can be found in [21].

Conformal Riemannian maps as a generalization of Riemannian maps and the harmonicity of such maps have been introduced in [22,23]. Conformal anti-invariant Riemannian maps have been studied in [2]. In this article, we expand this concept to almost contact metric manifolds as a generalization of semi-invariant Riemannian maps and totally real submanifolds.

The paper is organized as follows. Section 2 contains preliminaries. Section 3 includes conformal semi-invariant Riemannian maps from Riemannian manifolds to Sasakian manifolds and provides this notion by non-trivial example. Then, we get a decomposition theorem by using the existence of conformal

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semi-invariant Riemannian maps. Moreover, conformal semi-invariant Riemannian maps allow us to obtain new conditions for a map to be harmonic. We also investigate the total geodesicity of conformal semi-invariant maps. In Section 4, we give scope for future studies.

2. Preliminaries

Let S be an odd-dimensional smooth manifold. Then, S has an almost contact structure [21] if there exist a tensor field F of type$-(1,1)$, a vector field ξ, and 1-form η on S such that

$$F^2E_1 = -E_1 + \eta(E_1)\xi, F\xi = 0, \eta \circ F = 0, \eta(\xi) = 1. \tag{2}$$

If there exists a Riemannian metric g_S on an almost contact manifold S satisfying:

$$\begin{aligned} g_S(FE_1, FE_2) &= g_S(E_1, E_2) - \eta(E_1)\eta(E_2), \\ g_S(E_1, FE_2) &= -g_S(FE_1, E_2), \end{aligned} \tag{3}$$

$$\eta(E_1) = g_S(E_1, \xi), \tag{4}$$

where E_1, E_2 are any vector fields on S, then S is called an almost contact metric manifold with an almost contact structure (F, ξ, η, g_S) and is symbolized by (S, F, ξ, η, g_S).

A manifold S with the structure (F, ξ, η, g_S) is said to be Sasakian structure given by [4]

$$(\overset{S}{\triangledown}_{E_1} F)E_2 = g_S(E_1, E_2)\xi - \eta(E_2)E_1, \tag{5}$$

for any vector fields E_1, E_2 on S, where $\triangledown$ stands for the Riemannian connection of the metric g_S on S. For a Sasakian manifold, we get

$$\overset{S}{\triangledown}_{E_1} \xi = -FE_1, \tag{6}$$

for any vector field E_1 on S.

ϑ_* can be considered as a part of bundle $\hom(TS_1, \vartheta^{-1}TS_2) \to S_1$, where $\vartheta^{-1}TS_2$ is the pullback bundle. The bundle has a connection $\triangledown$ induced from the pullback connection $\overset{S_2}{\triangledown}{}^{\vartheta}$ and the Levi-Civita connection $\triangledown^{S_1}$. Then the second fundamental form $(\triangledown\vartheta_*)(A_1, A_2)$ of ϑ is given by [14]

$$(\triangledown\vartheta_*)(A_1, A_2) = \overset{S_2}{\triangledown}{}^{\vartheta}_{A_1} \vartheta_* A_2 - \vartheta_*(\triangledown^{S_1}_{A_1} A_2), \tag{7}$$

for all $A_1, A_2 \in \Gamma(TS_1)$, where $\overset{S_2}{\triangledown}{}^{\vartheta}_{A_1} \vartheta_* A_2 \circ \vartheta = \overset{S_2}{\triangledown}{}^{\vartheta}_{\vartheta_* A_1} \vartheta_* A_2$. It is known that $(\triangledown\vartheta_*)(A_1, A_2)$ is symmetric and $(\triangledown\vartheta_*)(A_1, A_2)$ has no component in $range\vartheta_*$, for all $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$ [21]. It means that, we get

$$(\triangledown\vartheta_*)(A_1, A_2) \in \Gamma(range\vartheta_*)^\perp.$$

The tension field of ϑ is defined to be the trace of the second fundamental form of ϑ, i.e. $\tau(\vartheta) = trace(\triangledown\vartheta_*) = \sum\limits_{i=1}^{m} (\triangledown\vartheta_*)(e_i, e_i)$, where $m = \dim(S_1)$ and $\{e_1, e_2, ..., e_m\}$ is the orthonormal frame on S_1. Moreover, a map $\vartheta : (S_1, g_{S_1}) \to (S_2, g_{S2})$ is harmonic if and only if the tension field of ϑ vanishes at each point $q \in S_1$.

For any section B_1 of $(range\vartheta_*)^\perp$ and vector field A_1 on S_1, we get $\triangledown^{\vartheta\perp}_{A_1} B_1$, which is the orthogonal projection of $\triangledown^{S_2}_{A_1} B_1$ on $(range\vartheta_*)^\perp$, where $\triangledown^{\vartheta\perp}$ is linear connection on $(range\vartheta_*)^\perp$ such that $\triangledown^{\vartheta\perp} g_{S_2} = 0$. For a Riemannian map ϑ we describe S_{B_1} as ([21], p. 188)

$$\triangledown^{S_2}_{\vartheta_* A_1} B_1 = -S_{B_1}\vartheta_* A_1 + \triangledown^{\vartheta\perp}_{A_1} B_1, \tag{8}$$

where $S_{B_1}\vartheta_* A_1$ is the tangential component of $\triangledown^{S_2}_{\vartheta_* A_1} B_1$ and $\triangledown^{S_2}$is Levi-Civita connection on S_2. Therefore, we have $\triangledown^{S_2}_{\vartheta_* A_1} B_1(q) \in T_{\vartheta(q)} S_2, S_{B_1}\vartheta_* A_1 \in \vartheta_{*q}(T_q S_1)$ and $\triangledown^{\vartheta\perp}_{A_1} B_1 \in (\vartheta_{*q}(T_q S_1))^\perp$ at $q \in S_1$. We know that $S_{B_1}\vartheta_* A_1$ is bilinear in B_1, and ϑA_1 at q depends only on B_{1q} and $\vartheta_{*q} A_{1q}$. From here, using (7) and (8) we have

$$g_{S_2}(S_{B_1}\vartheta_* A_1, \vartheta_* A_2) = g_{S_2}(B_1, (\triangledown\vartheta_*)(A_1, A_2)), \tag{9}$$

where S_{B_1} is self adjoint operator for $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$.

For all $B_1, B_2 \in \Gamma(range\vartheta_*)^\perp$ we define

$$\triangledown^{S_2}_{B_1} B_2 = R(\triangledown^{S_2}_{B_1} B_2) + \triangledown^{\vartheta\perp}_{B_1} B_2,$$

where $R(\nabla^{S_2}_{B_1}B_2)$) and $\nabla^{\vartheta\perp}_{B_1}B_2$ denote $range\vartheta_*$ and $(range\vartheta_*)^\perp$ part of $\nabla^{S_2}_{B_1}B_2$, respectively. Therefore $(range\vartheta_*)^\perp$ is totally geodesic if and only if

$$\nabla^{S_2}_{B_1}B_2 = \nabla^{\vartheta\perp}_{B_1}B_2. \tag{10}$$

3. Conformal Semi-Invariant Riemannian Maps to Sasakian Manifolds

Definition 1. *[23] Let $\vartheta : (S_1, g_{S_1}) \to (S_2, g_{S_2})$ be a conformal Riemannian map (CRM). Then, ϑ is a horizontally homothetic map if $H(grad\lambda) = 0$.*

Definition 2. *[22] Let $\vartheta : (S_1, g_{S_1}) \to (S_2, g_{S_2})$ be a smooth map between Riemannian manifolds. Then, ϑ is a CRM at $q \in S_1$ if $0 < rank\vartheta_{*q} \leq \min\{\dim(S_1), \dim(S_2)\}$ and ϑ_{*q} maps the horizontal space $H(q) = (\ker\vartheta_{*q})^\perp$ conformally into $range\vartheta_{*q}$, it means that there exists a number $\lambda^2(q) \neq 0$ such that*

$$g_{S_2}(\vartheta_{*q}A_1, \vartheta_{*q}A_2) = \lambda^2(q)g_{S_1}(A_1, A_2),$$

for $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$. Moreover, if ϑ is CRM at any $q \in S_1$, then ϑ is called CRM.

Lastly, the second fundamental form of ϑ is given by [22]

$$(\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} = A_1(\ln\lambda)\,\vartheta_*A_2 + A_2(\ln\lambda)\,\vartheta_*A_1 - g_{S_1}(A_1, A_2)\vartheta_*(grad\ln\lambda). \tag{11}$$

Therefore, if we state the $(range\vartheta_*)^\perp$ component of $(\nabla\vartheta_*)(A_1, A_2)$ by $(\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}$, then we can write

$$(\nabla\vartheta_*)(A_1, A_2) = (\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} + (\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}, \tag{12}$$

for $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$. Therefore we get

$$\begin{aligned}(\nabla\vartheta_*)(A_1, A_2) \;=\;& A_1(\ln\lambda)\,\vartheta_*A_2 + A_2(\ln\lambda)\,\vartheta_*A_1 \\ & - g_{S_1}(A_1, A_2)\vartheta_*(grad\ln\lambda) + (\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}.\end{aligned} \tag{13}$$

Definition 3. *Let ϑ be a CRM from a Riemannian manifold (S_1, g_{S_1}) to an almost contact metric manifold $(S_2, F, \xi, \eta, g_{S_2})$. Then ϑ is a conformal semi-invariant Riemannian map (CSIRM) at $q \in S_1$ if there is a subbundle $D_1 \subseteq (range\vartheta_*)$ such that*

$$range\vartheta_{*q} = D_1 \oplus D_2, F(D_1) = D_1, F(D_2) \subseteq (range\vartheta_{*q})^\perp,$$

where D_2 is orthogonal complementary to D_1 in $range\vartheta_*$. If ϑ is a CSIRM for any $q \in S_1$, then ϑ is called a CSIRM.

For $\vartheta_*A_1 \in \Gamma(range\vartheta_*)$, then we write

$$F\vartheta_*A_1 = \phi\vartheta_*A_1 + \omega\vartheta_*A_1, \tag{14}$$

where $\phi\vartheta_*A_1 \in \Gamma(D_1)$ and $\omega\vartheta_*A_1 \in \Gamma(FD_2)$. Also, for $\vartheta_*A_1 \in \Gamma(D_1)$ and $\vartheta_*A_2 \in \Gamma(D_2)$, we have $g_{S_2}(\vartheta_*A_1, \vartheta_*A_2) = 0$. Thus we have two orthogonal distributions $\bar{D}_1$ and $\bar{D}_2$ such that

$$(\ker\vartheta_{*_q})^\perp = \bar{D}_1 \oplus \bar{D}_2.$$

On the other hand, for $B_1 \in \Gamma((range\vartheta_*)^\perp)$, then we have

$$FB_1 = \beta_1B_1 + \alpha_1B_1, \tag{15}$$

where $\beta_1B_1 \in \Gamma(D_1)$ and $\alpha_1B_1 \in \Gamma(\eta)$. Here η is the complementary orthogonal distribution to $\omega(D_2)$ in $(range\vartheta_*)^\perp$. It is easy to see that η is invariant with respect to F.

Example 1. *Let S_1 be an Euclidean space given by*

$$S_1 = \left\{(u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5 : u_1 \neq 0, u_2 \neq 0, u_5 \neq 0\right\}.$$

We describe the Riemannian metric g_{S_1} on S_1 given by

$$g_{S_1} = du_1^2 + du_2^2 + du_3^2 + du_4^2 + du_5^2.$$

Let $S_2 = \left\{(v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5\right\}$ be a Euclidean space with metric g_{S_2} on S_2 given by

$$g_{S_2} = e^{2u_1}dv_1^2 + e^{2u_1}dv_2^2 + e^{2u_1}dv_3^2 + dv_4^2 + dv_5^2.$$

Usual Sasakian structure (F,ξ,η) *on* (S_2,g_{S_2}) *can be choosen as [5]*

$$\begin{aligned} F(\frac{\partial}{\partial v_1}) &= \frac{\partial}{\partial v_2}, F(\frac{\partial}{\partial v_2}) = -\frac{\partial}{\partial v_1}, F(\frac{\partial}{\partial v_3}) = \frac{\partial}{\partial v_4}, F(\frac{\partial}{\partial v_4}) = -\frac{\partial}{\partial v_3}, \\ \eta &= dv_5, \xi = \frac{\partial}{\partial v_5}, F(\xi) = 0. \end{aligned}$$

Then a basis of T_qS_1 *is*

$$\left\{ e_i = e^{u_1}\frac{\partial}{\partial u_i} \text{ for } 1 \le i \le 5 \right\},$$

and a F*-basis on* $T_{\vartheta(q)}S_2$ *is*

$$\left\{ e_j^* = \frac{\partial}{\partial v_j} \text{ for } 1 \le j \le 4, e_4^* = e^{u_1}\frac{\partial}{\partial v_4}, \xi = e_5^* = \frac{\partial}{\partial v_5} \right\},$$

for all $q \in S_1$*. Now, we define a map* $\vartheta : (S_1,g_{S_1}) \to (S_2,g_{S_2},F)$ *by*

$$\vartheta(u_1,u_2,u_3,u_4,u_5) = (u_1,u_2,u_5,0,0).$$

Then, we have

$$\begin{aligned} ker\vartheta_* &= Span\{U_1 = e_3, U_2 = e_4\}, \\ (ker\vartheta_*)^\perp &= Span\{A_1 = e_1, A_2 = e_2, A_3 = e_5\}. \end{aligned}$$

Hence it is easy to see that $\vartheta_*A_1 = e^{u_1}e_1^*, \vartheta_*A_2 = e^{u_1}e_2^*, \vartheta_*A_3 = e^{u_1}e_3^*$ *and* $g_{S_2}(\vartheta_*(A_{1i}),\vartheta_*(A_{1j})) = e^{2u_1}g_{S_1}(A_{1i},A_{1j})$ *for* $i,j = 1,2,3$*. Thus* ϑ *is a CRM with* $\lambda = e^{2u_1}$ *and we get*

$$\begin{aligned} range\vartheta_* &= Span\{e^{u_1}e_1^*, e^{u_1}e_2^*, e^{u_1}e_3^*\}, \\ (range\vartheta_*)^\perp &= Span\{e_4^*, \xi\}, \\ D_1 = Span\{e^{u_1}e_1^*, &e^{u_1}e_2^*\}, D_2 = Span\{e^{u_1}e_3^*\}. \end{aligned}$$

Moreover it is easy to see that $F\vartheta_*A_1 = e^{u_1}e_2^*, F\vartheta_*A_2 = -e^{u_1}e_1^*, F\vartheta_*A_3 = e^{u_1}e_4^*$*. Thus* ϑ *is a CSIRM.*

Remark 1. *Throughout this article* $\xi \in (range\vartheta_*)^\perp$ *will be taken as the Reeb vector field.*

We obtain the following theorem for the geometry of the leaves of D_1.

Theorem 1. *Let* ϑ *be a CSIRM from a Riemannian manifold* (S_1,g_{S_1}) *to a Sasakian manifold* (S_2,F,ξ,η,g_{S_2})*. Then* D_1 *describes a totally geodesic foliation on* S_2 *if and only if*

(i).

$$\begin{aligned} g_{S_2}(A_1,\vartheta_*A_2)\eta(\alpha_1B_1) &= \eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\eta(B_1) + g_{S_2}(\beta_1B_1(\ln\lambda)\vartheta_*A_1 \\ &\quad +\vartheta_*(\nabla^{S_1}_{A_1}A_3), F\vartheta_*A_2) + g_{S_2}(S_{\alpha_1B_1}\vartheta_*A_1, F\vartheta_*A_2) \end{aligned}$$

(ii). $\phi S_{F\vartheta_*B_2}\vartheta_*A_1 - \eta(\omega\vartheta_*B_2)A_1$ *has no components in* $\Gamma(D_1)$*, for any* $A_1,A_2,A_3,B_2 \in \Gamma(ker\vartheta_*)^\perp$ *such that* $\vartheta_*A_1,\vartheta_*A_2 \in \Gamma(D_1), \vartheta_*B_2 \in \Gamma(D_2)$ *and* $B_1 \in \Gamma(range\vartheta_*)^\perp$ *such that* $\vartheta_*A_3 = \beta_1B_1$*.*

Proof. For $\vartheta_*A_1,\vartheta_*A_2 \in \Gamma(D_1), B_1 \in \Gamma(range\vartheta_*)^\perp$ and $\vartheta_*B_2 \in \Gamma(D_2)$, since ϑ is a CRM, using (2) and (3) we have

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, B_1) = g_{S_2}(F\nabla^{S_2}_{A_1}\vartheta_*A_2, FB_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\eta(B_1).$$

From (4), (5) and (6) we get

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, B_1) &= -g_{S_2}(A_1,\vartheta_*A_2)\eta(FB_1) + g_{S_2}(A_1,FB_1)\underbrace{\eta(\vartheta_*A_2)}_{0} \\ &\quad +\eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\eta(B_1) - g_{S_2}(\nabla^{S_2}_{A_1}FB_1, F\vartheta_*A_2) \end{aligned}$$

From (15)

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, B_1) &= -g_{S_2}(A_1,\vartheta_*A_2)\eta(\alpha_1B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\eta(B_1) \\ &\quad -g_{S_2}(\nabla^{S_2}_{A_1}\beta_1B_1, F\vartheta_*A_2) - g_{S_2}(\nabla^{S_2}_{A_1}\alpha_1B_1, F\vartheta_*A_2). \end{aligned}$$

Using (8), we have

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, B_1) = -g_{S_2}(A_1,\vartheta_*A_2)\eta(\alpha_1B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\eta(B_1)$$

$$-g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_3, F\vartheta_* A_2) - g_{S_2}(-S_{\alpha_1 B_1}\vartheta_* A_1 + \nabla^{\vartheta\perp}_{A_1}\vartheta_* A_1, F\vartheta_* A_2)$$

where $\beta_1 B_1 = \vartheta_* A_3 \in \Gamma(D_2)$ for $A_3 \in \Gamma(ker\vartheta_*)^\perp$. From (7) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) &= -g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) \\ &\quad -g_{S_2}((\nabla^{S_2}\vartheta_*)(A_1, A_3) + \vartheta_*(\nabla^{S_1}_{A_1} A_3), F\vartheta_* A_2) \\ &\quad +g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, F\vartheta_* A_2) - \underbrace{g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\vartheta_* A_1, F\vartheta_* A_2)}_{0}. \end{aligned}$$

Using (12) in the above equation we get

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) &= -g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) \\ &\quad -g_{S_2}((\nabla^{S_2}\vartheta_*)(A_1, A_3)^{range\vartheta_*} + \vartheta_*(\nabla^{S_1}_{A_1} A_3), F\vartheta_* A_2) \\ &\quad +g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, F\vartheta_* A_2). \end{aligned}$$

From (**??**)

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) &= -g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) \\ &\quad -g_{S_2}(A_1 (\ln\lambda)\vartheta_* A_3 + A_3 (\ln\lambda)\vartheta_* A_1 \\ &\quad -g_{S_1}(A_1, A_3)\vartheta_*(grad \ln\lambda) + \vartheta_*(\nabla^{S_1}_{A_1} A_3), F\vartheta_* A_2) \\ &\quad +g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, F\vartheta_* A_2). \end{aligned}$$

Since $grad(\ln\lambda) \in (range\vartheta_*)^\perp$, using (3) and $\vartheta_* A_3 = \beta_1 B_1$ we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) &= -g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) \\ &\quad -g_{S_2}(\beta_1 B_1 (\ln\lambda)\vartheta_* A_1 + \vartheta_*(\nabla^{S_1}_{A_1} A_3), F\vartheta_* A_2) \\ &\quad +g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, F\vartheta_* A_2). \end{aligned}$$

This implies the proof of (i).
On the other hand, by using (3) we get

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, \vartheta_* B_2) &= g_{S_2}(F\,\nabla^{S_2}_{A_1}\,\vartheta_* A_2, F\vartheta_* B_2) \\ &\quad +\eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\underbrace{\eta(\vartheta_* B_2)}_{0}. \end{aligned}$$

From (4), (5) and (6) we get

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, \vartheta_* B_2) &= -g_{S_2}(A_1, \vartheta_* A_2)\eta(F\vartheta_* B_2) + g_{S_2}(A_1, F\vartheta_* B_2)\underbrace{\eta(\vartheta_* A_2)}_{0} \\ &\quad +g_{S_2}(F\,\nabla^{S_2}_{A_1}\,F\vartheta_* B_2, \vartheta_* A_2). \end{aligned}$$

From (14) and (8) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, \vartheta_* B_2) &= -g_{S_2}(A_1, \vartheta_* A_2)\eta(\omega\vartheta_* B_2) \\ &\quad +g_{S_2}(-\phi S_{F\vartheta_* B_2}\vartheta_* A_1 + \phi\,\nabla^{\vartheta\perp}_{A_1}\,F\vartheta_* B_2, \vartheta_* A_2) \\ &= -g_{S_2}(\eta(\omega\vartheta_* B_2)A_1 - \phi S_{F\vartheta_* B_2}\vartheta_* A_1, \vartheta_* A_2). \end{aligned}$$

This implies the proof of (ii). □

We obtain the following theorem for the geometry of the leaves of D_2.

Theorem 2. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. Then D_2 describes a totally geodesic foliation on S_2 if and only if*
(i).

$$\begin{aligned} \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2) &= g_{S_2}((\nabla^{S_2}\vartheta_*)(B_3, A_4)^{(range\vartheta_*)^\perp} \\ &\quad +\nabla^{\varphi\perp}_{B_3}\,\alpha_1 B_2, F\vartheta_* B_4) + g_{S_2}(B_3, \vartheta_* B_4)\eta(\alpha_1 B_2), \end{aligned}$$

(ii). $\beta_1(\nabla^{S_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}$ has no components in $\Gamma(D_2)$,

for any $A_3, A_4, B_3, B_4 \in \Gamma(ker\vartheta_*)^\perp$ *such that* $\vartheta_* A_3, \vartheta_* B_3, \vartheta_* B_4 \in \Gamma(D_2), B_2 \in \Gamma(range\vartheta_*)^\perp$ *such that* $\vartheta_* A_4 = \beta_1 B_2$.

Proof. For $\vartheta_* A_3, \vartheta_* B_3, \vartheta_* B_4 \in \Gamma(D_2), B_2 \in \Gamma(range\vartheta_*)^\perp$, using (3),(5), (15) and since ϑ is a CRM, then we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_3}\vartheta_* B_4, B_2) &= g_{S_2}(F\,\nabla^{S_2}_{B_3}\vartheta_* B_4, FB_2) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2) \\ &= g_{S_2}(-g_{S_2}(B_3, \vartheta_* B_4)\xi + \eta(\vartheta_* B_4)B_3 \\ &\quad + \nabla^{S_2}_{B_3} F\vartheta_* B_4, FB_2) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2) \\ &= -g_{S_2}(B_3, \vartheta_* B_4)\eta(FB_2) + g_{S_2}(B_3, FB_2)\underbrace{\eta(\vartheta_* B_4)}_{0} \\ &\quad + g_{S_2}(\nabla^{S_2}_{B_3} F\vartheta_* B_4, FB_2) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2) \\ &= -g_{S_2}(B_3, \vartheta_* B_4)\eta(\alpha_1 B_2) - g_{S_2}(\nabla^{S_2}_{B_3}\beta_1 B_2, F\vartheta_* B_4) \\ &\quad - g_{S_2}(\nabla^{S_2}_{B_3}\alpha_1 B_2, F\vartheta_* B_4) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2) \end{aligned}$$

From (7), (8) and $\vartheta_* A_4 = \beta_1 B_2$ we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_3}\vartheta_* B_4, B_2) &= -g_{S_2}(B_3, \vartheta_* B_4)\eta(\alpha_1 B_2) - g_{S_2}((\nabla^{S_2}\vartheta_*)(B_3, A_4) + \vartheta_*(\nabla^{S_1}_{B_3} A_4) \\ &\quad - S_{\alpha_1 B_2}\vartheta_* B_3 + \nabla^{\vartheta\perp}_{B_3}\alpha_1 B_2, F\vartheta_* B_4) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2). \end{aligned}$$

Since D_2 defines a totally geodesic foliation on S_2, using (12) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_3}\vartheta_* B_4, B_2) &= -g_{S_2}((\nabla^{S_2}\vartheta_*)(B_3, A_4)^{(range\vartheta_*)^\perp} + \nabla^{\vartheta\perp}_{B_3}\alpha_1 B_2, F\vartheta_* B_4) \\ &\quad - g_{S_2}(B_3, \vartheta_* B_4)\eta(\alpha_1 B_2) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2). \end{aligned}$$

This implies the proof of (i).

On the other hand, by the virtue of (3), (8), (12) and (15) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_3}\vartheta_* A_3, \vartheta_* B_3) &= g_{S_2}(F(\nabla^{S_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}, \vartheta_* B_3) + \eta(\nabla^{S_2}_{B_3}\vartheta_* A_3)\underbrace{\eta(\vartheta_* B_3)}_{0} \\ &= g_{S_2}(\beta_1(\nabla^{S_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}, \vartheta_* B_3). \end{aligned}$$

Since D_2 defines a totally geodesic foliation on S_2 then we can say that $\beta_1(\nabla^{S_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}$ has no components in $\Gamma(D_2)$. This completes the proof of (ii). □

Theorem 3. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. If $(range\vartheta_*)$ defines a totally geodesic foliation on S_2 and ϑ is a horizontally homothetic CRM then we have*

$$\begin{aligned} g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, \phi\vartheta_* A_2) - g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_1} A_3), \phi\vartheta_* A_2) &= g_{S_2}(S_{\omega\vartheta_* A_2}\vartheta_* A_1, \beta_1 B_1) - g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_2, \alpha_1 B_1) \\ &\quad - \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) + g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) \quad (16) \end{aligned}$$

for any $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$ *such that* $\vartheta_* A_1, \vartheta_* A_2 \in \Gamma(range\vartheta_*), B_1 \in \Gamma(range\vartheta_*)^\perp$ *such that* $\vartheta_* A_3 = \beta_1 B_1$.

Proof. For $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$, using (3) and (5) we get

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) = g_{S_2}(\nabla^{S_2}_{A_1} F\vartheta_* A_2, FB_1) - g_{S_2}(A_1, \vartheta_* A_2)\eta(FB_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1).$$

From (15) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) &= g_{S_2}(\nabla^{S_2}_{A_1} F\vartheta_* A_2, \beta_1 B_1) + g_{S_2}(\nabla^{S_2}_{A_1} F\vartheta_* A_2, \alpha_1 B_1) \\ &\quad - g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) \\ &= -g_{S_2}(\nabla^{S_2}_{A_1}\beta_1 B_1, F\vartheta_* A_2) - g_{S_2}(\nabla^{S_2}_{A_1}\alpha_1 B_1, F\vartheta_* A_2) \\ &\quad - g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned}$$

From (14) and $\vartheta_* A_3 = \beta_1 B_1$ then we have

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) = -g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_3, \phi\vartheta_* A_2) + g_{S_2}(\nabla^{S_2}_{A_1}\omega\vartheta_* A_2, \vartheta_* A_3)$$

$$-g_{S_2}(\bigtriangledown^{S_2}_{A_1}\alpha_1 B_1, \phi\vartheta_* A_2) + g_{S_2}(\bigtriangledown^{S_2}_{A_1}\omega\vartheta_* A_2, \alpha_1 B_1)$$
$$-g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\bigtriangledown^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1).$$

Since ϑ is a CRM, using (7)

$$\begin{aligned} g_{S_2}(\bigtriangledown^{S_2}_{A_1}\vartheta_* A_2, B_1) &= -g_{S_2}((\bigtriangledown^{S_2}\vartheta_*)(A_1, A_3) + \vartheta_*(\bigtriangledown^{S_1}_{A_1}A_3), \phi\vartheta_* A_2) \\ &+g_{S_2}(\bigtriangledown^{S_2}_{A_1}\omega\vartheta_* A_2, \vartheta_* A_3) - g_{S_2}(\bigtriangledown^{S_2}_{A_1}\alpha_1 B_1, \phi\vartheta_* A_2) \\ &+g_{S_2}(\bigtriangledown^{S_2}_{A_1}\omega\vartheta_* A_2, \alpha_1 B_1) - g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) \\ &+\eta(\bigtriangledown^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned}$$

Using (13), (8) and $\vartheta_* A_3 = \beta_1 B_1$ we get

$$\begin{aligned} g_{S_2}(\bigtriangledown^{S_2}_{A_1}\vartheta_* A_2, B_1) &= -g_{S_2}(A_1 (\ln\lambda)\, \vartheta_* A_3 + A_3 (\ln\lambda)\, \vartheta_* A_1 \\ &-g_{S_1}(A_1, A_3)\vartheta_*(grad(\ln\lambda)), \phi\vartheta_* A_2) \\ &-g_{S_2}(\vartheta_*(\bigtriangledown^{S_1}_{A_1}A_3), \phi\vartheta_* A_2) - g_{S_2}(S_{\omega\vartheta_* A_2}\vartheta_* A_1, \vartheta_* A_3) \\ &+g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, \phi\vartheta_* A_2) + g_{S_2}(\bigtriangledown^{\vartheta\perp}_{A_1}\omega\vartheta_* A_2, \alpha_1 B_1) \\ &-g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\bigtriangledown^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned}$$

If we take $A_1 (\ln\lambda) = g_{S_1}(A_1, Hgrad(\ln\lambda))$ and $A_3 (\ln\lambda) = g_{S_1}(A_3, Hgrad(\ln\lambda))$, then we obtain

$$\begin{aligned} g_{S_2}(\bigtriangledown^{S_2}_{A_1}\vartheta_* A_2, B_1) &= -g_{S_1}(A_1, Hgrad(\ln\lambda))g_{S_2}(\vartheta_* A_3, \phi\vartheta_* A_2) \\ &-g_{S_1}(A_3, Hgrad(\ln\lambda))g_{S_2}(\vartheta_* A_1, \phi\vartheta_* A_2) \\ &-g_{S_1}(A_1, A_3)g_{S_2}(\vartheta_*(grad(\ln\lambda)), \phi\vartheta_* A_2) \\ &-g_{S_2}(\vartheta_*(\bigtriangledown^{S_1}_{A_1}A_3), \phi\vartheta_* A_2) - g_{S_2}(S_{\omega\vartheta_* A_2}\vartheta_* A_1, \beta_1 B_1) \\ &+g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, \phi\vartheta_* A_2) + g_{S_2}(\bigtriangledown^{\vartheta\perp}_{A_1}\omega\vartheta_* A_2, \alpha_1 B_1) \\ &-g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\bigtriangledown^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned} \tag{17}$$

Since $(range\vartheta_*)$ describes a totally geodesic foliation on S_2 and ϑ is a horizontally homothetic CRM, then from (17) we obtain (16). □

Theorem 4. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. Then $(range\vartheta_*)^\perp$ defines a totally geodesic foliation on S_2 if and only if*

$$\begin{aligned} g_{S_2}(\alpha_1 B_1, (\bigtriangledown\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}) &= g_{S_2}(B_2, [B_1, \vartheta_* A_1] + \bigtriangledown^{\vartheta\perp}_{\vartheta_* A_1}F\beta_1 B_1 \\ &+ \alpha_1 \bigtriangledown^{\vartheta\perp}_{\vartheta_* A_1} F\alpha_1 B_1) + B_1\eta(\omega\vartheta_* A_1), \end{aligned}$$

for any $B_1, B_2 \in \Gamma(range\vartheta_*)^\perp$ and $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$ such that $\vartheta_* A_2 = \beta_1 B_2$.

Proof. For any $B_1, B_2 \in \Gamma(range\vartheta_*)^\perp$ and $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$, using (3), (5) and since S_2 is a Sasakian manifold,

$$\begin{aligned} g_{S_2}(\bigtriangledown^{S_2}_{B_1}B_2, \vartheta_* A_1) &= -g_{S_2}(B_2, [B_1, \vartheta_* A_1]) - g_{S_2}(FB_2, \bigtriangledown^{S_2}_{\vartheta_* A_1}FB_1) \\ &-g_{S_2}(B_2, B_1)\eta(F\vartheta_* A_1) + \eta(\bigtriangledown^{S_2}_{B_1}B_2)\underbrace{\eta(\vartheta_* A_1)}_{0}. \end{aligned}$$

Then using (7), (8), (14) and (15) we have

$$\begin{aligned} g_{S_2}(\bigtriangledown^{S_2}_{B_1}B_2, \vartheta_* A_1) &= -g_{S_2}(B_2, [B_1, \vartheta_* A_1]) - g_{S_2}(B_2, \bigtriangledown^{\vartheta\perp}_{\vartheta_* A_1}F\beta_1 B_1) + g_{S_2}(\alpha_1 B_1, (\bigtriangledown\vartheta_*)(A_1, A_2)) \\ &-g_{S_2}(B_2, \alpha_1 \bigtriangledown^{\vartheta\perp}_{\vartheta_* A_1} F\alpha_1 B_1) - g_{S_2}(B_2, B_1\eta(\omega\vartheta_* A_1)). \end{aligned}$$

From (12), (10) and since $(range\vartheta_*)^\perp$ defines a totally geodesic foliation we have

$$\begin{aligned} g_{S_2}(\alpha_1 B_1, (\bigtriangledown\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}) &= g_{S_2}(B_2, [B_1, \vartheta_* A_1] + \bigtriangledown^{\vartheta\perp}_{\vartheta_* A_1}F\beta_1 B_1 + \alpha_1 \bigtriangledown^{\vartheta\perp}_{\vartheta_* A_1} F\alpha_1 B_1) \\ &+ B_1\eta(\omega\vartheta_* A_1). \end{aligned}$$

This completes the proof. □

Remark 2. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. From the second fundamental form, one can easly see that $ker\vartheta_*$ and $(ker\vartheta_*)^\perp$ define a totally geodesic foliation on S_1.*

From the above fact we can state following theorem.

Theorem 5. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. Then ϑ is totally geodesic foliation if and only if*

$$\begin{aligned}\phi((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} - \vartheta_*(\nabla^{S_1}_{A_1}A_2) - S_{\omega\vartheta_*A_3}\vartheta_*A_1) &= -\beta_1((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp} \\ &+ \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) - \vartheta_*(\nabla^{S_1}_{A_1}A_3), \quad (18)\\ \omega((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} - \vartheta_*(\nabla^{S_1}_{A_1}A_2) - S_{\omega\vartheta_*A_3}\vartheta_*A_1) &= -\alpha_1((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp} \\ &+ + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi, \quad (19)\end{aligned}$$

for any $A_1, A_2, A_3 \in \Gamma(ker\vartheta_*)^\perp$ such that $\vartheta_*A_2 = \phi\vartheta_*A_3$.

Proof. For $A_1, A_3 \in \Gamma(ker\vartheta_*)^\perp$, using (2), (5), (7) and (14) we have

$$\begin{aligned}(\nabla^{S_2}\vartheta_*)(A_1, A_3) &= \nabla^{S_2}_{A_1}\vartheta_*A_3 - \vartheta_*(\nabla^{S_1}_{A_1}A_3)\\ &= -F(\nabla^{S_2}_{A_1}\phi\vartheta_*A_3 + \nabla^{S_2}_{A_1}\omega\vartheta_*A_3)\\ &\quad -\vartheta_*(\nabla^{S_1}_{A_1}A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi.\end{aligned}$$

From (8) and (12) we have

$$\begin{aligned}(\nabla^{S_2}\vartheta_*)(A_1, A_3) &= -F((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*}) - F((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp})\\ &\quad -F(\vartheta_*(\nabla^{S_1}_{A_1}A_2)) + F(S_{\omega\vartheta_*A_3}\vartheta_*A_1) - F(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3)\\ &\quad -\vartheta_*(\nabla^{S_1}_{A_1}A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi.\end{aligned}$$

Since ϑ is a CRM, from (14) and (15) we have

$$\begin{aligned}(\nabla^{S_2}\vartheta_*)(A_1, A_3) &= -\phi((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*}) - \omega((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*})\\ &\quad -\beta_1((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}) - \alpha_1((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp})\\ &\quad -\phi(\vartheta_*(\nabla^{S_1}_{A_1}A_2)) - \omega(\vartheta_*(\nabla^{S_1}_{A_1}A_2)) + \phi(S_{\omega\vartheta_*A_3}\vartheta_*A_1) + \omega(S_{\omega\vartheta_*A_3}\vartheta_*A_1)\\ &\quad -\beta_1(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) - \alpha_1(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) - \vartheta_*(\nabla^{S_1}_{A_1}A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi.\end{aligned}$$

Taking $range\vartheta_*$ and $(range\vartheta_*)^\perp$ components we have

$$\begin{aligned}\phi((\nabla\vartheta_*)(A_1, A_3)^{range\vartheta_*}) &= -\phi((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*}) + \vartheta_*(\nabla^{S_1}_{A_1}A_2) - S_{\omega\vartheta_*A_3}\vartheta_*A_1)\\ &\quad -\beta_1((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp} + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) - \vartheta_*(\nabla^{S_1}_{A_1}A_3)\end{aligned}$$

$$\begin{aligned}(\nabla\vartheta_*)(A_1, A_3)^{(range\vartheta_*)^\perp} &= -\omega((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} + \vartheta_*(\nabla^{S_1}_{A_1}A_2) - S_{\omega\vartheta_*A_3}\vartheta_*A_1)\\ &\quad -\alpha_1((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp} + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi.\end{aligned}$$

Thus $(\nabla\vartheta_*)(A_1, A_3) = 0$ if and only if (18) and (19) are satisfied. This completes the proof. □

Proposition 1. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$ such that* $\dim(range\vartheta_*) > 1$. *Then the following statements are true.*

(i). $\bar{D}_1$ defines a totally geodesic foliation if and only if $(\nabla\vartheta_*)(A_1, U_1)$ has no component in D_1 such that

$$g_{S_2}(-S_{F\vartheta_*A_1'}\vartheta_*A_1 + g_{S_2}(A_1', \xi)\vartheta_*A_1, F\vartheta_*A_2) = \eta(\nabla^{S_1}_{A_1}A_2)\eta(A_1')$$

for $A_1, A_2 \in \Gamma(\bar{D}_1), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1' \in \Gamma(\bar{D}_2)$.

(ii). $\bar{D}_2$ defines a totally geodesic foliation if and only if $(\nabla\vartheta_*)(A_2, U_1)$ has no component in D_2 such that

$$g_{S_2}(S_{F\vartheta_*A_3'}\vartheta_*A_3, F\vartheta_*A_4) = g_{S_2}(g_{S_2}(A_3', \xi)\vartheta_*A_3, F\vartheta_*A_4) - \eta(\nabla^{S_1}_{\vartheta_*A_3}\vartheta_*A_4)\eta(\vartheta_*A_3')$$

for $A_2, A_3, A_4 \in \Gamma(\bar{D}_2), U_1 \in \Gamma(ker\vartheta_*)$ and $A_3' \in \Gamma(\bar{D}_2)$.

Proof. We know that $\bar{D}_1$ defines totally geodesic foliation if and only if $g_{S_1}(\nabla^{S_1}_{A_1}A_2, U_1) = 0$ and $g_{S_1}(\nabla^{S_1}_{A_1}A_2, A_1^{'}) = 0$ for $A_1, A_2 \in \Gamma(\bar{D}_1), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1^{'} \in \Gamma(\bar{D}_2)$. Now, since ϑ is Riemannian map, using (1), (7) and (8) we have

$$\begin{aligned} g_{S_1}(\nabla^{S_1}_{A_1}A_2, U_1) &= -g_{S_1}(\nabla^{S_1}_{A_1}U_1, A_2) \\ &= -g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_1}U_1), \vartheta_*A_2) \\ &= g_{S_2}((\nabla\vartheta_*)(A_1, U_1), \vartheta_*A_2), \end{aligned}$$

and similarly

$$\begin{aligned} g_{S_1}(\nabla^{S_1}_{A_1}A_2, A_1^{'}) &= -g_{S_1}(\nabla^{S_1}_{A_1}A_1^{'}, A_2) \\ &= -g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_1}A_1^{'}), \vartheta_*A_2) \\ &= -g_{S_2}(\nabla^{\vartheta}_{A_1}\vartheta_*A_1^{'}, \vartheta_*A_2) \\ &= -g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1^{'}, \vartheta_*A_2). \end{aligned}$$

Since S_2 is Sasakian manifold, using (3), (5) and then (8), we have

$$g_{S_1}(\nabla^{S_1}_{A_1}A_2, A_1^{'}) = -g_{S_2}(-S_{F\vartheta_*A_1^{'}}\vartheta_*A_1, F\vartheta_*A_2) + \eta(\nabla^{S_1}_{A_1}A_2)\eta(A_1^{'}) - g_{S_2}(g_{S_2}(A_1^{'}, \xi)\vartheta_*A_1, F\vartheta_*A_2)$$

This completes the proof of (i).

On the other hand, we know that $\bar{D}_2$ defines a totally geodesic foliation if and only if $g_{S_1}(\nabla^{S_1}_{A_3}A_4, U_1) = 0$ and $g_{S_1}(\nabla^{S_1}_{A_3}A_4, A_3^{'}) = 0$ for $A_3, A_4 \in \Gamma(\bar{D}_2), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1^{'} \in \Gamma(\bar{D}_1)$. Now, since ϑ is Riemannian map, using (1) and (7) we have

$$\begin{aligned} g_{S_1}(\nabla^{S_1}_{A_3}A_4, U_1) &= -g_{S_1}(\nabla^{S_1}_{A_3}U_1, A_4) \\ &= -g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_3}U_1), \vartheta_*A_4) \\ &= g_{S_2}((\nabla^{S_2}\vartheta_*)(A_3, U_1), \vartheta_*A_4), \end{aligned}$$

and similarly

$$\begin{aligned} g_{S_1}(\nabla^{S_1}_{A_3}A_4, A_3^{'}) &= g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_3}A_4), \vartheta_*A_3^{'}) \\ &= g_{S_2}(\nabla^{\vartheta}_{A_3}\vartheta_*A_4, \vartheta_*A_3^{'}) \\ &= g_{S_2}(\nabla^{S_2}_{\vartheta_*A_3}\vartheta_*A_4, \vartheta_*A_3^{'}). \end{aligned}$$

Since S_2 is Sasakian manifold, using (3),(5) and then (8), we have

$$g_{S_1}(\nabla^{S_1}_{A_3}A_4, A_3^{'}) = -g_{S_2}(-S_{F\vartheta_*A_3^{'}}\vartheta_*A_3, F\vartheta_*A_4) + \eta(\nabla^{S_1}_{\vartheta_*A_3}\vartheta_*A_4)\eta(\vartheta_*A_3^{'}) - g_{S_2}(g_{S_2}(A_3^{'}, \xi)\vartheta_*A_3, F\vartheta_*A_4).$$

This completes the proof of (ii). □

Definition 4. *[19] Let (S_1, g_{S_1}) be a Riemannian manifold and assume that the canonical foliations K_1 and K_2 such that $K_1 \cap K_2 = \{0\}$ everywhere. Then (S_1, g_{S_1}) is a locally product manifold if and only if K_1 and K_2 are totally geodesic foliations.*

Theorem 6. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$ such that $\dim(range\vartheta_*) > 1$. Then $(ker\vartheta_*)^{\perp}$ is a locally product manifold of $\bar{D}_1$ and $\bar{D}_2$ if and only if*

(i). $(\nabla\vartheta_*)(A_1, U_1)$ has no component in D_1 such that −

$$g_{S_2}(S_{F\vartheta_*A_3^{'}}\vartheta_*A_1, F\vartheta_*A_2) = g_{S_2}(g_{S_2}(A_1^{'}, \xi)\vartheta_*A_1, F\vartheta_*A_4) - \eta(\nabla^{S_1}_{A_1}A_2)\eta(A_1^{'})$$

for $A_1, A_2 \in \Gamma(\bar{D}_1), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1^{'} \in \Gamma(\bar{D}_2)$,

(ii). $(\nabla\vartheta_*)(A_3, U_1)$ has no component in D_2 such that

$$g_{S_2}(S_{F\vartheta_*A_3^{'}}\vartheta_*A_3, F\vartheta_*A_4) = g_{S_2}(g_{S_2}(A_3^{'}, \xi)\vartheta_*A_3, F\vartheta_*A_4) - \eta(\nabla^{S_1}_{\vartheta_*A_3}\vartheta_*A_4)\eta(A_3^{'})$$

for $A_2, A_3, A_4 \in \Gamma(\bar{D}_2), U_1 \in \Gamma(ker\vartheta_*)$ and $A_3^{'} \in \Gamma(\bar{D}_2)$.

Proof. The proof is clear by Proposition (1) and Definition (4). □

Theorem 7. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$ such that $\dim(range\vartheta_*) > 1$. Then the base manifold is locally product manifold $S_2 \times S_2$ if and only if*

$$g_{S_2}(\vartheta_*(\nabla^{S_2}_{A_1}\phi\vartheta_* A_1), \beta_1 B_1) + g_{S_2}(\vartheta_*(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1), \alpha_1 B_1) + g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, \phi\vartheta_* A_1)$$
$$+\eta(\nabla^{S_2}_{\vartheta_* A_1}\vartheta_* A_1)\eta(B_1) = |\vartheta_* A_1|^2\,\Gamma(\alpha_1 B_1),$$

for $A_1 \in \Gamma(\bar{D}_1)$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$.

Proof. Since S_2 is Sasakian manifold, using (3) and (3) we have

$$g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}\vartheta_* A_1, B_1) = g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}F\vartheta_* A_1, FB_1) + \eta(\nabla^{S_2}_{\vartheta_* A_1}\vartheta_* A_1)\eta(B_1)$$
$$- |\vartheta_* A_1|^2\,\Gamma(\alpha_1 B_1) + \Gamma(\vartheta_* A_1)\underbrace{g_{S_2}(\vartheta_* A_1, FB_1)}_{0},$$

for $\vartheta_* A_1 \in \Gamma(range\vartheta_*)$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$. Using (14) and (15) then we have

$$g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}\vartheta_* A_1, B_1) = g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}\phi\vartheta_* A_1, \beta_1 B_1) + g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}\omega\vartheta_* A_1, \beta_1 B_1) + g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}\phi\vartheta_* A_1, \alpha_1 B_1)$$
$$+g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}\omega\vartheta_* A_1, \alpha_1 B_1) + \eta(\nabla^{S_2}_{\vartheta_* A_1}\vartheta_* A_1)\eta(B_1) - |\vartheta_* A_1|^2\,\Gamma(\alpha_1 B_1),$$

using (8) in above equation, we get

$$g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}\vartheta_* A_1, B_1) = g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}\phi\vartheta_* A_1, \beta_1 B_1) + g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, \phi\vartheta_* A_1)$$
$$+g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1, \alpha_1 B_1) + \eta(\nabla^{S_2}_{\vartheta_* A_1}\vartheta_* A_1)\eta(B_1) - |\vartheta_* A_1|^2\,\Gamma(\alpha_1 B_1).$$

Then, using (7) we obtain

$$g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1}\vartheta_* A_1, B_1) = g_{S_2}((\nabla\vartheta_*)(A_1, {}^*\vartheta_*(\phi\vartheta_* A_1)), \beta_1 B_1) + g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, \phi\vartheta_* A_1)$$
$$+g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1, \alpha_1 B_1) + \eta(\nabla^{S_2}_{\vartheta_* A_1}\vartheta_* A_1)\eta(B_1) - |\vartheta_* A_1|^2\,\Gamma(\alpha_1 B_1),$$

from Definition (4), the proof is completed. □

Now, we will examine the harmonicity of CSIRM from a Riemannian manifold (S_1, g_{S_1}) to Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$ in the following theorem.

Theorem 8. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. Then ϑ is harmonic if and only if the following conditions are satisfied*

(i). The fibres are minimal,

(ii).

$$trace\phi S_{\omega\vartheta_* A_1}A_1 - \beta_1\,\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1 - \vartheta_*(\nabla^{S_1}_{A_1}A_1) - (\nabla^{S_2}F\phi\vartheta_* A_1)^{range\vartheta_*} = 0,$$

(iii).

$$trace\omega S_{\omega\vartheta_* A_1}A_1 - \alpha_1\,\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1 - (\nabla^{S_2}F\phi\vartheta_* A_1)^{(range\vartheta_*)^\perp} + \eta((\nabla\vartheta_*)(A_1, A_1)^{(range\vartheta_*)^\perp})\xi = 0$$

for $A_1 \in (ker\vartheta_*)^\perp$.

Proof. For $U_1 \in ker\vartheta_*$ using (7) we get

$$(\nabla\vartheta_*)(U_1, U_1) = \nabla^{S_2}_{U_1}\vartheta_* U_1 - \vartheta_*(\nabla^{S_1}_{U_1}U_1) = -\vartheta_*(\nabla^{S_1}_{U_1}U_1), \tag{20}$$

since $\vartheta_* U_1 = 0$. For $A_1 \in (ker\vartheta_*)^\perp$ using (3), (7), (15), (12) and (8) we have

$$(\nabla\vartheta_*)(A_1, A_1) = \nabla^{S_2}_{A_1}\vartheta_* A_1 - \vartheta_*(\nabla^{S_1}_{A_1}A_1)$$
$$= -\nabla^{S_2}_{A_1}F\phi\vartheta_* A_1 - F(\nabla^{S_2}_{A_1}\omega\vartheta_* A_1) - \vartheta_*(\nabla^{S_1}_{A_1}A_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_1)\xi$$
$$= -\nabla^{S_2}_{A_1}F\phi\vartheta_* A_1 - F(-S_{\omega\vartheta_* A_1}\vartheta_* A_1 + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1) - \vartheta_*(\nabla^{S_1}_{A_1}A_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_1)\xi.$$

Since ϑ is a CRM, from (14) and (15) we have

$$(\nabla\vartheta_*)(A_1, A_1) = -\nabla^{S_2}_{A_1}F\phi\vartheta_* A_1 + \phi S_{\omega\vartheta_* A_1}\vartheta_* A_1 + \omega S_{\omega\vartheta_*\vartheta_* A_1}\vartheta_* A_1$$
$$-\beta_1\,\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1 - \alpha_1\,\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1 - \vartheta_*(\nabla^{S_1}_{A_1}A_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_1)\xi.$$

Taking $range\vartheta_*$ and $(range\vartheta_*)^\perp$ components we have

$$(\nabla\vartheta_*)(A_1,A_1)^{range\vartheta} = \phi S_{\omega\vartheta_* A_1}\vartheta_* A_1 - \beta_1 \nabla^{\vartheta\perp}_{A_1} \omega\vartheta_* A_1 - \vartheta_*(\nabla^{S_1}_{A_1} A_1) - (\nabla^{S_2}_{A_1} F\phi\vartheta_* A_1)^{range\vartheta} \quad (21)$$

and

$$\begin{aligned}(\nabla\vartheta_*)(A_1,A_1)^{(range\vartheta_*)^\perp} &= \omega S_{\omega\vartheta_*\vartheta_* A_1}\vartheta_* A_1 - \alpha_1 \nabla^{\vartheta\perp}_{A_1} \omega\vartheta_* A_1 - (\nabla^{S_2}_{A_1} F\phi\vartheta_* A_1)^{(range\vartheta_*)^\perp} \\ &+ \eta((\nabla\vartheta_*)(A_1,A_1)^{(range\vartheta_*)^\perp})\xi. \quad (22)\end{aligned}$$

Thus the proof is completed from (20), (21) and (22). □

4. Future Studies

The Clairaut Riemannian maps are particular Riemannian maps having important applications in the geometry [13,26]. The notions of invariant, anti-invariant, and semi-invariant Clairaut Riemannian maps with almost hermitian manifolds have been studied by the first author and other authors in [9,18,30,31]. Recently, the notions of Clairaut conformal submersions and Clairaut conformal Riemannian maps have been introduced in [11,12] and showed that these smooth maps generate a lot of interest due to their associated geometric properties. Our paper combined the notions of conformal Riemannian maps and semi-invariant Riemannian maps to Sasakian manifolds. Therefore in future it will be interesting to combine more notions of Clairaut Riemannian maps to these two notions and study Clairaut conformal semi-invariant Riemannian maps (and in particular Clairaut conformal semi-invariant submersions) to Kähler and/or to Sasakian manifolds.

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