International Electronic Journal of Algebra
Published Online: May 2024
DOI: 10.24330/ieja.1488471

ADDITIVITY OF MULTIPLICATIVE GENERALIZED JORDAN MAPS ON TRIANGULAR RINGS

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Received: 12 October 2023; Revised: 24 January 2024; Accepted: 27 January 2024
Communicated by Burcu Üngör

Abstract. This paper presents three different conditions for the additivity of a map on a triangular ring $\mathcal{T}$. First, we prove a map δ on $\mathcal{T}$ satisfying $\delta(a_1b_1+b_1a_1) = \delta(a_1)b_1+a_1\tau(b_1)+\delta(b_1)a_1+b_1\tau(a_1)$ for all $a_1, b_1 \in \mathcal{T}$ and for some maps τ over $\mathcal{T}$ satisfying $\tau(a_1b_1+b_1a_1) = \tau(a_1)b_1+a_1\tau(b_1)+\tau(b_1)a_1+b_1\tau(a_1)$, is additive. Secondly, it is shown that a map T on $\mathcal{T}$ satisfying $T(a_1b_1) = T(a_1)b_1 = a_1T(b_1)$ for all $a_1, b_1 \in \mathcal{T}$ is additive. Finally, we show that if a map D over $\mathcal{T}$ satisfies $(m+n)D(a_1b_1) = 2mD(a_1)b_1 + 2na_1D(b_1)$ for all $a_1, b_1 \in \mathcal{T}$ and integers $m, n \geq 1$, then D is additive.

Mathematics Subject Classification (2020): 16W10, 16S50
Keywords: Additivity, Jordan derivation, two-sided centralizer, (m, n)-derivation, triangular ring

1. Introduction

Let $\mathcal{R}$ be a ring. A map $f : \mathcal{R} \to \mathcal{R}$ is said to be additive if $f(a_1 + b_1) = f(a_1) + f(b_1)$ for all $a_1, b_1 \in \mathcal{R}$. In 1957, Herstein [9] introduced Jordan derivation over rings as an additive map $\tau : \mathcal{R} \to \mathcal{R}$ satisfies $\tau(a^2) = \tau(a)a + a\tau(a)$ for all $a \in \mathcal{R}$. He also proved that a Jordan derivation over some prime ring becomes a derivation with some torsion restrictions. In 2003, Jing and Lu [10] introduced generalized Jordan derivation and proved that every generalized Jordan derivation is a generalized derivation over a 2-torsion-free prime ring. Recall that an additive map $\delta : \mathcal{R} \to \mathcal{R}$ is known as a generalized Jordan derivation if it follows the condition $\delta(a^2) = \delta(a)a + a\tau(a)$ for any $a \in \mathcal{R}$ and for some Jordan derivation $\tau : \mathcal{R} \to \mathcal{R}$. In 1952, Wendel [15] introduced the concept of a centralizer. An additive map $T : \mathcal{R} \to \mathcal{R}$ is said to be a left centralizer if $T(a_1b_1) = T(a_1)b_1$ for every $a_1, b_1 \in \mathcal{R}$. Similarly, we define the right centralizer over a ring. An additive map $T : \mathcal{R} \to \mathcal{R}$ is said to be a two-sided centralizer if T is both left and right centralizer. In 2014, Ali and Fošner [1] introduced the concept of (m, n)-derivation. Let $m, n \geq 0$ be integers. Then an additive map $D : \mathcal{R} \to \mathcal{R}$ is called an (m, n)-derivation if $(m + n)D(a_1b_1) = 2mD(a_1)b_1 + 2na_1D(b_1)$, for every $a_1, b_1 \in \mathcal{R}$. In fact, for $m = 1$ and $n = 1$, D is a normal derivation on a 2-torsion free ring $\mathcal{R}$. For more results, reader can see [7,8].

Let $n \geq 2$ be any integer. Then a ring $\mathcal{R}$ is called n-torsion free if $na = 0$ for some $a \in \mathcal{R}$ implies $a = 0$. It is an interesting question that when a map over a ring (satisfying some functional equations) is additive. The question was first raised by Rickart [12] in 1948. He showed that any bijective and multiplicative mapping of a Boolean ring B onto any arbitrary ring S is additive. In 1969, Martindale III [11] proved that any multiplicative isomorphism of a ring $\mathcal{R}$ onto an arbitrary ring S is additive under the existence of a family of idempotent elements in $\mathcal{R}$ satisfying certain conditions. In 1991, by assuming Martindale's conditions, Daif [3] proved that any multiplicative derivation of a ring $\mathcal{R}$ is additive. Later, in 2009, Wang [13] proved that any n-multiplicative derivation d of $\mathcal{R}$ is additive by using Martindale's conditions where $n > 1$ is an integer. He proved that any n-multiplicative isomorphism or n-multiplicative derivation of a standard operator algebra A over a Banach space X with $dim(X) \geq 2$, is additive. Again, in 2011, Wang [14] proved that any n-multiplicative derivation d of a triangular ring $\mathcal{T}$ is additive, with some assumptions on $\mathcal{T}$. In 2014, Ferreira [5] proved that every m-multiplicative isomorphism from an n-triangular matrix ring $\mathcal{T}$ onto any ring $\mathcal{S}$ is additive with some assumptions on $\mathcal{T}$ where $n, m > 1$ are integers. He also showed that every m-multiplicative derivation of $\mathcal{T}$ is additive. In 2015, Ferreira [6] again revisit and proved that a map τ on a triangular ring $\mathcal{T}$ satisfying $\tau(a_1b_1 + b_1a_1) = \tau(a_1)b_1 + a_1\tau(b_1) + \tau(b_1)a_1 + b_1\tau(a_1)$ for all $a_1, b_1 \in \mathcal{T}$, is additive with some assumptions on $\mathcal{T}$. Moreover, if $\mathcal{T}$ is 2-torsion free, then τ is a Jordan derivation. Ferreira's result [6] motivated us to work to find out the conditions under which a multiplicative generalized Jordan derivation is additive on a triangular ring. This paper provides an affirmative answer to the above question. In 2023, Aziz et al. [2] showed that under certain conditions, a generalized skew semi-derivation over a ring is additive. In 2014, El-Sayiad et al. [4] proved that every map T on a prime ring with a non-trivial idempotent satisfying

$$T(ab) = T(a)b \tag{1}$$

for every $a, b \in \mathcal{R}$, is additive. Note that a ring $\mathcal{R}$ is said to be a prime ring if $aRb = 0$ for some $a, b \in \mathcal{R}$ implies either $a = 0$ or $b = 0$ and is said to be a semi-prime ring if $aRa = 0$ for some $a \in \mathcal{R}$ implies that $a = 0$. Thus, every prime ring is semi-prime, but the converse is not true. Motivated by the work of El-Sayiad et al. [4], we prove that a map T on $\mathcal{T}$ satisfying

$$T(a_1b_1) = T(a_1)b_1 = a_1T(b_1),$$

for all $a_1, b_1 \in \mathcal{T}$, is additive.

Further, we prove that any map D on $\mathcal{T}$ which satisfies

$$(m+n)D(a_1b_1) = 2mD(a_1)b_1 + 2na_1D(b_1),$$

for all $a_1, b_1 \in \mathcal{T}$ and for some integers $m, n \geq 1$, is additive.

Throughout the work, we use the definition of triangular algebra given by Wang [14].

Definition 1.1. Let M be an (A, B)-bimodule where A and B are two rings such that

(1) M is faithful as a left A-module and a right B-module;
(2) If $AmB = 0$ for some $m \in M$, then $m = 0$.

Then the ring

$$\mathcal{T} = \text{Tri}(A, M, B) = \left\{ \begin{pmatrix} a_1 & m \\ & b_1 \end{pmatrix} \mid a_1 \in A,\, b_1 \in B,\, m \in M \right\},$$

under usual matrix addition and multiplication is said to be a triangular ring.

Let $\mathcal{T}_{11} = \left\{ \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix} \mid a \in A \right\}$, $\mathcal{T}_{12} = \left\{ \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \mid m \in M \right\}$ and $\mathcal{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix} \mid b \in B \right\}$.

Then $\mathcal{T} = \mathcal{T}_{11} \oplus \mathcal{T}_{12} \oplus \mathcal{T}_{22}$. Henceforth, $t_{ij} \in \mathcal{T}_{ij}$. Also, for $j \neq k$, we have $t_{ij}t_{kl} = 0$ where $j, k \in \{1, 2\}$.

In ring theory, various maps have been defined and studied to understand the structure of rings. Among them, generalized Jordan derivations, two-sided centralizers, and (m, n)-derivations are important tools for investigating the properties of rings. However, when we remove the additivity condition from their definitions, we get multiplicative generalized Jordan derivation, multiplicative two-sided centralizer, and multiplicative (m, n)-derivation. This raises the question of whether these multiplicative maps become additive over some rings.

In this paper, we address this question and provide an affirmative answer for triangular rings under certain conditions. In Section 2, we prove that every multiplicative generalized Jordan derivation over a triangular ring with certain conditions becomes additive. We also explore an example where a particular condition is not satisfied and leave the general case as a conjecture. In Section 3, we prove that every multiplicative two-sided centralizer over a triangular ring with some conditions is additive. Finally, in Section 4, we prove that every multiplicative (m, n)-derivation over a triangular ring with certain conditions is additive. Our results provide a better understanding of the structures of rings and the behavior of these maps.

2. Generalized Jordan derivations on triangular rings

Theorem 2.1. *Let $\mathcal{T} = Tri(A, M, B)$ be a triangular ring such that A and B are rings with identity.* $(*)$
If a function $\delta : \mathcal{T} \to \mathcal{T}$ satisfies

$$\delta(a_1b_1 + b_1a_1) = \delta(a_1)b_1 + a_1\tau(b_1) + \delta(b_1)a_1 + b_1\tau(a_1),$$

for all $a_1, b_1 \in \mathcal{T}$ where $\tau : \mathcal{T} \to \mathcal{T}$ satisfies

$$\tau(a_1b_1 + b_1a_1) = \tau(a_1)b_1 + a_1\tau(b_1) + \tau(b_1)a_1 + b_1\tau(a_1),$$

for all $a_1, b_1 \in \mathcal{T}$, then δ is additive. Moreover, if $\mathcal{T}$ is 2-torsion free, then δ is a generalized Jordan derivation.

Since $\tau : \mathcal{T} \to \mathcal{T}$ satisfies all the conditions in Theorem 3.1 of [6], τ is additive. In this section, we frequently use all three conditions for the triangular ring $\mathcal{T}$, the conditions on the maps δ and τ described in Theorem 2.1, and the additivity of τ without mentioning them. Note that $\delta(0) = 0$. Before proving Theorem 2.1, we first discuss several needful lemmas.

Lemma 2.2. *Let $a_1 \in \mathcal{T}_{11}$, $b_2 \in \mathcal{T}_{22}$ and $m \in \mathcal{T}_{12}$. Then*

$$\begin{aligned} &(i)\ \delta(a_1 + m) = \delta(a_1) + \delta(m);\\ &(ii)\ \delta(b_2 + m) = \delta(b_2) + \delta(m). \end{aligned} \tag{2}$$

Proof. Let $t_2 \in \mathcal{T}_{22}$. Then

$$\begin{aligned} &\delta[(a_1 + m)t_2 + t_2(a_1 + m)]\\ &= \delta(a_1 + m)t_2 + (a_1 + m)\tau(t_2) + \delta(t_2)(a_1 + m) + t_2\tau(a_1 + m). \end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned} &\delta[(a_1 + m)t_2 + t_2(a_1 + m)]\\ &= \delta(mt_2) = \delta(0) + \delta(mt_2 + 0)\\ &= \delta(a_1t_2 + t_2a_1) + \delta(mt_2 + t_2m)\\ &= \delta(a_1)t_2 + a_1\tau(t_2) + \delta(t_2)a_1 + t_2\tau(a_1)\\ &+ \delta(m)t_2 + m\tau(t_2) + \delta(t_2)m + t_2\tau(m). \end{aligned} \tag{4}$$

Since τ is additive, comparing (3) and (4), we have

$$\begin{aligned} &[\delta(a_1 + m) - \delta(a_1) - \delta(m)]t_2 = 0\\ &\implies [\delta(a_1 + m) - \delta(a_1) - \delta(m)]_{12}t_2 = 0\\ &\text{and } [\delta(a_1 + m) - \delta(a_1) - \delta(m)]_{22}t_2 = 0. \end{aligned} \tag{5}$$

Using condition (2) of Definition 1.1 and ($*$) of Theorem 2.1, we have

$$\begin{aligned}&[\delta(a_1+m)-\delta(a_1)-\delta(m)]_{12}=0\\ &\text{and } [\delta(a_1+m)-\delta(a_1)-\delta(m)]_{22}=0.\end{aligned}\tag{6}$$

Now, in order to prove $[\delta(a_1+m)-\delta(a_1)-\delta(m)]_{11}=0$, let $n\in\mathcal{T}_{12}$. Then

$$\begin{aligned}&\delta[(a_1+m)n+n(a_1+m)]=\delta(a_1n)\\ &=\delta(a_1n+na_1)+\delta(mn+nm)\\ &=\delta(a_1)n+a_1\tau(n)+\delta(n)a_1+n\tau(a_1)+\delta(m)n+m\tau(n)+\delta(n)m+n\tau(m).\end{aligned}\tag{7}$$

Also, we have

$$\begin{aligned}&\delta[(a_1+m)n+n(a_1+m)]\\ &=\delta(a_1+m)n+(a_1+m)\tau(n)+\delta(n)(a_1+m)+n\tau(a_1+m)\\ &=\delta(a_1+m)n+(a_1+m)\tau(n)+\delta(n)(a_1+m)+n(\tau(a_1)+\tau(m)) \text{ (By additivity of } \tau).\end{aligned}\tag{8}$$

Comparing the equalities (7) and (8),

$$\begin{aligned}&[\delta(a_1+m)-\delta(a_1)-\delta(m)]_{11}n=0\\ &\implies [\delta(a_1+m)-\delta(a_1)-\delta(m)]_{11}=0 \text{ (by condition (1) of Definition 1.1).}\end{aligned}\tag{9}$$

Hence, $\delta(a_1+m)=\delta(a_1)+\delta(m)$. Similarly, we can prove that $\delta(b_2+m)=\delta(b_2)+\delta(m)$. □

Lemma 2.3. *Let $a_1\in\mathcal{T}_{11}$, $b_2\in\mathcal{T}_{22}$ and $m,n\in\mathcal{T}_{12}$. Then*

$$\delta(a_1m+nb_2)=\delta(a_1m)+\delta(nb_2).\tag{10}$$

Proof. Since $a_1m+nb_2=(a_1+n)(m+b_2)+(m+b_2)(a_1+n)$, by Lemma 2.2 and the additivity of τ, we have

$$\begin{aligned}\delta(a_1m+nb_2)&=\delta[(a_1+n)(m+b_2)+(m+b_2)(a_1+n)]\\ &=\delta(a_1+n)(m+b_2)+(a_1+n)\tau(m+b_2)\\ &+\delta(m+b_2)(a_1+n)+(m+b_2)\tau(a_1+n)\\ &=(\delta(a_1)+\delta(n))(m+b_2)+(a_1+n)(\tau(m)+\tau(b_2))\\ &+(\delta(m)+\delta(b_2))(a_1+n)+(m+b_2)(\tau(a_1)+\tau(n))\\ &=\delta(a_1m+ma_1)+\delta(nb_2+b_2n)+\delta(mn+nm)+\delta(a_1b_2+b_2a_1)\\ &=\delta(a_1m)+\delta(nb_2).\end{aligned}\tag{11}$$

□

Lemma 2.4. *Let $m,n\in\mathcal{T}_{12}$. Then*

$$\delta(m+n)=\delta(m)+\delta(n).\tag{12}$$

Proof. Since we assume both A and B are rings with identity in Theorem 2.1, putting $a=1$ and $b=1$ in Lemma 2.3, we have the desired result. □

Lemma 2.5. *Let $a_1, a_2 \in \mathcal{T}_{11}$ and $b_1, b_2 \in \mathcal{T}_{22}$. Then*

$$\begin{aligned} &(i)\ \delta(a_1 + a_2) = \delta(a_1) + \delta(a_2);\\ &(ii)\ \delta(b_1 + b_2) = \delta(b_1) + \delta(b_2). \end{aligned} \tag{13}$$

Proof. Let $t_2 \in \mathcal{T}_{22}$. Then

$$\begin{aligned} &\delta(a_1 + a_2)t_2 + (a_1 + a_2)\tau(t_2) + \delta(t_2)(a_1 + a_2) + t_2\tau(a_1 + a_2)\\ &= \delta[(a_1 + a_2)t_2 + t_2(a_1 + a_2)] = \delta(0) = 0\\ &= \delta(a_1t_2 + t_2a_1) + \delta(a_2t_2 + t_2a_2)\\ &= \delta(a_1)t_2 + a_1\tau(t_2) + \delta(t_2)a_1 + t_2\tau(a_1)\\ &+ \delta(a_2)t_2 + a_2\tau(t_2) + \delta(t_2)a_2 + t_2\tau(a_2). \end{aligned} \tag{14}$$

Since τ is additive, from (14), we have

$$\begin{aligned} &[\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)]t_2 = 0\\ &\implies [\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)]_{12}t_2 = 0\\ &\text{and } [\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)]_{22}t_2 = 0. \end{aligned} \tag{15}$$

Using condition (2) of Definition 1.1 and $(*)$ of Theorem 2.1, we have

$$\begin{aligned} &[\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)]_{12} = 0\\ &\text{and } [\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)]_{22} = 0. \end{aligned} \tag{16}$$

Now, to prove $[\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)]_{11} = 0$, let $m \in \mathcal{T}_{12}$. Then

$$\begin{aligned} &\delta(a_1 + a_2)m + (a_1 + a_2)\tau(m) + \delta(m)(a_1 + a_2) + m\tau(a_1 + a_2)\\ &= \delta[(a_1 + a_2)m + m(a_1 + a_2)]\\ &= \delta[(a_1m + ma_1) + (a_2m + ma_2)]\\ &= \delta(a_1m + ma_1) + \delta(a_2m + ma_2) \text{ (By Lemma 2.4)}\\ &= \delta(a_1)m + a_1\tau(m) + \delta(m)a_1 + m\tau(a_1) + \delta(a_2)m + a_2\tau(m) + \delta(m)a_2 + m\tau(a_2). \end{aligned} \tag{17}$$

By (17) and the additivity of τ, we see that

$$\begin{aligned} &[\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)]m = 0\\ &\implies [\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)]_{11}m = 0\\ &\implies [\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)]_{11} = 0 \text{ (by condition (1) of Definition 1.1).} \end{aligned} \tag{18}$$

Hence, $\delta(a_1 + a_2) = \delta(a_1) + \delta(a_2)$. In a similar way, we can prove that $\delta(b_1 + b_2) = \delta(b_1) + \delta(b_2)$. □

Lemma 2.6. *Let $a_1 \in \mathcal{T}_{11}$, $b_2 \in \mathcal{T}_{22}$ and $m \in \mathcal{T}_{12}$. Then*

$$\delta(a_1 + m + b_2) = \delta(a_1) + \delta(m) + \delta(b_2). \tag{19}$$

Proof. Let $t_2 \in \mathcal{T}_{22}$. Then

$$\begin{aligned}
&\delta(a_1+m+b_2)t_2+(a_1+m+b_2)\tau(t_2)+\delta(t_2)(a_1+m+b_2)+t_2\tau(a_1+m+b_2)\\
&=\delta[(a_1+m+b_2)t_2+t_2(a_1+m+b_2)]\\
&=\delta(mt_2+b_2t_2+t_2b_2)\\
&=\delta(b_2t_2+t_2b_2)+\delta(mt_2)\text{ (By Lemma 2.2)}\\
&=\delta(b_2t_2+t_2b_2)+\delta(t_2m+mt_2)+\delta(a_1t_2+t_2a_1)\\
&=\delta(b_2)t_2+b_2\tau(t_2)+\delta(t_2)b_2+t_2\tau(b_2)+\delta(t_2)m+t_2\tau(m)\\
&+\delta(m)t_2+m\tau(t_2)+\delta(a_1)t_2+a_1\tau(t_2)+\delta(t_2)a_1+t_2\tau(a_1)\\
&\implies[\delta(a_1+m+b_2)-\delta(a_1)-\delta(m)-\delta(b_2)]t_2=0\text{ (Using additivity of }\tau)\\
&\implies[\delta(a_1+m+b_2)-\delta(a_1)-\delta(m)-\delta(b_2)]_{12}t_2=0\\
&\text{and }[\delta(a_1+m+b_2)-\delta(a_1)-\delta(m)-\delta(b_2)]_{22}t_2=0.
\end{aligned}\tag{20}$$

Using condition (2) of Definition 1.1 and $(*)$ of Theorem 2.1, we have

$$\begin{aligned}
&[\delta(a_1+m+b_2)-\delta(a_1)-\delta(m)-\delta(b_2)]_{12}=0\\
&\text{and }[\delta(a_1+m+b_2)-\delta(a_1)-\delta(m)-\delta(b_2)]_{22}=0.
\end{aligned}\tag{21}$$

Let $t_1 \in \mathcal{T}_{11}$. Then

$$\begin{aligned}
&\delta(a_1+m+b_2)t_1+(a_1+m+b_2)\tau(t_1)+\delta(t_1)(a_1+m+b_2)+t_1\tau(a_1+m+b_2)\\
&=\delta[(a_1+m+b_2)t_1+t_1(a_1+m+b_2)]\\
&=\delta(a_1t_1+t_1a_1+t_1m)\\
&=\delta(a_1t_1+t_1a_1)+\delta(t_1m)\text{ (By Lemma 2.2)}\\
&=\delta(a_1t_1+t_1a_1)+\delta(t_1m+mt_1)+\delta(b_2t_1+t_1b_2)\\
&=\delta(a_1)t_1+a_1\tau(t_1)+\delta(t_1)a_1+t_1\tau(a_1)+\delta(t_1)m+t_1\tau(m)\\
&+\delta(m)t_1+m\tau(t_1)+\delta(b_2)t_1+b_2\tau(t_1)+\delta(t_1)b_2+t_1\tau(b_2)\\
&\implies[\delta(a_1+m+b_2)-\delta(a_1)-\delta(m)-\delta(b_2)]t_1=0\\
&\implies[\delta(a_1+m+b_2)-\delta(a_1)-\delta(m)-\delta(b_2)]_{11}t_1=0\\
&\implies[\delta(a_1+m+b_2)-\delta(a_1)-\delta(m)-\delta(b_2)]_{11}=0\\
&\text{(By condition }(*)\text{ of Theorem 2.1).}
\end{aligned}\tag{22}$$

Hence, by (21) and (22), $\delta(a_1+m+b_2)=\delta(a_1)+\delta(m)+\delta(b_2)$. □

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $t \in \mathcal{T}$ and $u \in \mathcal{T}$. Then $t = t_{11} + t_{12} + t_{22}$ and $u = u_{11} + u_{12} + u_{22}$ where $t_{ij}, u_{ij} \in \mathcal{T}_{ij}$ and $i,\ j \in \{1, 2\}$. Now,

$$\begin{aligned}
\delta(t+u) &= \delta((t_{11}+t_{12}+t_{22})+(u_{11}+u_{12}+u_{22})) \\
&= \delta((t_{11}+u_{11})+(t_{12}++u_{12})+(t_{22}+u_{22})) \\
&= \delta(t_{11}+u_{11})+\delta(t_{12}+u_{12})+\delta(t_{22}+u_{22}) \text{ (By Lemma 2.6)} \\
&= \delta(t_{11})+\delta(u_{11})+\delta(t_{12})+\delta(u_{12})+\delta(t_{22})+\delta(u_{22}) \\
&\quad \text{(By Lemma 2.4 and 2.5)} \\
&= \delta(t_{11})+\delta(t_{12})+\delta(t_{22})+\delta(u_{11})+\delta(u_{12})+\delta(u_{22}) \\
&= \delta(t_{11}+t_{12}+t_{22})+\delta(u_{11}+u_{12}+u_{22}) \text{ (By Lemma 2.6)} \\
&= \delta(t)+\delta(u).
\end{aligned} \tag{23}$$

Therefore, δ is additive.

Let $\mathcal{T}$ be 2-torsion free. By Theorem 3.1 of [6], τ is a Jordan derivation. For any $t \in \mathcal{T}$,

$$\begin{aligned}
&2\delta(t^2) = \delta(2t^2) = \delta(tt+tt) = 2[\delta(t)t + t\tau(t)] \\
&\implies \delta(t^2) = \delta(t)t + t\tau(t).
\end{aligned} \tag{24}$$

Thus, δ is a generalized Jordan derivation. □

The above motivates us to post Theorem 2.1 without assuming condition $(*)$ as a conjecture.

Conjecture 2.7. *Let $\mathcal{T}$ be a triangular ring. If a map $\delta : \mathcal{T} \to \mathcal{T}$ satisfies*

$$\delta(a_1b_1 + b_1a_1) = \delta(a_1)b_1 + a_1\tau(b_1) + \delta(b_1)a_1 + b_1\tau(a_1),$$

for every $a_1, b_1 \in \mathcal{T}$ where $\tau : \mathcal{T} \to \mathcal{T}$ satisfies

$$\tau(a_1b_1 + b_1a_1) = \tau(a_1)b_1 + a_1\tau(b_1) + \tau(b_1)a_1 + b_1\tau(a_1),$$

for every $a_1, b_1 \in \mathcal{T}$, then δ is additive. Moreover, if $\mathcal{T}$ is 2-torsion free, then δ becomes a generalized Jordan derivation.

3. Two-sided centralizers on triangular rings

Theorem 3.1. *Let $\mathcal{T}$ be a triangular ring with conditions:*

(i) $Aa_1 = 0$ *for some* $a_1 \in A$ *implies* $a_1 = 0$*;*
(ii) $b_1B = 0$ *for some* $b_1 \in B$ *implies* $b_1 = 0$*;*
(iii) $a_1A = 0$ *for some* $a_1 \in A$ *implies* $a_1 = 0$.

If a map $T : \mathcal{T} \to \mathcal{T}$ satisfies

$$T(a_1b_1) = T(a_1)b_1 = a_1T(b_1),$$

for all $a_1, b_1 \in \mathcal{T}$, then T is additive. Moreover, T is a two-sided centralizer.

Lemma 3.2. *Let $a_1 \in \mathcal{T}_{11}$, $b_2 \in \mathcal{T}_{22}$ and $m \in \mathcal{T}_{12}$. Then*

$$\begin{aligned} &(i)\ T(a_1+m) = T(a_1) + T(m); \\ &(ii)\ T(b_2+m) = T(b_2) + T(m). \end{aligned} \tag{25}$$

Proof. Let $t_2 \in \mathcal{T}_{22}$. Now, we have

$$T[(a_1+m)t_2] = T(a_1+m)t_2. \tag{26}$$

Also,

$$\begin{aligned} T[(a_1+m)t_2] = T(mt_2) &= 0 + T(mt_2) \\ &= T(a_1t_2) + T(mt_2) \\ &= T(a_1)t_2 + T(m)t_2. \end{aligned} \tag{27}$$

Comparing identities (26) and (27),

$$\begin{aligned} &[T(a_1+m) - T(a_1) - T(m)]t_2 = 0 \\ &\implies [T(a_1+m) - T(a_1) - T(m)]_{12}t_2 = 0 \\ &\text{and } [T(a_1+m) - T(a_1) - T(m)]_{22}t_2 = 0 \\ &\implies [T(a_1+m) - T(a_1) - T(m)]_{12} = 0 \text{ (By condition (2) of Definition 1.1)} \\ &\text{and } [T(a_1+m) - T(a_1) - T(m)]_{22} = 0 \text{ (By condition (ii) of Theorem 3.1).} \end{aligned} \tag{28}$$

Let $n \in \mathcal{T}_{12}$. We have

$$T[(a_1+m)n] = T(a_1+m)n. \tag{29}$$

Also,

$$T[(a_1+m)n] = T(a_1n) + 0 = T(a_1n) + T(mn) = T(a_1)n + T(m)n. \tag{30}$$

Comparing (29) and (30), we have

$$\begin{aligned} &[T(a_1+m) - T(a_1) - T(m)]n = 0 \\ &\implies [T(a_1+m) - T(a_1) - T(m)]_{11}n = 0 \\ &\implies [T(a_1+m) - T(a_1) - T(m)]_{11} = 0 \text{ (By condition (1) of Definition 1.1).} \end{aligned} \tag{31}$$

Hence, by using (28) and (31), we get (i) of Lemma 3.2. Similarly, we can prove (ii) of Lemma 3.2. □

Lemma 3.3. *Let $a_1 \in \mathcal{T}_{11}, b_2 \in \mathcal{T}_{22}$ and $m, n \in \mathcal{T}_{12}$. Then*

$$T(a_1m + nb_2) = T(a_1m) + T(nb_2). \tag{32}$$

Proof. Since $a_1m + nb_2 = (a_1+n)(b_2+m)$, we have

$$\begin{aligned} T(a_1m + nb_2) &= T((a_1+n)(b_2+m)) \\ &= (a_1+n)T(b_2+m) \\ &= (a_1+n)(T(b_2) + T(m)) \text{ (By (ii) of Lemma 3.2).} \end{aligned} \tag{33}$$

Also,

$$\begin{aligned} T(a_1m)+T(nb_2) &= T(a_1m)+T(nm)+T(nb_2)+T(a_1b_2) \\ &= a_1T(m)+nT(m)+nT(b_2)+a_1T(b_2) \\ &= (a_1+n)T(m)+(a_1+n)T(b_2) \\ &= (a_1+n)(T(m)+T(b_2)). \end{aligned} \tag{34}$$

Comparing (33) and (34), we have the desired result. □

Lemma 3.4. *Let $b_2 \in \mathcal{T}_{22}$ and $m, n \in \mathcal{T}_{12}$. Then*

$$T(mb_2+nb_2) = T(mb_2+T(nb_2). \tag{35}$$

Proof. Let $a_1 \in \mathcal{T}_{11}$. Then

$$T[a_1((m+n)b_2)] = a_1T((m+n)b_2). \tag{36}$$

Also,

$$\begin{aligned} T[a_1((m+n)b_2)] &= T(a_1(mb_2)+(a_1n)b_2) \\ &= T(a_1(mb_2))+T(a_1(nb_2)) \text{ (By Lemma 3.3)} \\ &= a_1T(mb_2)+a_1T(nb_2). \end{aligned} \tag{37}$$

Comparing (36) and (37), we have

$$\begin{aligned} & a_1[T(mb_2+nb_2)-T(mb_2)-T(nb_2)] = 0 \\ & \implies a_1[T(mb_2+nb_2)-T(mb_2)-T(nb_2)]_{11} = 0 \\ & \& \ a_1[T(mb_2+nb_2)-T(mb_2)-T(nb_2)]_{12} = 0 \\ & \implies [T(mb_2+nb_2)-T(mb_2)-T(nb_2)]_{11} = 0 \text{ (By condition (i) of Theorem 3.1)} \\ & \& \ [T(mb_2+nb_2)-T(mb_2)-T(nb_2)]_{12} = 0 \text{ (By condition (2) of Definition 1.1).} \end{aligned} \tag{38}$$

Let $p \in \mathcal{T}_{12}$. Then

$$T[p((m+n)b_2)] = pT((m+n)b_2). \tag{39}$$

Also,

$$\begin{aligned} T[p((m+n)b_2)] &= T[p(mb_2+nb_2)] \\ &= T(0) = 0 \\ &= T(p(mb_2))+T(p(nb_2)) \\ &= pT(mb_2)+pT(nb_2). \end{aligned} \tag{40}$$

Comparing (39) and (40), we get

$$\begin{aligned} & p[T(mb_2+nb_2)-T(mb_2)-T(nb_2)] = 0 \\ & \implies p[T(mb_2+nb_2)-T(mb_2)-T(nb_2)]_{22} = 0 \\ & \implies [T(mb_2+nb_2)-T(mb_2)-T(nb_2)]_{22} = 0 \text{ (By condition (1) of Definition 1.1).} \end{aligned} \tag{41}$$

Thus, the result follows from (38) and (41). □

Lemma 3.5. *Let $m_1, n_1 \in \mathcal{T}_{12}$. Then*

$$T(m_1 + n_1) = T(m_1) + T(n_1). \tag{42}$$

Proof. Let $b_2 \in \mathcal{T}_{22}$. Then

$$T((m_1 + n_1)b_2) = T(m_1 + n_1)b_2. \tag{43}$$

Now,

$$\begin{aligned} T((m_1 + n_1)b_2) &= T(m_1 b_2 + n_1 b_2) \\ &= T(m_1 b_2) + T(n_1 b_2) \text{ (By Lemma 3.4)} \\ &= T(m_1)b_2 + T(n_1)b_2. \end{aligned} \tag{44}$$

By (43) and (44),

$$\begin{aligned} &[T(m_1 + n_1) - T(m_1) - T(n_1)]b_2 = 0 \\ &\implies [T(m_1 + n_1) - T(m_1) - T(n_1)]_{12} b_2 = 0 \\ &\& \ [T(m_1 + n_1) - T(m_1) - T(n_1)]_{22} b_2 = 0 \\ &\implies [T(m_1 + n_1) - T(m_1) - T(n_1)]_{12} = 0 \text{ (By condition (2) of Definition 1.1)} \\ &\& \ [T(m_1 + n_1) - T(m_1) - T(n_1)]_{22} = 0 \text{ (By condition (ii) of Theorem 3.1).} \end{aligned} \tag{45}$$

Let $p \in \mathcal{T}_{12}$. Then

$$T((m_1 + n_1)p) = T(m_1 + n_1)p. \tag{46}$$

Also,

$$T((m_1 + n_1)p) = T(0) = 0 = T(m_1 p) + T(n_1 p) = T(m_1)p + T(n_1)p. \tag{47}$$

Comparing (46) and (47),

$$\begin{aligned} &[T(m_1 + n_1) - T(m_1) - T(n_1)]p = 0 \\ &\implies [T(m_1 + n_1) - T(m_1) - T(n_1)]_{11} p = 0 \\ &\implies [T(m_1 + n_1) - T(m_1) - T(n_1)]_{11} = 0 \text{ (By condition (1) of Definition 1.1).} \end{aligned} \tag{48}$$

By (45) and (48), we get the result. □

Lemma 3.6. *Let $a_1, a_2 \in \mathcal{T}_{11}$ and $b_1, b_2 \in \mathcal{T}_{22}$. Then*

$$\begin{aligned} &(i)\ T(a_1 + a_2) = T(a_1) + T(a_2); \\ &(ii)\ T(b_1 + b_2) = T(b_1) + T(b_2). \end{aligned} \tag{49}$$

Proof. Let $t_2 \in \mathcal{T}_{22}$. Then

$$T((a_1 + a_2)t_2) = T(a_1 + a_2)t_2. \tag{50}$$

Now,

$$T((a_1 + a_2)t_2) = T(0) = T(a_1 t_2) + T(a_2 t_2) = T(a_1)t_2 + T(a_2)t_2. \tag{51}$$

Comparing (50) and (51), we have

$$\begin{aligned}
&[T(a_1+a_2)-T(a_1)-T(a_2)]t_2=0\\
&\implies [T(a_1+a_2)-T(a_1)-T(a_2)]_{12}t_2=0\\
&\&\ [T(a_1+a_2)-T(a_1)-T(a_2)]_{22}t_2=0\\
&\implies [T(a_1+a_2)-T(a_1)-T(a_2)]_{12}=0 \text{ (By condition (2) of Definition 1.1)}\\
&\&\ [T(a_1+a_2)-T(a_1)-T(a_2)]_{22}=0 \text{ (By condition (ii) of Theorem 3.1).}
\end{aligned} \tag{52}$$

Let $m \in \mathcal{T}_{12}$. Then

$$T((a_1+a_2)m)=T(a_1+a_2)m. \tag{53}$$

Also,

$$\begin{aligned}
T((a_1+a_2)m) &= T(a_1m+a_2m)\\
&= T(a_1m)+T(a_2m) \text{ (By Lemma 3.5)}\\
&= T(a_1)m+T(a_2)m.
\end{aligned} \tag{54}$$

Comparing (53) and (54), we get

$$\begin{aligned}
&[T(a_1+a_2)-T(a_1)-T(a_2)]m=0\\
&\implies [T(a_1+a_2)-T(a_1)-T(a_2)]_{11}m=0\\
&\implies [T(a_1+a_2)-T(a_1)-T(a_2)]_{11}=0 \text{ (By condition (1) of Definition 1.1).}
\end{aligned} \tag{55}$$

By (52) and (55), we have (i) of Lemma 3.6. Similarly, we can prove (ii) of Lemma 3.6. □

Lemma 3.7. *Let $a_1 \in \mathcal{T}_{11}, b_2 \in \mathcal{T}_{22}$ and $m \in \mathcal{T}_{12}$. Then*

$$T(a_1+m+b_2)=T(a_1)+T(m)+T(b_2). \tag{56}$$

Proof. Let $x_1 \in \mathcal{T}_{11}$. Then

$$T((a_1+m+b_2)x_1)=T(a_1+m+b_2)x_1. \tag{57}$$

Also,

$$\begin{aligned}
T((a_1+m+b_2)x_1) &= T(a_1x_1+mx_1+b_2x_1)\\
&= T(a_1x_1)\\
&= T(a_1x_1)+T(mx_1)+T(b_2x_1)\\
&= T(a_1)x_1+T(m)x_1+T(b_2)x_1.
\end{aligned} \tag{58}$$

By (57) and (58), we have

$$\begin{aligned}
&[T(a_1+m+b_2)-T(a_1)-T(m)-T(b_2)]x_1=0\\
&\implies [T(a_1+m+b_2)-T(a_1)-T(m)-T(b_2)]_{11}x_1=0\\
&\implies [T(a_1+m+b_2)-T(a_1)-T(m)-T(b_2)]_{11}=0\\
&\text{(By condition (iii) of Theorem 3.1).}
\end{aligned} \tag{59}$$

Let $t_2 \in \mathcal{T}_{22}$. Then

$$T((a_1 + m + b_2)t_2) = T(a_1 + m + b_2)t_2 \tag{60}$$

Also,

$$\begin{aligned} T((a_1 + m + b_2)t_2) &= T(a_1t_2 + mt_2 + b_2t_2) \\ &= T(mt_2 + b_2t_2) \\ &= T(mt_2) + T(b_2t_2) \quad \text{[By (ii) of Lemma 3.2]} \\ &= T(a_1t_2) + T(mt_2) + T(b_2t_2) \\ &= T(a_1)t_2 + T(m)t_2 + T(b_2)t_2. \end{aligned} \tag{61}$$

Comparing (60) and (61), we get

$$\begin{aligned} &[T(a_1 + m + b_2) - T(a_1) - T(m) - T(b_2)]t_2 = 0 \\ &\implies [T(a_1 + m + b_2) - T(a_1) - T(m) - T(b_2)]_{12}t_2 = 0 \\ &\& \ [T(a_1 + m + b_2) - T(a_1) - T(m) - T(b_2)]_{22}t_2 = 0 \\ &\implies [T(a_1 + m + b_2) - T(a_1) - T(m) - T(b_2)]_{12} = 0 \text{ (By condition (2) of Definition 1.1)} \\ &\& \ [T(a_1 + m + b_2) - T(a_1) - T(m) - T(b_2)]_{22} = 0 \text{ (By condition (ii) of Theorem 3.1).} \end{aligned} \tag{62}$$

Thus, by (59) and (62), we have $T(a_1 + m + b_2) = T(a_1) + T(m) + T(b_2)$. □

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $t \in \mathcal{T}$ and $u \in \mathcal{T}$. Then $t = t_{11} + t_{12} + t_{22}$ and $u = u_{11} + u_{12} + u_{22}$ where $t_{ij}, u_{ij} \in \mathcal{T}_{ij}$ and $i,\ j \in \{1, 2\}$. We have

$$\begin{aligned} T(t + u) &= T((t_{11} + t_{12} + t_{22}) + (u_{11} + u_{12} + u_{22})) \\ &= T((t_{11} + u_{11}) + (t_{12} + u_{12}) + (t_{22} + u_{22})) \\ &= T(t_{11} + u_{11}) + T(t_{12} + u_{12}) + T(t_{22} + u_{22}) \text{ (By Lemma 3.7)} \\ &= T(t_{11}) + T(u_{11}) + T(t_{12}) + T(u_{12}) + T(t_{22}) + T(u_{22}) \\ &\quad \text{(By Lemma 3.5 and 3.6)} \\ &= T(t_{11}) + T(t_{12}) + T(t_{22}) + T(u_{11}) + T(u_{12}) + T(u_{22}) \\ &= T(t_{11} + t_{12} + t_{22}) + T(u_{11} + u_{12} + u_{22}) \text{ (By Lemma 3.7)} \\ &= T(t) + T(u). \end{aligned} \tag{63}$$

Hence, T is additive and T is a two-sided centralizer. □

4. (m, n)-Derivations on triangular rings

Theorem 4.1. *Let $m > 0, n > 0$ be integers and $\mathcal{T}$ be a triangular ring with conditions:*

(i) $a_1A = 0$ *for some* $a_1 \in A$ *implies* $a_1 = 0$*;*

(ii) $b_1 B = 0$ *for some* $b_1 \in B$ *implies* $b_1 = 0$;

(iii) $Aa_1 = 0$ *for some* $a_1 \in A$ *implies* $a_1 = 0$;

(iv) $\mathcal{T}$ *is* $mn(m+n)$*-torsion free.*

If a mapping $D : \mathcal{T} \to \mathcal{T}$ *satisfies*

$$(m+n)D(a_1 b_1) = 2mD(a_1)b_1 + 2na_1 D(b_1)$$

for all $a_1, b_1 \in \mathcal{T}$*, then* D *is additive. Moreover,* D *is an* (m,n)*-derivation.*

In this section, we frequently use all four conditions for the triangular ring $\mathcal{T}$, the condition on the map D described in Theorem 4.1 without mentioning them. Note that $D(0) = 0$. Before proving Theorem 4.1, we have some lemmas.

Lemma 4.2. *Let* $a_1 \in \mathcal{T}_{11}, p \in \mathcal{T}_{12}$ *and* $b_2 \in \mathcal{T}_{22}$*. Then*

$$\begin{aligned} &\text{(i) } D(a_1 + p) = D(a_1) + D(p); \\ &\text{(ii) } D(b_2 + p) = D(b_2) + D(p). \end{aligned} \tag{64}$$

Proof. Let $c \in \mathcal{T}_{22}$. Then

$$(m+n)D((a_1 + p)c) = 2mD(a_1 + p)c + 2n(a_1 + p)D(c). \tag{65}$$

Also,

$$\begin{aligned} (m+n)D((a_1 + p)c) &= (m+n)D(pc) \\ &= (m+n)[D(a_1 c) + D(pc)] \\ &= 2mD(a_1)c + 2na_1 D(c) + 2mD(p)c + 2npD(c). \end{aligned} \tag{66}$$

Comparing (65) and (66), we get

$$\begin{aligned} &2m[D(a_1 + p) - D(a_1) - D(p)]c = 0 \\ &\implies [D(a_1 + p) - D(a_1) - D(p)]c = 0 \text{ (By condition (iv) of Theorem 4.1)} \\ &\implies [D(a_1 + p) - D(a_1) - D(p)]_{12} c = 0 \\ &\& \ [D(a_1 + p) - D(a_1) - D(p)]_{22} c = 0 \\ &\implies [D(a_1 + p) - D(a_1) - D(p)]_{12} = 0 \text{ (By condition (2) of Definition 1.1)} \\ &\& \ [D(a_1 + p) - D(a_1) - D(p)]_{22} = 0 \text{ (By condition (ii) of Theorem 4.1).} \end{aligned} \tag{67}$$

Let $x_1 \in \mathcal{T}_{11}$. Then

$$(m+n)D((a_1 + p)x_1) = 2mD(a_1 + p)x_1 + 2n(a_1 + p)D(x_1). \tag{68}$$

Also,

$$\begin{aligned} (m+n)D((a_1 + p)x_1) &= (m+n)D(a_1 x_1) \\ &= (m+n)[D(a_1 x_1) + D(px_1)] \\ &= 2mD(a_1)x_1 + 2na_1 D(x_1) + 2mD(p)x_1 + 2npD(x_1). \end{aligned} \tag{69}$$

Comparing (68) and (69), we have

$$\begin{aligned}
&2m[D(a_1+p)-D(a_1)-D(p)]x_1=0\\
&\implies [D(a_1+p)-D(a_1)-D(p)]x_1=0 \text{ (By condition (iv) of Theorem 4.1)}\\
&\implies [D(a_1+p)-D(a_1)-D(p)]_{11}x_1=0\\
&\implies [D(a_1+p)-D(a_1)-D(p)]_{11}=0\\
&\text{(By condition (i) of Theorem 4.1).}
\end{aligned} \tag{70}$$

By (67) and (70), we have (i) of Lemma 4.2. Similarly, we can prove (ii) of Lemma 4.2. □

Lemma 4.3. *Let $a_1 \in \mathcal{T}_{11}$, $b_2 \in \mathcal{T}_{22}$ and $p, q \in \mathcal{T}_{12}$. Then*

$$D(a_1p+qb_2)=D(a_1p)+D(qb_2). \tag{71}$$

Proof. Since $(a_1p+qb_2)=(a_1+q)(p+b_2)+(p+b_2)(a_1+q)$,

$$\begin{aligned}
(m+n)D(a_1p+qb_2) &= (m+n)D((a_1+q)(p+b_2)+(p+b_2)(a_1+q))\\
&= 2mD(a_1+q)(p+b_2)+2n(a_1+q)D(p+b_2)\\
&= 2m(D(a_1)+D(q))(p+b_2)+2n(a_1+q)(D(p)+D(b_2))\\
&\text{(By Lemma 4.2)}\\
&= 2mD(a_1)p+2na_1D(p)+2mD(q)b_2+2nqD(b_2)\\
&+2mD(q)p+2nqD(p)+2mD(a_1)b+2naD(b_2)\\
&= (m+n)[D(a_1p)+D(qb_2)+D(qp)+D(a_1b_2)]\\
&= (m+n)[D(a_1p)+D(qb_2)].
\end{aligned} \tag{72}$$

Using condition (iv) of Theorem 4.1 and the above identity (72), we get the desired result. □

Lemma 4.4. *Let $b_2 \in \mathcal{T}_{22}$ and $p, q \in \mathcal{T}_{12}$. Then*

$$D(pb_2+qb_2)=D(pb_2)+D(qb_2). \tag{73}$$

Proof. Let $a_1 \in \mathcal{T}_{11}$. Then

$$(m+n)D[a_1((p+q)b_2)]=2mD(a_1)(p+q)b_2+2naD((p+q)b_2). \tag{74}$$

Also,

$$\begin{aligned}
(m+n)D[a_1((p+q)b_2)] &= (m+n)D[a_1(pb_2)+(a_1q)b_2]\\
&= (m+n)[D(a_1pb_2)+D(a_1qb_2)] \text{ (By Lemma 4.3)}\\
&= 2mD(a_1)(pb_2)+2na_1D(pb_2)+2mD(a_1)(qb_2)+2na_1D(qb_2).
\end{aligned} \tag{75}$$

Comparing (74) and (75), we have

$$\begin{aligned}
&2na_1[D((p+q)b_2)-D(pb_2)-D(qb_2)]=0\\
&\implies a_1[D((p+q)b_2)-D(pb_2)-D(qb_2)]=0 \text{ (By condition (iv) of Theorem 4.1)}\\
&\implies a_1[D(pb_2+qb_2)-D(pb_2)-D(qb_2)]_{11}=0\\
&\& \ a_1[D((pb_2+qb_2)-D(pb_2)-D(qb_2)]_{12}=0\\
&\implies [D(pb_2+qb_2)-D(pb_2)-D(qb_2)]_{11}=0 \text{ (By condition (iii) of Theorem 4.1)}\\
&\& \ [D((pb_2+qb_2)-D(pb_2)-D(qb_2)]_{12}=0 \text{ (By condition (2) of Definition 1.1).}
\end{aligned} \tag{76}$$

Let $s \in \mathcal{T}_{12}$. Then

$$(m+n)D[s(pb_2+qb_2)] = 2mD(s)(pb_2+qb_2)+2nsD(pb_2+qb_2). \tag{77}$$

Also,

$$\begin{aligned}
(m+n)D[s(pb_2+qb_2)] &= (m+n)D(0)\\
&= (m+n)D(spb_2)+(m+n)D(sqb_2)\\
&= 2mD(s)pb_2+2nsD(pb_2)+2mD(s)qb_2+2nsD(qb_2).
\end{aligned} \tag{78}$$

Comparing (77) and (78),

$$\begin{aligned}
&2ns[D(pb_2+qb_2)-D(pb_2)-D(qb_2)]=0\\
&\implies s[D(pb_2+qb_2)-D(pb_2)-D(qb_2)]=0 \text{ (By condition (iv) of Theorem 4.1)}\\
&\implies s[D(pb_2+qb_2-D(pb_2)-D(qb_2)]_{22}=0\\
&\implies [D(pb_2+qb_2)-D(pb_2)-D(qb_2)]_{22}=0 \text{ (By condition (1) of Definition 1.1).}
\end{aligned} \tag{79}$$

By (76) and (79), we have the desired result. □

Lemma 4.5. *Let $p, q \in \mathcal{T}_{12}$. Then*

$$D(p+q) = D(p)+D(q). \tag{80}$$

Proof. Let $b_2 \in \mathcal{T}_{22}$. Then

$$(m+n)D((p+q)b_2) = 2mD(p+q)b_2+2n(p+q)D(b_2). \tag{81}$$

Also,

$$\begin{aligned}
(m+n)D((p+q)b_2) &= (m+n)(D(pb_2)+D(qb_2)) \text{ (By Lemma 4.4)}\\
&= 2mD(p)b_2+2npD(b_2)+2mD(q)b_2+2nqD(b_2).
\end{aligned} \tag{82}$$

Comparing (81) and (82), we get

$$\begin{aligned}
&2m[D(p+q)-D(p)-D(q)]b_2=0\\
&\implies [D(p+q)-D(p)-D(q)]b_2=0 \text{ (By condition (iv) of Theorem 4.1)}\\
&\implies [D(p+q)-D(p)-D(q)]_{12}b_2=0\\
&\& \ [D(p+q)-D(p)-D(q)]_{22}b_2=0\\
&\implies [D(p+q)-D(p)-D(q)]_{12}=0 \text{ (By condition (2) of Definition 1.1)}\\
&\& \ [D(p+q)-D(p)-D(q)]_{22}=0 \text{ (By condition (ii) of Theorem 4.1).}
\end{aligned} \tag{83}$$

Let $s \in \mathcal{T}_{12}$. Then

$$(m+n)D((p+q)s) = 2mD(p+q)s + 2n(p+q)D(s). \tag{84}$$

Also,

$$\begin{aligned}(m+n)D((p+q)s) &= (m+n)D(0)\\ &= (m+n)(D(ps)+D(qs))\\ &= 2mD(p)s + 2npD(s) + 2mD(q)s + 2nqD(s).\end{aligned} \tag{85}$$

By (84) and (85), we have

$$\begin{aligned}&2m[D(p+q)-D(p)-D(q)]s = 0\\ &\implies [D(p+q)-D(p)-D(q)]s = 0 \text{ (By condition (iv) of Theorem 4.1)}\\ &\implies [D(p+q)-D(p)-D(q)]_{11}s = 0\\ &\implies [D(p+q)-D(p)-D(q)]_{11} = 0 \text{ (By condition (1) of Definition 1.1).}\end{aligned} \tag{86}$$

By (83) and (86), we get the desired result. □

Lemma 4.6. *Let $a_1, a_2 \in \mathcal{T}_{11}$ and $b_1, b_2 \in \mathcal{T}_{22}$. Then*

$$\begin{aligned}&\text{(i) } D(a_1+a_2) = D(a_1)+D(a_2);\\ &\text{(ii) } D(b_1+b_2) = D(b_1)+D(b_2).\end{aligned} \tag{87}$$

Proof. Let $t_2 \in \mathcal{T}_{22}$. Then

$$(m+n)D((a_1+a_2)t_2) = 2mD(a_1+a_2)t_2 + 2n(a_1+a_2)D(t_2). \tag{88}$$

Also,

$$\begin{aligned}(m+n)D((a_1+a_2)t_2) &= (m+n)D(0)\\ &= (m+n)(D(a_1t_2)+D(a_2t_2))\\ &= 2mD(a_1)t_2 + 2na_1D(t_2) + 2mD(a_2)t_2 + 2na_2D(t_2).\end{aligned} \tag{89}$$

Comparing (88) and (89), we get

$$\begin{aligned}&2m[D(a_1+a_2)-D(a_1)-D(a_2)]t_2 = 0\\ &\implies [D(a_1+a_2)-D(a_1)-D(a_2)]t_2 = 0 \text{ (By condition (iv) of Theorem 4.1)}\\ &\implies [D(a_1+a_2)-D(a_1)-D(a_2)]_{12}t_2 = 0\\ &\ \&\ [D(a_1+a_2)-D(a_1)-D(a_2)]_{22}t_2 = 0\\ &\implies [D(a_1+a_2)-D(a_1)-D(a_2)]_{12} = 0 \text{ (By condition (2) of Definition 1.1)}\\ &\ \&\ [D(a_1+a_2)-D(a_1)-D(a_2)]_{22} = 0 \text{ (By condition (ii) of Theorem 4.1).}\end{aligned} \tag{90}$$

Let $p \in \mathcal{T}_{12}$. Then

$$(m+n)D((a_1+a_2)p) = 2mD(a_1+a_2)p + 2n(a_1+a_2)D(p). \tag{91}$$

Also,

$$\begin{aligned}(m+n)D((a_1+a_2)p) &= (m+n)D(a_1p+a_2p)\\ &= (m+n)[D(a_1p)+D(a_2p)] \text{ (By Lemma 4.5)}\\ &= 2mD(a_1)p+2na_1D(p)+2mD(a_2)p+2na_2D(p).\end{aligned} \tag{92}$$

Using (91) and (92), we have

$$\begin{aligned}&2m[D(a_1+a_2)-D(a_1)-D(a_2)]p=0\\ &\implies [D(a_1+a_2)-D(a_1)-D(a_2)]p=0 \text{ (By condition (iv) of Theorem 4.1)}\\ &\implies [D(a_1+a_2)-D(a_1)-D(a_2)]_{11}p=0\\ &\implies [D(a_1+a_2)-D(a_1)-D(a_2)]_{11}=0 \text{ (By condition (1) of Definition 1.1)}\end{aligned} \tag{93}$$

By (90) and (93), we have (i) of Lemma 4.6. Similarly, we can prove (ii) of Lemma 4.6. □

Lemma 4.7. *Let $a_1 \in \mathcal{T}_{11}, p \in \mathcal{T}_{12}$ and $b_2 \in \mathcal{T}_{22}$. Then*

$$D(a_1+p+b_2) = D(a_1)+D(p)+D(b_2). \tag{94}$$

Proof. Let $x_1 \in \mathcal{T}_{11}$. Then

$$(m+n)D((a_1+p+b_2)x_1) = 2mD(a_1+p+b_2)x_1+2n(a_1+p+b_2)D(x_1). \tag{95}$$

Also,

$$\begin{aligned}(m+n)D((a_1+p+b_2)x_1) &= (m+n)D(a_1x_1)\\ &= (m+n)[D(a_1x_1)+D(px_1)+D(b_2x_1)]\\ &= 2mD(a_1)x_1+2na_1D(x_1)+2mD(p)x_1\\ &\quad +2npD(x_1)+2mD(b_2)x_1+2nb_2D(x_1)\end{aligned} \tag{96}$$

Comparing (95) and (96), we have

$$\begin{aligned}&2m[D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]x_1=0\\ &\implies [D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]x_1=0\\ &\text{(By condition (iv) of Theorem 4.1)}\\ &\implies [D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]_{11}x_1=0\\ &\implies [D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]_{11}=0\\ &\text{(By condition (i) of Theorem 4.1).}\end{aligned} \tag{97}$$

Let $t_2 \in \mathcal{T}_{22}$. Then

$$(m+n)D((a_1+p+b_2)t_2) = 2mD(a_1+p+b_2)t_2+2n(a_1+p+b_2)D(t_2). \tag{98}$$

Also,

$$\begin{aligned}(m+n)D((a_1+p+b_2)t_2) &= (m+n)D(pt_2+b_2t_2)\\ &= (m+n)[D(pt_2)+D(b_2t_2)] \text{ (By Lemma 4.2)}\\ &= (m+n)[D(a_1t_2)+D(pt_2)+D(b_2t_2)]\\ &= 2mD(a_1)t_2+2na_1D(t_2)+2mD(p)t_2\\ &\quad + 2npD(t_2)+2mD(b_2)t_2+2nb_2D(t_2).\end{aligned} \tag{99}$$

Comparing (98) and (99), we have

$$\begin{aligned}&2m[D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]t_2=0\\ &\implies [D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]t_2=0\\ &\text{(By condition (iv) of Theorem 4.1)}\\ &\implies [D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]_{12}t_2=0\\ &\ \&\ [D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]_{22}t_2=0\\ &\implies [D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]_{12}=0\\ &\text{(By condition (2) of Definition 1.1)}\\ &\ \&\ [D(a_1+p+b_2)-D(a_1)-D(p)-D(b_2)]_{22}=0\\ &\text{(By condition (ii) of Theorem 4.1).}\end{aligned} \tag{100}$$

By (97) and (100), we get the desired result. □

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $a, b \in \mathcal{T}$. Then $a = a_{11} + a_{12} + a_{22}$ and $b = b_{11} + b_{12} + b_{22}$ where a_{ij}, $b_{ij} \in \mathcal{T}_{ij}$.

$$\begin{aligned}D(a+b) &= D(a_{11}+a_{12}+a_{22}+b_{11}+b_{12}+b_{22})\\ &= D(a_{11}+b_{11}+a_{12}+b_{12}+a_{22}+b_{22})\\ &= D(a_{11}+b_{11})+D(a_{12}+b_{12})+D(a_{22}+b_{22}) \text{ (By Lemma 4.7)}\\ &= D(a_{11})+D(b_{11})+D(a_{12})+D(b_{12})+D(a_{22})+D(b_{22})\\ &\text{(By Lemma 4.5 and 4.6)}\\ &= D(a_{11})+D(a_{12})+D(a_{22})+D(b_{11})+D(b_{12})+D(b_{22})\\ &= D(a_{11}+a_{12}+a_{22})+D(b_{11}+b_{12}+b_{22}) \text{ (By Lemma 4.7)}\\ &= D(a)+D(b).\end{aligned} \tag{101}$$

Hence, D is additive. Thus, D is an (m, n)-derivation. □

Acknowledgement. The first author is thankful to the University Grants Commission (UGC), Govt. of India for financial supports under UGC Ref. No. 1256 dated 16/12/2019 and all authors are thankful to the Indian Institute of Technology Patna for providing the research facilities. Also, the authors would like to thank the anonymous referee(s) and the Editor of this journal for their valuable comments to improve the presentation of the paper.

Declarations

Data Availability Statement: The authors declare that [the/all other] data supporting the findings of this study are available within the article.
Disclosure statement: The authors report that there are no competing interests to declare.

References

[1] S. Ali and A. Fošner, *On generalized (m, n)-derivations and generalized (m, n)-Jordan derivations in rings*, Algebra Colloq., 21(3) (2014), 411-420.

[2] S. Aziz, A. Ghosh and O. Prakash, *Additivity of multiplicative (generalized) skew semi-derivations on rings*, Georgian Math. J., (2023), doi.org/10.1515/gmj-2023-2100.

[3] M. N. Daif, *When is a multiplicative derivation additive?*, Internat. J. Math. Math. Sci., 14(3) (1991), 615-618.

[4] M. S. T. El-Sayiad, M. N. Daif and V. D. Filippis, *Multiplicativity of left centralizers forcing additivity*, Bol. Soc. Parana. Mat. (3), 32(1) (2014), 61-69.

[5] B. L. M. Ferreira, *Multiplicative maps on triangular n-matrix rings*, Int. J. Math. Game Theory Algebr., 23(2) (2014), 1-14.

[6] B. L. M. Ferreira, *Jordan derivations on triangular matrix rings*, Extracta Math., 30(2) (2015), 181-190.

[7] A. Ghosh and O. Prakash, *New results on generalized (m,n)-Jordan derivations over semiprime rings*, Southeast Asian Bull. Math., 43(3) (2019), 323-331.

[8] A. Ghosh and O. Prakash, *Characterization of Jordan {g,h}-derivations over matrix algebras*, J. Algebr. Syst., 11(1) (2023), 77-95.

[9] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc., 8 (1957), 1104-1110.

[10] W. Jing and S. Lu, *Generalized Jordan derivations on prime rings and standard operator algebras*, Taiwanese J. Math., 7(4) (2003), 605-613.

[11] W. S. Martindale, III, *When are multiplicative mappings additive?*, Proc. Amer. Math. Soc., 21(3) (1969), 695-698.

[12] C. E. Rickart, *One-to-one mappings of rings and lattices*, Bull. Amer. Math. Soc., 54 (1948), 758-764.

[13] Y. Wang, *The additivity of multiplicative maps on rings*, Comm. Algebra, 37(7) (2009), 2351-2356.

[14] Y. Wang, *Additivity of multiplicative maps on triangular rings*, Linear Algebra Appl., 434(3) (2011), 625-635.

[15] J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math., 2 (1952), 251-261.

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