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# Position Vectors of Curves in Isotropic Space $I^3$ and Their Relation to the Frenet Frame

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Abstract. This paper investigates position vectors of arbitrary curves in isotropic 3-space (denoted by $I^3$). We first establish the relationship between a curve's position vector and the Frenet frame. Then, we derive a natural representation of any curve's position vector using curvature and torsion. Furthermore, we define various curves within isotropic space, including straight lines, plane curves, helices, general helices, Salkowski curves, and anti-Salkowski curves. Finally, graphical illustrations accompany illustrative examples to elucidate the discussed concepts.

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**Keywords:** Isotropic space, Frenet frame, position vector.

## 1. Introduction

The concept that our universe exhibits homogeneity and isotropy implies that its progression can be depicted as a sequential arrangement of three-dimensional space-like hypersurfaces, each displaying uniformity and isotropy. These hypersurfaces naturally serve as the surfaces of constant time. Homogeneity denotes uniform physical conditions across every point on a given hypersurface, while isotropy implies identical physical conditions in all directions from any point on the hypersurface. Isotropy inherently ensures homogeneity, although the reverse isn't necessarily true.

Homogeneous and isotropic spaces, boasting the ultimate level of symmetry with free translations and rotations in all three dimensions, strictly limit their possible geometries. These spaces fall into just three categories, each with a distinct curvature: flat, spherical with constant positive curvature, and hyperbolic with constant negative curvature.

Meanwhile, helices, captivating curves found throughout science and nature, have long captivated scientists. From the nanoscale (nano-springs and carbon nanotubes) to the biological realm (DNA, collagen, and bacterial flagella), helices weave their way into countless structures. Their presence extends to horns, vines, screws, and seashells, showcasing their diverse applications. Helices even find a home in the realm of fractal geometry (hyperhelices) and practical applications like computer-aided design (tool paths and kinematic simulations) and highway design

The curves arising from the solution of significant physical problems also hold significance in the theory of curves and surfaces. The fundamental theorem of curves asserts that curvatures uniquely determine curves, making curvature functions invaluable for extracting special and crucial information about curves. In this paper, our study delves into some special curves within Isotropic Space.

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As indicated in [3], the challenge of determining the position vector of a space curve relative to the Frenet frame remains unresolved in Euclidean space, posing a generally difficult problem to solve. However, certain special curves such as plane curves, helices, and slant helices have seen successful resolutions, as documented in [1, 2, 8].

In contrast, within the Galilean space $G^3$, the aforementioned problem has been addressed comprehensively for all curves as outlined in [19]. This study aims to extend these solutions to encompass all curves on a surface in $I^3$ concerning the Frenet Frame. The study of curves in geometric spaces has been a subject of fascination and inquiry for mathematicians and physicists alike. In particular, the exploration of curves within the Isotropic 3-space $I^3$ presents intriguing challenges and opportunities due to its unique geometric properties. In this paper, we delve into the investigation of position vectors of arbitrary curves within Isotropic space and their relationship with the Frenet frame.

Building on this approach, researchers have used similar methods with different moving reference frames, like the Frenet frame, to find the position vectors of special curves in various spaces, including familiar Euclidean space, Minkowski spacetime, Galilean space and Isotropic space in [6, 7, 9, 13, 16, 19].

The concept of the Frenet frame provides a powerful framework for understanding the behavior of curves in Euclidean space. It characterizes the orientation and curvature of a curve at each point along its path, offering valuable insights into its geometric properties. Extending this framework to Isotropic space introduces new complexities and nuances, prompting a deeper exploration of curve analysis in this setting.

Our study begins by establishing the position vector of a curve with respect to the Frenet frame in Isotropic 3-space. This foundational understanding serves as the cornerstone for our subsequent investigations. By determining the position vector in relation to the Frenet frame, we gain valuable insights into the intrinsic geometry of curves within Isotropic space.

Building upon this framework, we then derive the natural representation of the position vector of an arbitrary curve using curvature and torsion. These intrinsic properties of curves play a fundamental role in characterizing their shape and behavior, allowing for a concise and elegant description of curve dynamics in Isotropic space.

In addition to exploring the position vectors of arbitrary curves, we define various types of curves within Isotropic space, including straight lines, plane curves, helices, general helices, Salkowski curves, and anti-Salkowski curves. Each of these curve types exhibits distinct geometric properties, contributing to the richness and diversity of curves in Isotropic 3-space.

Throughout this paper, we provide examples and graphical illustrations to elucidate the concepts and methods discussed. These examples serve to illustrate the practical applicability of our findings and showcase the versatility of our approach in analyzing and understanding curves in Isotropic space.

In summary, this paper aims to contribute to the theoretical framework of curve analysis in Isotropic 3-space $I^3$. By investigating the position vectors of curves with respect to the Frenet frame and exploring various curve types, we endeavor to deepen our understanding of curves in this unique geometric setting. We hope that this study will inspire further research and exploration in the field of differential geometry and spatial mathematics.

## 2. Preliminaries

**2.1. The Simple Isotropic Space ($I_3^{(1)}$).** The simple isotropic geometry belongs to the real Cayley-Klein geometries in [10]. Its absolute consists of an ordered triple $w, f_1, f_2$, where $w$ represents the ideal (absolute) plane and $f_1$, $f_2$ denote a pair of complex conjugate lines in $w$.

The geometry of simple isotropic space $I_3^{(1)}$ has been extensively elucidated in [4, 17]. In particular, the scalar product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $I_3^{(1)}$ is defined as follows:

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} a_1.b_1 + a_2.b_2, & \text{if } a_i \neq 0 \text{ or } b_i \neq 0\,, (i = 1, 2), \\ a_3.b_3, & \text{if } a_i = 0 \text{ and } b_i = 0\,, (i = 1, 2). \end{cases}$$

For an admissible curve $\mathbf{c}\colon I \to I_3^{(1)}$, where $I \subseteq \mathbb{R}$ is parameterized by arc length, given by

$$\mathbf{c} = (x(s), y(s), z(s)),$$
$$\widetilde{\mathbf{c}} = (x(s), y(s)),$$

the curvature $\kappa(s)$ and torsion $\tau(s)$ are defined as follows:

$$\kappa(s) = \det(\widetilde{\mathbf{c}}'(s), \widetilde{\mathbf{c}}''(s)),$$
$$\tau(s) = \frac{\det(\mathbf{c}'(s), \mathbf{c}''(s), \mathbf{c}'''(s))}{\kappa^2(s)}.$$

The associated trihedron is given by

$$\mathbf{T} = (x'(s), y'(s), z'(s)),$$
$$\mathbf{N} = \frac{1}{\kappa(s)}(x''(s), y''(s), z''(s)),$$
$$\mathbf{B} = (0, 0, 1).$$

For these vector fields, the following Frenet's formulas hold:

$$\mathbf{T}' = \kappa \cdot \mathbf{N}, \quad \mathbf{N}' = -\kappa \cdot \mathbf{T} + \tau \cdot \mathbf{B}, \quad \mathbf{B}' = \mathbf{0}. \tag{2.1}$$

The general solution of Frenet's system in the simple isotropic space is provided in [15]. The geometry of simple isotropic space is detailed in [17].

2.2. **The Double Isotropic Space ($I_3^2$).** The double isotropic geometry is another instance of real Cayley-Klein geometries. Its absolute consists of an ordered triple $\{w, f, U\}$, where $w$ represents the ideal (absolute) plane, $f \subset w$ is an (absolute) line, and $U \in f$ denotes a point (the absolute point).

The geometry of the double isotropic space $I_3^2$ has been expounded in [5]. In particular, the scalar product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $I_3^2$ is defined as follows:

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} a_1 \cdot b_1, & a_1 \neq 0 \text{ or } b_1 \neq 0, \\ a_2 \cdot b_2, & a_1 = b_1 = 0 \text{ and } (a_2 \neq 0 \text{ or } b_2 \neq 0), \\ a_3 \cdot b_3, & a_1 = b_1 = a_2 = b_2 = 0. \end{cases}$$

For an admissible curve $\mathbf{c}\colon I \to I_3^{(2)}$, where $I \subseteq \mathbb{R}$ is parameterized by arc length $ds = dx$, given by

$$\mathbf{c}(x) = (x, y(x), z(x)),$$

the curvature $\kappa(x)$ and torsion $\tau(x)$ are defined as follows:

$$\kappa(x) = y''(x),$$
$$\tau(x) = \left(\frac{z''(x)}{y''(x)}\right)'.$$

The associated trihedron is given by

$$\mathbf{T} = (1, y'(x), z'(x)), \quad \mathbf{N} = (0, 1, \frac{z''}{y''}), \quad \mathbf{B} = (0, 0, 1).$$

For these vector fields, the following Frenet's formulas hold:

$$\mathbf{T}' = \kappa\mathbf{N}, \quad \mathbf{N}' = \tau\mathbf{B}, \quad \mathbf{B}' = 0.$$

The general solution of Frenet's system in the double isotropic space is given in [14].

## 3. Position Vector of a Curve in Isotropic Space

In this section, we will consider an arbitrary curve on a surface in $I^3$. We aim to analyze the position vector of the curve with respect to the Frenet frame in $I^3$.

**Theorem 3.1.** *The position vector $\alpha(s)$ of an arbitrary curve with respect to the Frenet frame in the Isotropic space $I^3$ is computed from the natural representation form:*

$$\begin{aligned}\alpha(\mathbf{s}) = &\Bigg(\int\Big[\int \kappa C_\kappa\, ds\Big]ds, \int\Big[\int \kappa S_\kappa\, ds\Big]ds,\\ &-\int\Big[\int\Big(\kappa C_\kappa \int S_\kappa\Big(\frac{\tau}{\kappa}\Big)' ds\Big)ds\Big]ds + \int\Big[\int\Big(\kappa S_\kappa \int C_\kappa\Big(\frac{\tau}{\kappa}\Big)' ds\Big)ds\Big]ds\Bigg),\end{aligned} \tag{3.1}$$

*where* $C_\kappa = \cos[\int \kappa\, ds]$ *and* $S_\kappa = \sin[\int \kappa\, ds]$.

*Proof.* If $\alpha(s)$ is an arbitrary curve in Isotropic space $I^3$, then the Frenet equations (2.1) hold.

$$\begin{aligned} \mathbf{T}' &= \kappa N, \\ \mathbf{N}' &= \tau B - \kappa T, \\ \mathbf{B}' &= 0. \end{aligned} \tag{3.2}$$

From Eq. (3.2), we obtain

$$\kappa T = \tau B - N'. \tag{3.3}$$

Dividing Eq. (3.3) by $\kappa$, we get

$$\mathbf{T} = \frac{\tau}{\kappa} B - \frac{1}{\kappa} N'.$$

Taking the derivative of Eq. (3.3) with respect to $s$, we obtain

$$\mathbf{T}' = \left(\frac{\tau}{\kappa}\right)' B - \left(\frac{1}{\kappa} N'\right)'. \tag{3.4}$$

By substituting Eq. (3.4) into Eq. (3.2), we get

$$\kappa \mathbf{N} = \left(\frac{\tau}{\kappa}\right)' B - \left(\frac{1}{\kappa} N'\right)'.$$

The above equation can be rewritten as

$$\mathbf{N}'' + \mathbf{N} = \left(\frac{\tau}{\kappa}\right)' B, \tag{3.5}$$

where $t$ is the new variable that equals $\int \kappa ds$. We know that $\mathbf{B} = (0, 0, 1)$. Therefore, Eq. (3.5) becomes

$$\begin{aligned} &(N_1)'' + (N_1) = 0, \\ &(N_2)'' + (N_2) = 0, \\ &(N_3)'' + (N_3) = \left(\frac{\tau}{\kappa}\right)'. \end{aligned} \tag{3.6}$$

Let $g(t) = \left(\frac{\tau}{\kappa}\right)'$. Solving Eq. (3.6), we find

$$\begin{aligned} N_1 &= \cos(t), \\ N_2 &= \sin(t), \\ N_3 &= -\cos(t)\left(\int \sin(t) g(t) dt\right) + \sin(t)\left(\int \cos(t) g(t) dt\right). \end{aligned}$$

Thus, we can rewrite $\mathbf{N}$ as follows:

$$\mathbf{N}(t) = \left(\cos(t), \sin(t), -\cos(t)\left(\int \sin(t) g(t) dt\right) + \sin(t)\left(\int \cos(t) g(t) dt\right)\right). \tag{3.7}$$

Substituting $t = \int \kappa ds$ and $g(t) = \left(\frac{\tau}{\kappa}\right)'$ into Eq. (3.7), we obtain

$$\mathbf{N} = \left(\cos\left[\int \kappa ds\right], \sin\left[\int \kappa ds\right], -\cos\left[\int \kappa ds\right]\left[\int \sin\left[\int \kappa ds\right]\left(\frac{\tau}{\kappa}\right)' ds\right] + \sin\left[\int \kappa ds\right]\left[\int \cos\left[\int \kappa ds\right]\left(\frac{\tau}{\kappa}\right)' ds\right]\right).$$

Integrating Equation (3.2) with respect to $s$, we have

$$\mathbf{T} = \left(\int \kappa C_\kappa ds, \int \kappa S_\kappa ds, -\int \kappa C_\kappa \left[\int S_\kappa \left(\frac{\tau}{\kappa}\right)' ds\right] + \int \kappa S_\kappa \left[\int C_\kappa \left(\frac{\tau}{\kappa}\right)' ds\right]\right) + \mathbf{d},$$

where $\mathbf{d}$ is a constant vector. Setting $\mathbf{d} = (0, 0, 0)$, we get

$$\mathbf{T} = \left(\int \kappa C_\kappa ds, \int \kappa S_\kappa ds, -\int \kappa C_\kappa \left[\int S_\kappa \left(\frac{\tau}{\kappa}\right)' ds\right] + \int \kappa S_\kappa \left[\int C_\kappa \left(\frac{\tau}{\kappa}\right)' ds\right]\right). \tag{3.8}$$

Integrating Equation (3.8) with respect to $s$, we obtain

$$\alpha(\mathbf{s}) = \left(\int\left[\int \kappa C_\kappa ds\right]ds, \int\left[\int \kappa S_\kappa ds\right]ds, -\int\left[\int \kappa C_\kappa\left[\int S_\kappa\left(\frac{\tau}{\kappa}\right)' ds\right] + \int \kappa S_\kappa\left[\int C_\kappa\left(\frac{\tau}{\kappa}\right)' ds\right]\right]ds\right).$$

This leads to Equation (3.1), and the proof is complete. □

## 4. Applications

**Definition 4.1.** We can say that $\alpha$ is called:

$$\begin{array}{ccc} \kappa,\tau & & \alpha \\ \hline \kappa \equiv 0 & \Longrightarrow & \textbf{a straight line.} \\ \tau \equiv 0 & \Longrightarrow & \textbf{a plane curve.} \\ \kappa \equiv cons.{>}0, \tau \equiv cons.{>}0 & \Longrightarrow & \textbf{a circular helix} \\ \frac{\tau}{\kappa} \equiv cons. & \Longrightarrow & \textbf{a generalized helix.} \\ \kappa \equiv cons., \tau \neq cons. & \Longrightarrow & \textbf{Salkowski curve [11,18].} \\ \kappa \neq cons., \tau \equiv cons. & \Longrightarrow & \textbf{anti-Salkowski curve [18].} \end{array} \tag{4.1}$$

**Corollary 4.2.** *The position vector $\alpha(s)$ of a straight line in the Isotropic space $I^3$ is given by*

$$\alpha(\mathbf{s}) = \left(d_1 s + d_2, d_3 s + d_4, d_5 s + d_6\right),$$

*where $d_1, d_2, d_3, d_4, d_5$ and $d_6$ are constants.*

*Proof.* By using $\kappa \equiv 0$ in the equation (4.1), we obtain the above equation. □

**Corollary 4.3.** *The position vector $\alpha(s)$ of a plane curve in the Isotropic space $I^3$ is given by*

$$\alpha(\mathbf{s}) = \left(\int\left[\int \kappa cos(\int \kappa ds)\, ds\right]ds, \int\left[\int \kappa sin(\int \kappa ds)\, ds\right]ds,\right.$$
$$\left. -\int\left[\int \kappa cos(\int \kappa ds) d_1 ds\right]ds + \int\left[\int \kappa sin(\int \kappa ds) d_2 ds\right]ds,\right)$$

*where $d_1$ and $d_2$ are constants.*

*Proof.* By using $\tau \equiv 0$ in the equation (4.1), we obtain the above equation. □

**Corollary 4.4.** *The position vector $\alpha(s)$ of a circular helix in the Isotropic space $I^3$ is given by*

$$\alpha(\mathbf{s}) = \left(-\frac{1}{\kappa}cos[\kappa s + d_1] + d_2 s + d_3, -\frac{1}{\kappa}sin[\kappa s + d_4] + d_5 s + d_6, \frac{d_7}{\kappa}cos[\kappa s + d_8] - \frac{d_9}{\kappa}sin[\kappa s + d_{10}] + d_{11}s + d_{12}\right),$$

*where $d_1$, $d_2$ , $d_3$, $d_4$ , $d_5$, $d_6$, $d_7$, $d_8$, $d_9$, $d_{10}$, $d_{11}$ and $d_{12}$ are constants.*

*Proof.* By using $\kappa \equiv$cons. and $\tau \equiv$ cons. in the equation (4.1), we obtain the above equation. □

**Example 4.5.** Assuming $\kappa = 1$ and $\tau = 1$, and choosing the integration constants as $d_1 = d_3 = d_4 = d_6 = d_8 = d_{10} = d_{12} = 0, d_2 = d_5 = d_7 = d_9 = d_{11} = 1$, the curve is obtained as follows (Figure 1).

$$\alpha(s) = \left(-cos[s] + s, -sin[s] + s, cos[s] - sin[s] + s\right).$$

**Corollary 4.6.** *The position vector $\alpha$ of a generalized helix in the isotropic space $I^3$ is given by*

$$\alpha(\mathbf{s}) = \left(-\frac{1}{\kappa}cos[\int \kappa ds] + d_1 s + d_2, -\frac{1}{\kappa}sin[\int \kappa ds] + d_3 s + d_4, \frac{d_5}{\kappa}cos[\int \kappa ds] + d_6 s + d_7 - \frac{d_8}{\kappa}sin[\int \kappa ds]\right)$$

*where $d_1$, $d_2$, $d_3$, $d_4$, $d_5$,$d_6$, $d_7$ and $d_8$ are constants.*

*Proof.* By using $\left(\frac{\tau}{\kappa}\right)' \equiv 0$ in the equation (4.1), we obtain the equation. □

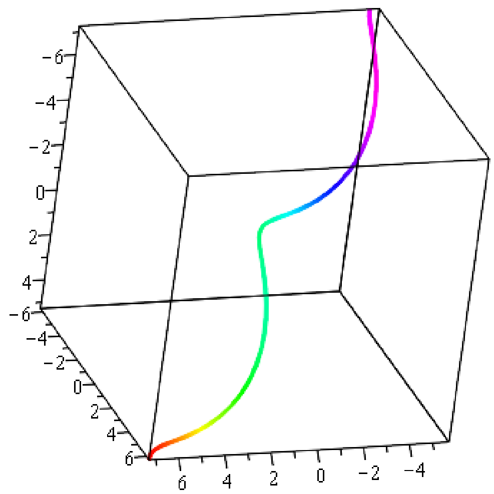


FIGURE 1. Curve with $\kappa \equiv 1$ and $\tau \equiv 1$

**Example 4.7.** If we take $\kappa \equiv s$ and $\tau \equiv s$, and the integration constants as $d_1 = d_3 = d_5 = d_6 = d_8 = 1, d_2 = d_4 = d_7 = 0$, the curve is as follows (Figure 2)

$$\alpha(s) = \left( -\frac{1}{s}cos[\frac{s^2}{2}] + s, -\frac{1}{s}sin[\frac{s^2}{2}] + s, \frac{1}{s}cos[\frac{s^2}{2}] - \frac{1}{s}sin[\frac{s^2}{2}] + s \right).$$

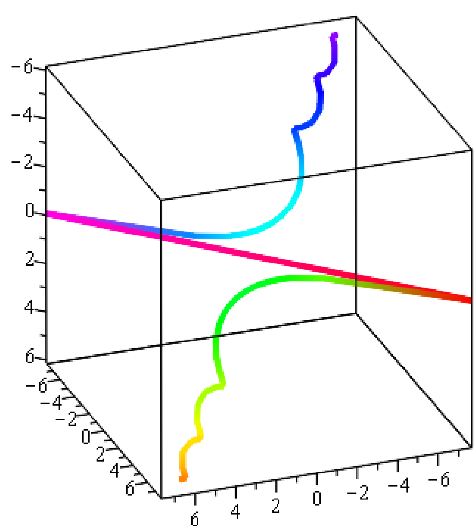


FIGURE 2. Curve with $\kappa$=s and $\tau$=s

**Corollary 4.8.** *The position vector $\alpha$ of a Salkowski curve in the isotropic space $I^3$ is given by*

$$\alpha(\mathbf{s}) = \Big( -\frac{1}{\kappa}cos[\kappa s + d_1] + d_2 s + d_3, -\frac{1}{\kappa}sin[\kappa s + d_4] + d_5 s + d_6, \\ -\int\Big[\int cos[\kappa s + d_7]\Big[\int sin[\kappa s + d_8]\tau' ds\Big]ds\Big]ds \\ +\int\Big[\int sin[\kappa s + d_9]\Big[\int cos[\kappa s + d_{10}]\tau' ds\Big]ds\Big]ds\Big),$$

*where $d_1$, $d_2$, $d_3$, $d_4$, $d_5$, $d_6$, $d_7$, $d_8$, $d_9$ and $d_{10}$ are integral constants.*

*Proof.* If we take $\kappa$ as constant and $\tau$ as non-constant in (4.1), then the equation above is obtained. □

**Example 4.9.** If we take $\kappa = 1$ and $\tau = s$, and the integration constants as $d_1$=$d_3$=$d_4$=$d_6$=$d_7$=$d_9$=0 and $d_2$=$d_5$=1, the curve is as follows (Figure 3)

$$\alpha(s) = \Big( -cos[s] + s, -sin[s] + s, \frac{s^2}{2}\Big).$$

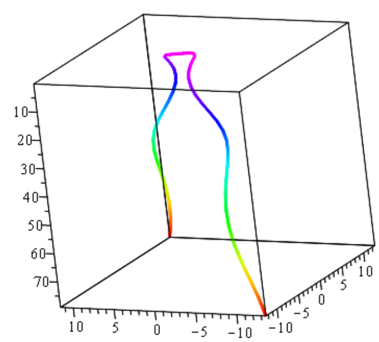


Figure 3. Curve with $\kappa$=1 and $\tau$=s

**Corollary 4.10.** *The position vector $\alpha$ of a anti-salkowski curve in the isotropic space $I^3$ is given by*

$$\alpha(\mathbf{s}) = \Big( -\frac{1}{\kappa}cos[\int \kappa ds] + d_1 s + d_2, -\frac{1}{\kappa}sin[\int \kappa ds] + d_3 s + d_4, \\ -\tau\int\Big[\kappa cos[\int \kappa ds]\Big[\int sin[\int \kappa ds]\Big(\frac{1}{\kappa}\Big)' ds\Big]ds\Big]ds \\ +\tau\int\Big[\kappa sin[\int \kappa ds]\Big[\int cos[\int \kappa ds]\Big(\frac{1}{\kappa}\Big)' ds\Big]ds\Big]ds\Big),$$

*where $d_1$ , $d_2$, $d_3$ and $d_4$ are constants.*

*Proof.* If we take $\kappa \not\equiv$ *cons.* and $\tau \equiv$cons. in equation (3.1), the above equation is obtained. □

**Definition 4.11.** Let $\alpha$ be a Frenet curve of order 3 of $I_3$. For $\tau(s) \neq 0$, $\alpha$ is a Bertrand curve if and only if there exist a linear relation

$$A\kappa(s) + B\tau(s) = 1, \tag{4.2}$$

where $A$, $B$ are non-zero constant and $\kappa(s)$ and $\tau(s)$ are the curvature functions of $\alpha$ [12].

**Corollary 4.12.** *The position vector $\alpha$ of a Bertrand curve in the isotropic space $I^3$ is given by*

$$\alpha(s) = \Big( \int \Big[ \int \kappa C_\kappa \, ds \Big] ds, \int \Big[ \int \kappa S_\kappa \, ds \Big] ds,$$
$$- \int \Big[ \int \Big( \kappa C_\kappa \int S_\kappa \Big( \frac{1 - A\kappa}{B\kappa} \Big)' ds \Big) ds \Big] ds$$
$$+ \int \Big[ \int \Big( \kappa S_\kappa \int C_\kappa \Big( \frac{1 - A\kappa}{B\kappa} \Big)' ds \Big) ds \Big] ds \Big),$$

*where A and B are constants other than zero.*

*Proof.* If we use equation (4.2) in (3.1), the above equation is obtained. □

## 5. Conclusion

In this study, we have explored the position vectors of arbitrary curves within the Isotropic 3-space $I^3$ and their relationship with the Frenet frame. Our investigation began with the determination of the position vector of a curve with respect to the Frenet frame, providing a foundational understanding of how curves behave within this unique space.

Furthermore, we elucidated the natural representation of the position vector of an arbitrary curve in terms of curvature and torsion. This representation not only allows for a concise description of the curve's behavior but also provides insights into its geometric properties.

Moreover, we extended our exploration by defining various types of curves within Isotropic space, including straight lines, plane curves, helices, general helices, Salkowski curves, and anti-Salkowski curves. Each of these curve types offers distinct characteristics and geometric features, enriching our understanding of curves in Isotropic space.

Through the examination of examples and graphical illustrations, we have demonstrated the applicability of our findings and concepts discussed throughout this paper. These examples serve to illustrate the versatility and utility of our methods in analyzing and understanding curves in Isotropic space.

In conclusion, this study contributes to the theoretical framework of curve analysis within Isotropic 3-space $I^3$. By investigating the position vectors of curves with respect to the Frenet frame and exploring various curve types, we have advanced our understanding of curves in this unique geometric setting. We hope that this paper will serve as a valuable resource for further research in the field of differential geometry and spatial mathematics.

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## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

## Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

## References

[1] Ali, A.T., *Position vectors of spacelike general helices in Minkowski 3-space*, Nonlin. Anal. Theory Meth. Appl., **73**(2010), 1118–1126.
[2] Ali, A.T., *Position vectors of slant helices in Euclidean 3-space*, Journal of the Egyptian Mathematical Society, **20**(1)(2012), 1–6.
[3] Ali, A.T., *Position vectors of curves in the Galilean space $G^3$*, Matematički Vesnik, **64**(3)(2012), 200–210.
[4] Aydın, M.E., *A generalization of translation surfaces with constant curvature in the isotropic space*, J. Geom., **107**(2016), 603–615.
[5] Brauner, H., *Geometrie des Zweifach Isotropen Raumes I*, J. Reine Angew. Math., **224**(1966), 118–146.
[6] Erjavec, Z., Divjak, B., Horvat, D. *The general solutions of Frenet's system in the equiform geometry of the Galilean, pseudo-Galilean , simple isotropic and double isotropic space*, International Mathematical Forum, **6**(17)(2011), 837–856.
[7] Güzelkardeşler, G., Şahiner, B., *An alternative method for determination of the position vector of a slant helix*, Journal of New Theory, **44**(2023), 97–105.
[8] Izumiya, S., Takeuchi,N., *New special curves and developable surfaces*, Turk. J. Math., **28**(2004), 531–537.

[9] Kızıltuğ, S., Kaya, S., Tarakçı, Ö., *The slant helices according to type-2 bishop frame in Euclidean 3-space*, International Journal of Pure and Applied Mathematics, **85**(2)(2013), 211–222.
[10] Molnar, E., *The projective interpretation of the eight 3-dimensional homogeneous geometries*, Beitr. Algebra Geom., **38**(1997), 261–288.
[11] Monterde, J., *Salkowski curves revisted: A family of curves with constant curvature and non-constant torsion*, Comput. Aided Geomet. Design, **26**(2009), 271–278.
[12] Osman Öğrenmiş, A., Külahçı, M., Bektaş, M., *A Survey for some special curves in isotropic space*,Physical Review Research International, **3**(4)(2013), 321–329.
[13] Özyurt, G., Position Vectors and Characterization of Curves in the Isotropic Space, Master Thesis, Institute of Sciences, Amasya University, Amasya, 2021.
[14] Pavkovic, B.J., Kamenarovic, I., *The general solution of the Frenet's system in the doubly isotropic space* $I_3^{(2)}$, Rad JAZU, **428**(1987), 17–24.
[15] Pavković, B.J., Kamenarović, I., *The equiform differential geometry of curves in the Galilean space*, Glasnik Matematikči, **22**(42)(1987), 449–457.
[16] Pavković, B.J., *The general solution of the Frenet system of differential equations for curves in the Galilean space* $G^3$, Rad HAZU Math., **450** (1990), 123–128.
[17] Sachs, H., Isotrope Geometrie des Raumes, Vieweg, Braunschweig/ Wiesbaden, 1990.
[18] Salkowski, E., *Zur transformation von raumkurven*, Math. Ann., **66**(1909), 517–557.
[19] Şahin, T., Ceylan Dirişen, B., *Position vectors of curves with respect to Darboux frame in the Galilean space* $G^3$, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **68**(2)(2019), 2079–2093.