Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 54 (1) (2025), 291 – 318
DOI : 10.15672/hujms.1489598

Research Article

On the estimation of the stress-strength parameter for the inverse Lindley distribution based on lower record values

Bahareh Etemad Golestani[1], Ehsan Ormoz*[1], S.M.T.K. MirMostafaee[2]

[1] *Department of Mathematics and Statistics, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
[2] *Department of Statistics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*

Abstract

In this article, we consider the estimation of the stress-strength reliability parameter for the inverse Lindley distribution based on lower record values. The maximum likelihood estimator and its asymptotic distribution are obtained. An approximate classical confidence interval, as well as two bootstrap-type confidence intervals for the reliability parameter are derived. The Bayesian inference for the parameter has been considered using Tierney and Kadane's approximation method, as well as two Monte Carlo methods, namely the Metropolis-Hastings and importance sampling techniques under both symmetric and asymmetric loss functions. Besides, the Chen and Shao shortest width credible intervals are constructed for the stress-strength parameter. A simulation study and a real data example are conducted to explore and compare the performances of the presented results.

Mathematics Subject Classification (2020). 62F10, 62C10

Keywords. Bayes estimation, bootstrap confidence interval, inverse Lindley distribution, record values, stress-strength parameter

1. Introduction

Lower record values and upper record values deal with identifying the lowest and the highest values observed in a data set, respectively. These concepts are commonly used in statistical analysis to understand the extremes of a data set and can provide insights into the variability and distribution of the data. Applications of lower and upper record statistics can be found in various fields such as reliability, meteorology, sports science, finance, industrial stress testing, mortality studies, medicine and quality control. In sports science, these statistics are commonly used to track athletes' personal bests and performance records, see for example [55]. In mortality studies, lower record values can provide insights into extreme cases of mortality, identify vulnerable populations and help researchers find factors contributing to mortality rates. Lower record statistics can also be used in financial analysis to determine the lowest values of stock prices, interest rates, or other financial indicators, which can help investors and financial analysts understand the risks associated

*Corresponding Author.
Email addresses: etemad.bahar@gmail.com (B. Etemad Golestani), Ehsanormoz@mshdiau.ac.ir (E. Ormoz), m.mirmostatafaee@umz.ac.ir (S.M.T.K. MirMostafaee)
Received: 25.05.2024; Accepted: 24.12.2024

with investing in certain assets or markets. Moreover, lower record statistics are crucial in monitoring environmental conditions such as temperature, rainfall, pollution levels, etc. By tracking the lowest values in these parameters, researchers can discover trends, anomalies and potential environmental hazards. Let $\{X_j,\ j \geq 1\}$ be a sequence of independent identically distributed (iid) continuous random variables and $X_{L(m)} = \min\{X_1, \ldots, X_m\}$, $m \geq 1$, then an observation X_j is a lower record value of $\{X_{L(m)},\ m \geq 1\}$ if it becomes less than all its preceding observations. Equivalently, we can say that $X_j < X_i$ for all values i that are less than j and we have

$$L(1) = 1, \quad L(m) = \min\{j|j > L(m-1), X_j < X_{L(m-1)}\}.$$

The sequence $\{L(m),\ m \geq 1\}$ is called the record times. See [4] for more details on the theory and applications of record values.

The inverse Lindley distribution was introduced by [53]. Let Y be a Lindley distributed random variable with parameter θ whose density can be expressed as follows

$$f(y;\theta) = \frac{\theta^2}{1+\theta}(1+y)\mathrm{e}^{-\theta y}, \quad y > 0, \quad \theta > 0.$$

Then, the random variable $X = \dfrac{1}{Y}$ is said to have the inverse Lindley distribution and its probability density function (PDF) is then given by

$$f(x;\theta) = \frac{\theta^2}{1+\theta}\Big(\frac{1+x}{x^3}\Big)\mathrm{e}^{-\frac{\theta}{x}}, \quad x > 0, \quad \theta > 0. \tag{1.1}$$

We write $X \sim ILD(\theta)$ if the PDF of X can be written as (1.1). The corresponding cumulative distribution function (CDF) is also given by

$$F(x;\theta) = \Big(1 + \frac{\theta}{(1+\theta)x}\Big)\mathrm{e}^{-\frac{\theta}{x}}, \quad x > 0, \quad \theta > 0. \tag{1.2}$$

Moreover, the hazard rate function of X is given by

$$h(x;\theta) = \frac{\theta^2(1+x)\mathrm{e}^{-\frac{\theta}{x}}}{(1+\theta)x^3\left[1 - \left(1 + \frac{\theta}{(1+\theta)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right]}, \quad x > 0, \quad \theta > 0.$$

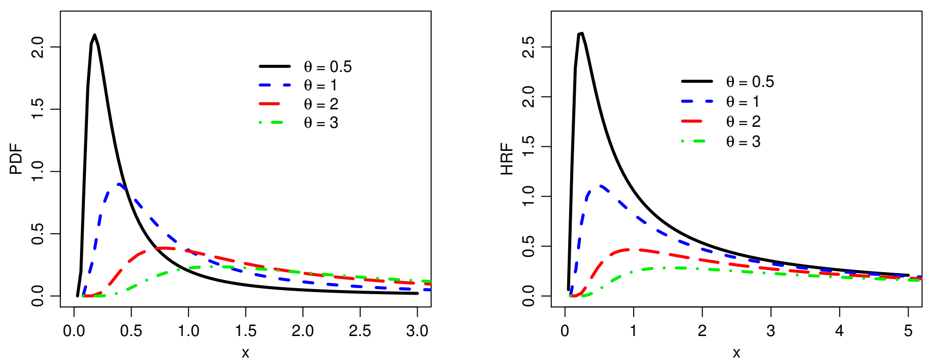

Figure 1. PDFs and HRFs of the inverse Lindley distribution for selected values of θ.

Figure 1 presents the PDFs and HRFs of the inverse Lindley model for selected values of θ. As it can be seen from Figure 1, the HRF of the inverse Lindley distribution involves an upside-down bathtub (UBT) shape. The UBT-shaped distributions may be applied to many real-life situations and experiments, see for example [24]. Langlands et al. [35] analyzed the breast carcinoma data and found out that the mortality rates peaked at a time and then exhibited a gradual decline. Bennett [13] studied lung cancer trial data which revealed the failure rates followed a UBT-shaped pattern. Joo and Mi [30] emphasized that the HRF of a series system formed by the two components could be UBT-shaped. The inverse Lindley distribution with a UBT-shaped HRF, also enjoys the advantage of possessing simple forms for its PDF and CDF with only one parameter. This feature was also highlighted by Alotaibi et al. [3] who underlined that this admirable characteristic smoothes out the mathematical difficulties and then focused on the inference for the inverse Lindley distribution based on adaptive Type II progressively censored data. Many other researchers also worked on the inferential problems related to the inverse Lindley distribution, see for example [11, 12, 17, 25, 28, 37] and [27]. Recently, Asgharzadeh et al. [5] addressed the problem of estimating the PDF and CDF of the inverse Lindley model based on different methods of estimation.

A system or a component works in a reliability stress-strength model, provided that the stress does not become bigger than the strength. Let X denote the strength and Y denote the stress, where X and Y are statistically independent, then the probability that the system or the component works becomes $R = P(Y < X)$. In addition, parameter R may be employed as a measure of comparison of two independent populations. For example, if we are interested in studying monthly rainfall related to two cities, say city A and city B, then we may be curious to know the probability that city A receives more rainfall amount than city B. In medicine, when the random variables X and Y are the number of cancer patients treated with two different chemotherapy methods, R can be used as a measure to identify the more effective treatment, see [44]. The parameter R is called the stress-strength parameter or the reliability parameter, which has many applications in many fields like life testing, reliability, military, medical sciences, economics, social sciences, psychology and engineering.

The term stress-strength was first used by [20], who worked on the estimation of R for normal variates. Since then, many studies on the estimation of R for various distributions have been accomplished by many authors, see [21, 31, 44, 48, 55] for examples of recent studies on this parameter. In recent years, the subject of estimating the stress-strength parameter based on record statistics has absorbed many scientists, see for example [8] and [62] for the generalized exponential distribution, [7, 9] and [10] for the two-parameter exponential distribution, [43] for the Kumaraswamy distribution, [39] for the Lomax distribution, [56] and [1] for the Chen distribution, [57] for the Burr Type X distribution, [6] for the two-parameter generalized exponential distribution, [34] and [19] for the Burr Type XII distribution, [32] for the Parero distribution, [59] for the Gompertz distribution, and [55] for the exponential power distribution.

Let X and Y be two independent random variables, where X follows an inverse Lindley distribution with parameter θ_1 and Y follows an inverse Lindley model with parameter θ_2, then the reliability parameter $R = P(Y < X)$ may be obtained as follows:

$$\begin{aligned}
R = \mathcal{R}(\theta_1, \theta_2) &= \int_0^\infty P(Y < X | X = x) f_X(x) \mathrm{d}x \\
&= \frac{\theta_1^2 \theta_2}{(1+\theta_1)(1+\theta_2)} \int_0^\infty \frac{1}{x^4} \mathrm{e}^{-\frac{\theta_1+\theta_2}{x}} \mathrm{d}x + \frac{\theta_1^2(1+2\theta_2)}{(1+\theta_1)(1+\theta_2)} \int_0^\infty \frac{1}{x^3} \mathrm{e}^{-\frac{\theta_1+\theta_2}{x}} \mathrm{d}x \\
&\quad + \frac{\theta_1^2}{1+\theta_1} \int_0^\infty \frac{1}{x^2} \mathrm{e}^{-\frac{\theta_1+\theta_2}{x}} \mathrm{d}x \\
&= \frac{\theta_1^2 \{2\theta_2 + (1+2\theta_2)(\theta_1+\theta_2) + (1+\theta_2)(\theta_1+\theta_2)^2\}}{(1+\theta_1)(1+\theta_2)(\theta_1+\theta_2)^3} = \frac{C\theta_1^2}{A_{113}}, \qquad (1.3)
\end{aligned}$$

where

$$A_{ijk} = (1+\theta_1)^i (1+\theta_2)^j (\theta_1+\theta_2)^k, \tag{1.4}$$

$$C = 2\theta_2 + (1+2\theta_2)A_{001} + A_{012}. \tag{1.5}$$

Sharma et al. [53] and Hassan et al. [28] discussed the problem of estimation of R for the inverse Lindley distribution based on simple random samples and based on ranked set samples, respectively. This paper aims to estimate the stress-strength parameter $R = P[Y < X]$ when the strength X and the stress Y are two independent random variables from the inverse Lindley distribution based on lower record values.

The outline of this paper is organized as follows: The maximum likelihood (ML) estimation of R is discussed in Section 2. In Section 3, the asymptotic confidence interval and the bootstrap confidence intervals of R are obtained. Section 4 is devoted to the Bayesian point and interval estimation of R. The Bayesian point estimates of R are found under one symmetric and two asymmetric loss functions. However, as it seems that the related integrals cannot be expressed in closed forms, three well-known methods are utilized to approximate the Bayes estimates. A simulation study is also conducted in Section 5 to check the performances of the proposed classical and approximate Bayes estimators. A real data example in Section 6 is presented for the purpose of illustration. Finally, several remarks end the paper in Section 7.

2. Maximum likelihood estimation

Let $\boldsymbol{R} = (R_1, \cdots, R_n)$ be the first n lower record values extracted from an inverse Lindley distribution with parameter θ_1 and $\boldsymbol{S} = (S_1, \cdots, S_m)$ be the first m lower record values extracted from an inverse Lindley distribution with parameter θ_2. Suppose that $\boldsymbol{r} = (r_1, \dots, r_n)$ is the observed set of $\boldsymbol{R}$ and $\boldsymbol{s} = (s_1, \dots, s_m)$ is the observed set of $\boldsymbol{S}$. Suppose further that $\boldsymbol{R}$ and $\boldsymbol{S}$ are statistically independent. Due to the independence of $\boldsymbol{R}$ and $\boldsymbol{S}$, the likelihood function of $\boldsymbol{\theta} = (\theta_1, \theta_2)$ given $\boldsymbol{r}$ and $\boldsymbol{s}$, can be written as

$$L(\theta|\boldsymbol{r}, \boldsymbol{s}) = L_1(\theta_1|\boldsymbol{r}) L_2(\theta_2|\boldsymbol{s}),$$

where

$$L_1(\theta_1|\boldsymbol{r}) = f(r_n;\theta_1) \prod_{i=1}^{n-1} \frac{f(r_i;\theta_1)}{F(r_i;\theta_1)},$$

and

$$L_2(\theta_2|\boldsymbol{s}) = g(s_m;\theta_2) \prod_{j=1}^{m-1} \frac{g(s_j;\theta_2)}{G(s_j;\theta_2)},$$

in which $f(\cdot;\theta_1)$ and $F(\cdot;\theta_1)$ are the PDF and CDF of $ILD(\theta_1)$, respectively, and $g(\cdot;\theta_2)$ and $G(\cdot;\theta_2)$ are the PDF and CDF of $ILD(\theta_2)$, respectively. Thus, from (1.1) and (1.2), we have

$$L_1(\theta_1|\boldsymbol{r}) = \frac{\theta_1^{2n}}{1+\theta_1} \cdot \frac{\mathrm{e}^{-\frac{\theta_1}{r_n}}}{r_n} \prod_{i=1}^{n} \frac{1+r_i}{r_i^2} \prod_{i=1}^{n-1} \left(\theta_1(1+r_i)+r_i\right)^{-1}, \tag{2.1}$$

$$L_2(\theta_2|\boldsymbol{s}) = \frac{\theta_2^{2m}}{1+\theta_2} \cdot \frac{\mathrm{e}^{-\frac{\theta_2}{s_m}}}{s_m} \prod_{j=1}^{m} \frac{1+s_j}{s_j^2} \prod_{j=1}^{m-1} \left(\theta_2(1+s_j)+s_j\right)^{-1}. \tag{2.2}$$

Therefore, the log-likelihood function of $\boldsymbol{\theta}$ given $\boldsymbol{r}$ and $\boldsymbol{s}$ denoted by $\ell(\theta_1, \theta_2|\boldsymbol{r}, \boldsymbol{s})$ is given by

$$\begin{aligned}\ell(\theta_1, \theta_2|\boldsymbol{r}, \boldsymbol{s}) = 2n \ln \theta_1 - \ln(1+\theta_1) - \frac{\theta_1}{r_n} - \sum_{i=1}^{n-1} \ln\left(\theta_1(1+r_i) + r_i\right) + 2m \ln \theta_2 \\ - \ln(1+\theta_2) - \frac{\theta_2}{s_m} - \sum_{j=1}^{m-1} \ln\left(\theta_2(1+s_j) + s_j\right) + A(\boldsymbol{r}) + B(\boldsymbol{s}),\end{aligned}$$

where

$$A(\boldsymbol{r}) = \sum_{i=1}^{n} \ln(1+r_i) - \ln r_n - 2\sum_{i=1}^{n} \ln r_i,$$

and

$$B(\boldsymbol{s}) = \sum_{j=1}^{m} \ln(1+s_j) - \ln s_m - 2\sum_{j=1}^{m} \ln s_j.$$

The ML estimates of θ_1 and θ_2 based on the observed lower records may be obtained by solving the following nonlinear equations

$$\frac{\partial \ell(\theta_1, \theta_2|\boldsymbol{r}, \boldsymbol{s})}{\partial \theta_1} = \frac{2n}{\theta_1} - \frac{1}{1+\theta_1} - \frac{1}{r_n} - \sum_{i=1}^{n-1} \frac{1+r_i}{\theta_1(1+r_i)+r_i} = 0, \tag{2.3}$$

$$\frac{\partial \ell(\theta_1, \theta_2|\boldsymbol{r}, \boldsymbol{s})}{\theta_2} = \frac{2m}{\theta_2} - \frac{1}{1+\theta_2} - \frac{1}{s_m} - \sum_{j=1}^{m-1} \frac{1+s_j}{\theta_2(1+s_j)+s_j} = 0. \tag{2.4}$$

From (2.3) and (2.4), the ML estimates of θ_1 and θ_2 can be obtained as the fixed point solutions of $h_1(\theta_1) = \theta_1$ and $h_2(\theta_2) = \theta_2$, respectively, where

$$h_1(\theta_1) = 2n\Big(\frac{1}{1+\theta_1} + \frac{1}{r_n} + \sum_{i=1}^{n-1} \frac{1+r_i}{\theta_1(1+r_i)+r_i}\Big)^{-1}, \tag{2.5}$$

and

$$h_2(\theta_2) = 2m\Big(\frac{1}{1+\theta_2} + \frac{1}{s_m} + \sum_{j=1}^{m-1} \frac{1+s_j}{\theta_2(1+s_j)+s_j}\Big)^{-1}. \tag{2.6}$$

Since the ML estimate of θ_1 is a fixed point solution of the nonlinear equation (2.5) and the ML estimate of θ_2 is a fixed point solution of the equation (2.6), therefore they can be obtained by using a simple iterative scheme. For the proof of the uniqueness of the resulting ML estimates, see [27].

Let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ be the ML estimators of θ_1 and θ_2, respectively. Then, by the invariance property of ML estimation, the ML estimator of R based on lower record values, denoted by $\widehat{R}$, is given by

$$\widehat{R} = \frac{\widehat{\theta}_1^2\{2\widehat{\theta}_2 + (1+2\widehat{\theta}_2)(\widehat{\theta}_1+\widehat{\theta}_2) + (1+\widehat{\theta}_2)(\widehat{\theta}_1+\widehat{\theta}_2)^2\}}{(1+\widehat{\theta}_1)(1+\widehat{\theta}_2)(\widehat{\theta}_1+\widehat{\theta}_2)^3}. \tag{2.7}$$

3. Classical confidence intervals for R

In this section, we present a modified asymptotic confidence interval, as well as two bootstrap-type approximate confidence intervals for R.

3.1. Asymptotic confidence interval for R

In this subsection, first, we obtain the asymptotic joint distribution of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ and then we focus on the asymptotic distribution of $\widehat{R}$. Then, a modified asymptotic confidence interval for R based on the asymptotic distribution of $\widehat{R}$ is constructed.

The expected Fisher information matrix of $\boldsymbol{R}$ and $\boldsymbol{S}$ about the parameter vector $\boldsymbol{\theta}$ is given by $I_{\boldsymbol{R},\boldsymbol{S}}(\theta_1,\theta_2) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$, where $I_{21} = I_{12} = 0$ due to the independency of $\boldsymbol{R}$ and $\boldsymbol{S}$ and

$$I_{11} = -E\left(\frac{\partial^2 \ln f_{R_1,\cdots,R_n,S_1,\cdots,S_m}(R_1,\cdots,R_n,S_1,\cdots,S_m)}{\partial\theta_1^2}\right),$$
$$I_{22} = -E\left(\frac{\partial^2 \ln f_{R_1,\cdots,R_n,S_1,\cdots,S_m}(R_1,\cdots,R_n,S_1,\cdots,S_m)}{\partial\theta_2^2}\right),$$

in which $f_{R_1,\cdots,R_n,S_1,\cdots,S_m}(r_1,\cdots,r_n,s_1,\cdots,s_m)$ is the joint distribution of $R_1,\cdots,R_n$, $S_1,\cdots,S_m$, provided that the above expectations exist. One can estimate I_{11} and I_{22} using the ML estimators of θ_1 and θ_2 as follows

$$\begin{aligned}\widehat{I}_{11} &= -\frac{\partial^2 \ln f_{R_1,\cdots,R_n,S_1,\cdots,S_m}(R_1,\cdots,R_n,S_1,\cdots,S_m)}{\partial\theta_1^2}\bigg|_{(\theta_1,\theta_2)=(\widehat{\theta}_1,\widehat{\theta}_2)} \\ &= \frac{2n}{\widehat{\theta}_1^2} - \frac{1}{(1+\widehat{\theta}_1)^2} - \sum_{i=1}^{n-1}\left(\frac{1+R_i}{\widehat{\theta}_1(1+R_i)+R_i}\right)^2, \qquad (3.1)\end{aligned}$$

$$\begin{aligned}\widehat{I}_{22} &= -\frac{\partial^2 \ln f_{R_1,\cdots,R_n,S_1,\cdots,S_m}(R_1,\cdots,R_n,S_1,\cdots,S_m)}{\partial\theta_2^2}\bigg|_{(\theta_1,\theta_2)=(\widehat{\theta}_1,\widehat{\theta}_2)} \\ &= \frac{2m}{\widehat{\theta}_2^2} - \frac{1}{(1+\widehat{\theta}_2)^2} - \sum_{j=1}^{m-1}\left(\frac{1+S_j}{\widehat{\theta}_2(1+S_j)+S_j}\right)^2. \qquad (3.2)\end{aligned}$$

Next, we define a vector $\boldsymbol{\eta}^T = (\eta_1,\eta_2)$, where (note that $R = \mathcal{R}(\theta_1,\theta_2)$)

$$\eta_1 = \frac{\partial\,\mathcal{R}(\theta_1,\theta_2)}{\partial\theta_1} = \frac{\theta_1(2C+\theta_1 B)}{A_{113}} - \frac{\theta_1^2 C(A_{013}+3A_{112})}{A_{113}^2}, \qquad (3.3)$$
$$\eta_2 = \frac{\partial\,\mathcal{R}(\theta_1,\theta_2)}{\partial\theta_2} = \frac{\theta_1^2 D}{A_{113}} - \frac{\theta_1^2 C(A_{103}+3A_{112})}{A_{113}^2}, \qquad (3.4)$$

in which A_{ijk} and C are given in (1.4) and (1.5), respectively and

$$B = 1 + 2\theta_2 + 2A_{011}, \qquad (3.5)$$
$$D = 3 + 2\theta_2 + A_{002} + 2A_{001}(2+\theta_2). \qquad (3.6)$$

It is clear that η_1 and η_2 are functions of θ_1 and θ_2, so we can write $\eta_1 = g_1(\theta_1,\theta_2)$ and $\eta_2 = g_2(\theta_1,\theta_2)$. Thus, using the invariance property of ML estimators, the ML estimators of η_1 and η_2 are given by $\widehat{\eta}_1 = g_1(\widehat{\theta}_1,\widehat{\theta}_2)$ and $\widehat{\eta}_2 = g_2(\widehat{\theta}_1,\widehat{\theta}_2)$, respectively.

Now, we can use the delta method (see for example [52]) and state that as $n\to\infty$, $m\to\infty$ and $\frac{m}{n}\to p$, where p is a constant, then the asymptotic distribution of $\widehat{R}$ is normal with the mean R and the variance δ, where

$$\delta = \boldsymbol{\eta}^T I_{\boldsymbol{R},\boldsymbol{S}}^{-1}(\theta_1,\theta_2)\boldsymbol{\eta},$$

in which $I_{\boldsymbol{R},\boldsymbol{S}}^{-1}(\theta_1,\theta_2)$ is the inverse matrix of $I_{\boldsymbol{R},\boldsymbol{S}}(\theta_1,\theta_2)$. So we have

$$\delta = \frac{\eta_1^2}{I_{11}} + \frac{\eta_2^2}{I_{22}},$$

where η_1 and η_2 are defined in (3.3) and (3.4), respectively.

As δ involves unknown parameters θ_1 and θ_2, we may estimate δ using (3.1), (3.2) and the ML estimators of θ_1 and θ_2. Therefore, an estimator for the asymptotic variance of $\widehat{R}$ is given by

$$\widehat{\delta} = \frac{\widehat{\eta}_1^2}{\widehat{I}_{11}} + \frac{\widehat{\eta}_2^2}{\widehat{I}_{22}},$$

where $\widehat{I}_{11}$ and $\widehat{I}_{22}$ are given in (3.1) and (3.2), respectively and $\widehat{\eta}_1$ and $\widehat{\eta}_2$ are the ML estimators of η_1 and η_2, respectively.

Therefore, a $100(1-\gamma)\%$ asymptotic confidence interval for R is given by

$$\left(\widehat{R} - z_{\frac{\gamma}{2}}\sqrt{\widehat{\delta}}, \widehat{R} + z_{\frac{\gamma}{2}}\sqrt{\widehat{\delta}}\right),$$

where z_γ is the 100γ-th upper quantile of $N(0,1)$. As $R \in [0,1]$, we propose the following $100(1-\gamma)\%$ modified asymptotic confidence interval (MACI) for R

$$\left(\max\left\{0, \widehat{R} - z_{\frac{\gamma}{2}}\sqrt{\widehat{\delta}}\right\}, \min\left\{\widehat{R} + z_{\frac{\gamma}{2}}\sqrt{\widehat{\delta}}, 1\right\}\right).$$

3.2. Bootstrap confidence intervals for R

In this subsection, two confidence intervals using parametric bootstrap techniques are suggested. The first confidence interval is the bootstrap percentile confidence interval (Boot-P CI) which is based on [26]. The second confidence interval, is the Normal bootstrap confidence interval (Norm-Boot CI) which is based on [61]. The following algorithm is used to generate parametric bootstrap samples.

Algorithm 1

Step 1 . Compute the ML estimates of θ_1, θ_2 and R, denoted by $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{R}$, respectively, from the two originally observed samples of lower records $(r_1, \cdots, r_n)$ and $(s_1, \cdots, s_m)$.

Step 2 . Generate a bootstrap lower record sample $\{r_1^*, r_2^*, \cdots, r_n^*\}$ from $ILD(\hat{\theta}_1)$ and a second bootstrap lower record sample $\{s_1^*, \cdots, s_m^*\}$ from $ILD(\hat{\theta}_2)$.

Step 3 . Compute the bootstrap estimates, say $\widehat{\theta}_1^*$, $\widehat{\theta}_2^*$ and $\widehat{R}^*$ based on $\{r_1^*, r_2^*, \cdots, r_n^*\}$ and $\{s_1^*, \cdots, s_m^*\}$.
Note: $\widehat{R}^*$ can be computed using equation (2.7).

Step 4 . Repeat Steps 2 and 3, B times to obtain the bootstrap sample $\{\widehat{R}_1^*, \cdots, \widehat{R}_B^*\}$, where B is a large number.

To construct the bootstrap confidence interval for R, the bootstrap sample generated by the aforementioned algorithm is used and two different bootstrap confidence intervals are obtained as follows:

(i) *Bootstrap percentile method*
An approximate $100(1-\gamma)\%$ Boot-P CI for R is given by (see for example [23], page 203)

$$\left(R^*_{((B+1)\frac{\gamma}{2})}, R^*_{((B+1)(1-\frac{\gamma}{2}))}\right),$$

where R_q^* is the q-th ordered value of the bootstrap sample $\{\widehat{R}_1^*, \cdots, \widehat{R}_B^*\}$.

(ii) *Normal bootstrap method*
An approximate $100(1-\alpha)\%$ confidence interval of R is given by (see, for example [61], page 110)

$$(\hat{R} - z_{\frac{\alpha}{2}}\sqrt{\text{var}(\hat{R}^*)}, \hat{R} + z_{\frac{\alpha}{2}}\sqrt{\text{var}(\hat{R}^*)}\ )$$

where $\sqrt{\text{var}(\hat{R}^*)}$ is the bootstrap estimate of the standard error and $\hat{R}$ is the ML estimate of R.

4. Bayesian estimation of R

In this section, the Bayesian estimation of R will be discussed under the assumption that the parameters θ_1 and θ_2 are independent gamma random variables, namely we have $\theta_1 \sim \Gamma(\alpha_1, \beta_1)$ and $\theta_2 \sim \Gamma(\alpha_2, \beta_2)$. Thus, the prior densities of θ_1 and θ_2 are given by

$$\pi(\theta_i) \quad = \quad \frac{\beta_i^{\alpha_i}\theta_i^{\alpha_i-1}\mathrm{e}^{-\beta_i\theta_i}}{\Gamma(\alpha_i)}, \qquad \theta_i > 0 \ \& \ i = 1, 2,$$

where a_1, b_1, a_2 and b_2 are positive hyperparameters that can be determined based on the prior knowledge of the researcher(s).

Due to the independency of θ_1 and θ_2, the joint prior density of θ_1 and θ_2 becomes

$$\pi(\theta_1, \theta_2) = \pi(\theta_1) \cdot \pi(\theta_2). \tag{4.1}$$

From (2.1), (2.2) and (4.1), the joint posterior density of θ_1 and θ_2 given $\boldsymbol{r}$ and $\boldsymbol{s}$ is given by

$$\begin{aligned}\pi^*(\theta_1, \theta_2|\boldsymbol{r}, \boldsymbol{s}) &= \frac{L_1(\theta_1|\boldsymbol{r})L_2(\theta_2|\boldsymbol{s})\pi(\theta_1, \theta_2)}{\int_0^\infty\int_0^\infty L_1(\theta_1|\boldsymbol{r})L_2(\theta_2|\boldsymbol{s})\pi(\theta_1, \theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2} \\ &= \frac{1}{K^*}\theta_1^{2n+\alpha_1-1}\theta_2^{2m+\alpha_2-1}\xi_1(\boldsymbol{r}, \theta_1)\xi_2(\boldsymbol{s}, \theta_2) \\ &\qquad \times \exp\Big(-\theta_1(\frac{1}{r_n}+\beta_1) - \theta_2(\frac{1}{s_m}+\beta_2)\Big),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}\xi_1(\boldsymbol{r}, \theta_1) &= \Big\{(1+\theta_1)\prod_{i=1}^{n-1}\left(\theta_1(1+r_i)+r_i\right)\Big\}^{-1}, \\ \xi_2(\boldsymbol{s}, \theta_2) &= \Big\{(1+\theta_2)\prod_{j=1}^{m-1}\left(\theta_2(1+s_j)+s_j\right)\Big\}^{-1},\end{aligned}$$

and

$$K^* = \int_0^\infty\int_0^\infty \theta_1^{2n+\alpha_1-1}\theta_2^{2m+\alpha_2-1}\xi_1(\boldsymbol{r}, \theta_1)\xi_2(\boldsymbol{s}, \theta_2)\exp\Big(-\theta_1(\frac{1}{r_n}+\beta_1) - \theta_2(\frac{1}{s_m}+\beta_2)\Big)\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Due to the independency of θ_1 and θ_2 and independency of $\boldsymbol{R}$ and $\boldsymbol{S}$, one can see that

$$\pi^*(\theta_1, \theta_2|\boldsymbol{r}, \boldsymbol{s}) = \pi_1^*(\theta_1|\boldsymbol{r}) \cdot \pi_2^*(\theta_2|\boldsymbol{s}),$$

where

$$\pi_1^*(\theta_1|\boldsymbol{r}) = \frac{1}{K_1^*}\theta_1^{2n+\alpha_1-1}\xi_1(\boldsymbol{r}, \theta_1)\exp\Big(-\theta_1(\frac{1}{r_n}+\beta_1)\Big), \tag{4.3}$$

and

$$\pi_2^*(\theta_2|\boldsymbol{s}) = \frac{1}{K_2^*}\theta_2^{2m+\alpha_2-1}\xi_2(\boldsymbol{s}, \theta_2)\exp\Big(-\theta_2(\frac{1}{s_m}+\beta_2)\Big), \tag{4.4}$$

in which

$$K_1^* = \int_0^\infty \theta_1^{2n+\alpha_1-1}\xi_1(\boldsymbol{r}, \theta_1)\exp\Big(-\theta_1(\frac{1}{r_n}+\beta_1)\Big)\mathrm{d}\theta_1,$$

and

$$K_2^* = \int_0^\infty \theta_2^{2m+\alpha_2-1}\xi_2(\boldsymbol{s}, \theta_2)\exp\Big(-\theta_2(\frac{1}{s_m}+\beta_2)\Big)\mathrm{d}\theta_2.$$

For point Bayesian estimation of R, we consider three loss functions. The first function is the squared error loss (SEL) function, which is a symmetric function, namely, it gives the same weights to the underestimation and overestimation. However, in many applications, the underestimation and overestimation do not have the same consequences, so using asymmetric loss functions seems logical in such situations. Here, we use two asymmetric loss functions, which are the linear-exponential loss (LEL) function and the general entropy loss (GEL) function. The LEL and GEL functions were proposed by [60] and [14], respectively. Let $\widehat{\theta}$ be an estimator of θ, then the LEL and GEL functions are defined as

$$\mathcal{L}_1(\widehat{\theta},\theta) = b\left(\exp\{c(\widehat{\theta}-\theta\} - c(\widehat{\theta}-\theta) - 1\right), \quad b>0, \quad c\neq 0, \tag{4.5}$$

and

$$\mathcal{L}_2(\widehat{\theta},\theta) \propto \left(\frac{\widehat{\theta}}{\theta}\right)^q - q\ln\left(\frac{\widehat{\theta}}{\theta}\right) - 1, \quad q\neq 0, \tag{4.6}$$

respectively.

Without loss of generality, one can take $b = 1$ in (4.5). The parameters c in (4.5) and q in (4.6) should be determined carefully, as the sign and magnitude of these parameters are important and affect the loss. A positive value of c in the LEL function will make the overestimation more serious than the underestimation and vice versa; see [63]. Similarly, a positive value of q in the GEL function causes the overestimation to be more serious than the underestimation and vice versa, see [14, 15].

From (1.3), the Bayes estimate of R under the SEL function is obtained from the following relation

$$\widehat{R}_S = \int_0^\infty\int_0^\infty \mathcal{R}(\theta_1,\theta_2)\ \pi^*(\theta_1,\theta_2|\boldsymbol{r},\boldsymbol{s})\mathrm{d}\theta_1\mathrm{d}\theta_2, \tag{4.7}$$

provided that the above integral exists.

In addition, the Bayes point estimates of R under the LEL and GEL functions are given by

$$\widehat{R}_L = -\frac{1}{c}\ln\left(\int_0^\infty\int_0^\infty \exp\{-c\,\mathcal{R}(\theta_1,\theta_2)\}\ \pi^*(\theta_1,\theta_2|\boldsymbol{r},\boldsymbol{s})\mathrm{d}\theta_1\mathrm{d}\theta_2\right), \tag{4.8}$$

and

$$\widehat{R}_G = \left(\int_0^\infty\int_0^\infty [\mathcal{R}(\theta_1,\theta_2)]^{-q}\ \pi^*(\theta_1,\theta_2|\boldsymbol{r},\boldsymbol{s})\mathrm{d}\theta_1\mathrm{d}\theta_2\right)^{-\frac{1}{q}}, \tag{4.9}$$

respectively, provided that the integrals given in (4.8) and (4.9) exist.

It seems that the integrals given in (4.7), (4.8) and (4.9) cannot be obtained analytically. Therefore, three different approaches are used to obtain the approximate Bayesian estimates of R, namely Tierney and Kadane's approximation, importance sampling (IS) and Metropolis-Hastings (M-H) methods.

4.1. Tierney and Kadane's approximation

In this subsection, we present Tierney and Kadane's (TK) method for approximating the required posterior means under the SEL, LEL and GEL functions. This method, introduced by [58], employs Laplace's formula to compute approximate posterior moments. Moreover, [58] demonstrated the superior accuracy of the proposed approach compared to Lindley's approximation method. Let $\delta(\theta_1,\theta_2|\boldsymbol{r},\boldsymbol{s}) = \ln\pi^*(\theta_1,\theta_2|\boldsymbol{r},\boldsymbol{s})$ be the log-posterior of (θ_1,θ_2), $g(\theta_1,\theta_2) = \frac{1}{N}\delta(\theta_1,\theta_2|\boldsymbol{r},\boldsymbol{s})$, where $N = n+m$ and $g^*(\theta_1,\theta_2) = g(\theta_1,\theta_2) + \frac{1}{N}\ln U(\theta_1,\theta_2)$. Then, the posterior moment of any function of $\boldsymbol{\theta} = (\theta_1,\theta_2)$ such as $U(\theta_1,\theta_2)$

may be approximated as follows

$$\begin{aligned}E(U(\theta_1,\theta_2)|\boldsymbol{r},\boldsymbol{s}) &= \frac{\int_0^\infty\int_0^\infty U(\theta_1,\theta_2)\exp\left(\delta(\theta_1,\theta_2|\boldsymbol{r},\boldsymbol{s})\right)\mathrm{d}\theta_1\mathrm{d}\theta_2}{\int_0^\infty\int_0^\infty \exp\left(\delta(\theta_1,\theta_2|\boldsymbol{r},\boldsymbol{s})\right)\mathrm{d}\theta_1\mathrm{d}\theta_2}\\ &= \frac{\int_0^\infty\int_0^\infty \exp\left(Ng^*(\theta_1,\theta_2)\right)\mathrm{d}\theta_1 d\theta_2}{\int_0^\infty\int_0^\infty \exp\left(Ng(\theta_1,\theta_2)\right)\mathrm{d}\theta_1\mathrm{d}\theta_2}\\ &\simeq \Big(\frac{\det(\Sigma^*)}{\det(\Sigma)}\Big)^{\frac{1}{2}}\exp\Big(N\big(g^*(\theta_1^*,\theta_2^*)-g(\tilde{\theta}_1,\tilde{\theta}_2)\big)\Big), \qquad (4.10)\end{aligned}$$

where $(\tilde{\theta}_1,\tilde{\theta}_2)$ represents the posterior mode of $g(\theta_1,\theta_2)$, (θ_1^*,θ_2^*) maximizes $g^*(\theta_1,\theta_2)$ and Σ^* and Σ denote the negative inverse Hessian matrice of $g^*(\theta_1,\theta_2)$ and $g(\theta_1,\theta_2)$, respectively, evaluated at (θ_1^*,θ_2^*) and $(\tilde{\theta}_1,\tilde{\theta}_2)$, respectively. Note that $(\tilde{\theta}_1,\tilde{\theta}_2)$ and (θ_1^*,θ_2^*) will be obtained from normal equations. In our case, we have

$$\begin{aligned}N\cdot g(\theta_1,\theta_2) = (2n+\alpha_1-1)\ln(\theta_1)-\ln(1+\theta_1)-\frac{\theta_1}{r_n}-\beta_1\theta_1\\ -\sum_{i=1}^{n-1}\ln\left(\theta_1(1+r_i)+r_i\right)+(2m+\alpha_2-1)\ln(\theta_2)-\ln(1+\theta_2)-\frac{\theta_2}{s_m}\\ -\beta_2\theta_2-\sum_{j=1}^{m-1}\ln\left(\theta_2(1+s_j)+s_j\right)-\ln(K^*).\end{aligned}$$

The maximizer arguments of $g(\theta_1,\theta_2)$, namely $(\tilde{\theta}_1,\tilde{\theta}_2)$, will be obtained from solving the following equations

$$\begin{aligned}\frac{\partial g(\theta_1,\theta_2)}{\partial\theta_1} &= \frac{1}{N}\Big[\frac{2n+\alpha_1-1}{\theta_1}-\frac{1}{1+\theta_1}-\frac{1}{r_n}-\beta_1-\sum_{i=1}^{n-1}\frac{1+r_i}{\theta_1(1+r_i)+r_i}\Big]=0,\\ \frac{\partial g(\theta_1,\theta_2)}{\partial\theta_2} &= \frac{1}{N}\Big[\frac{2m+\alpha_2-1}{\theta_2}-\frac{1}{1+\theta_2}-\frac{1}{s_m}-\beta_2-\sum_{i=1}^{m-1}\frac{1+s_j}{\theta_2(1+s_j)+s_j}\Big]=0.\end{aligned}$$

Using the second-order derivatives of $g(\theta_1,\theta_2)$, the determinant of the negative inverse Hessian of $g(\theta_1,\theta_2)$ at $(\tilde{\theta}_1,\tilde{\theta}_2)$ is given by $\det(\Sigma)=-\left(g_{11}g_{22}-g_{12}^2\right)^{-1}\Big|_{(\theta_1,\theta_2)=(\tilde{\theta}_1,\tilde{\theta}_2)}$, where

$$\begin{aligned}g_{11} &= \frac{1}{N}\Big[-\frac{2n+\alpha_1-1}{\theta_1^2}+\frac{1}{(1+\theta_1)^2}+\sum_{i=1}^{n-1}\frac{(1+r_i)^2}{(\theta_1(1+r_i)+r_i)^2}\Big],\\ g_{22} &= \frac{1}{N}\Big[-\frac{2m+\alpha_2-1}{\theta_2^2}+\frac{1}{(1+\theta_2)^2}+\sum_{j=1}^{m-1}\frac{(1+s_j)^2}{(\theta_2(1+s_j)+s_j)^2}\Big],\end{aligned}$$

and $g_{12}=g_{21}=0$.
Now, for computing the Bayes estimates of R based on SEL, LEL and GEL functions, we follow the following scenarios:

i) SEL
In this case, $U(\theta_1,\theta_2)=\mathcal{R}(\theta_1,\theta_2)$, so (θ_1^*,θ_2^*) can be computed by maximizing $g^*(\theta_1,\theta_2)=g(\theta_1,\theta_2)+\frac{1}{N}\ln\mathcal{R}(\theta_1,\theta_2)$. Therefore

$$\begin{aligned}g^{1*}(\theta_1,\theta_2) = \frac{1}{N}\big[2\ln(\theta_1)+\ln\left(2\theta_2+(1+2\theta_2)(\theta_1+\theta_2)+(1+\theta_2)(\theta_1+\theta_2)^2\right)\\ -\ln(1+\theta_1)-\ln(1+\theta_2)-3\ln(\theta_1+\theta_2)\big]+g(\theta_1,\theta_2).\end{aligned}$$

In Appendix, the first-order $g^*(\theta_1,\theta_2)$ with respect to (w.r.t.) θ_1 and θ_2 are computed. Setting the first-order derivatives of $g^*(\theta_1,\theta_2)$ equal to zero, we can obtain

(θ_1^*, θ_2^*) and then we have $\det(\Sigma^*) = -\left(g_{11}^{1*}g_{22}^{1*} - {g_{12}^{1*}}^2\right)^{-1}\Big|_{(\theta_1,\theta_2)=(\theta_1^*,\theta_2^*)}$, where g_{11}^{1*}, g_{22}^{1*} and g_{12}^{1*} are given in Appendix. To approximate the posterior mean of R under the SEL function, it is sufficient to substitute the above-mentioned elements in (4.10).

ii) LEL

In this case, we consider $U(\theta_1, \theta_2) = e^{-c\mathcal{R}(\theta_1,\theta_2)}$. Consequently, the function $g^{2*}(\theta_1, \theta_2)$ assumes the following form:

$$g^{2*}(\theta_1, \theta_2) = \frac{-c}{N}\Big[\frac{\theta_1^2(2\theta_2 + (1 + 2\theta_2)(\theta_1 + \theta_2) + (1 + \theta_2)(\theta_1 + \theta_2)^2)}{(1 + \theta_1)(1 + \theta_2)(\theta_1 + \theta_2)^3}\Big] + g(\theta_1, \theta_2).$$

The values of (θ_1^*, θ_2^*) are determined by equating the derivatives of $g^{2*}(\theta_1, \theta_2)$ w.r.t. θ_1 and θ_2 with zero. Then, we can find

$$\det(\Sigma^*) = -\left(g_{11}^{2*}g_{22}^{2*} - {g_{12}^{2*}}^2\right)^{-1}\Big|_{(\theta_1,\theta_2)=(\theta_1^*,\theta_2^*)},$$

where g_{11}^{2*}, g_{22}^{2*} and g_{12}^{2*} are derived in Appendix.

Subsequently, the approximate Bayes estimate of R under the LEL function takes the following form:

$$\widehat{R}_{LEL} = -\frac{1}{c}\ln\Big\{\Big(\frac{\det\Sigma^*}{\det\Sigma}\Big)^{\frac{1}{2}}\exp\Big(N\big(g^{2*}(\theta_1^*, \theta_2^*) - g(\tilde{\theta}_1, \tilde{\theta}_2)\big)\Big)\Big\}.$$

iii) GEL

In this scenario, we consider $U(\theta_1, \theta_2) = (\mathcal{R}(\theta_1, \theta_2))^{-q}$ and to obtain (θ_1^*, θ_2^*), the function $g^{3*}(\theta_1, \theta_2)$ needs to be considered, where

$$\begin{aligned}g^{3*}(\theta_1, \theta_2) = -\frac{q}{N}\Big[&2\ln(\theta_1) + \ln\left(2\theta_2 + (1 + 2\theta_2)(\theta_1 + \theta_2) + (1 + \theta_2)(\theta_1 + \theta_2)^2\right)\\ &- \ln(1 + \theta_1) - \ln(1 + \theta_2) - 3\ln(\theta_1 + \theta_2)\Big] + g(\theta_1, \theta_2).\end{aligned}$$

Additionally, by calculating the second-order derivatives of $g^{3*}(\theta_1, \theta_2)$ at (θ_1^*, θ_2^*), we derive the elements of $-\Sigma^*$, where the second-order derivatives of $g^{3*}(\theta_1, \theta_2)$ are given in Appendix.

As a result, the approximate Bayes estimate of R under the GEL function is as follows

$$\widehat{R}_{GEL} = \Big\{\Big(\frac{\det\Sigma^*}{\det\Sigma}\Big)^{\frac{1}{2}}\exp\Big(N\big(g^{3*}(\theta_1^*, \theta_2^*) - g(\tilde{\theta}_1, \tilde{\theta}_2)\big)\Big)\Big\}^{-\frac{1}{q}}.$$

4.2. Markov chain Monte Carlo (MCMC) method

The MCMC approach is essentially an iterative sampling algorithm, drawing values from the posterior distributions of the parameters of the concerned model. In this subsection, we consider an MCMC method to generate samples from the joint posterior distribution (4.2) and then compute the approximate Bayes estimates of R under SEL, LEL and GEL functions based on the observed lower record values from the inverse Lindley distributions. An important subclass of MCMC methods involves Metropolis-Hastings (M-H) algorithm (see [29, 42]). The M-H algorithm can be used to generate samples from any complex distribution that is known up to a normalizing constant. For more details about MCMC methods and the related methodologies, one can refer to [50]. Since the posterior density of $\boldsymbol{\theta} = (\theta_1, \theta_2)$ in (4.2) is not known, we may use the M-H method with positive truncated normal proposal distributions to generate samples from this distribution. The M-H algorithm is described as follows.

Algorithm 2

Step 1. Start with an initial guess $\boldsymbol{\theta}^{(0)} = (\theta_1^{(0)}, \theta_2^{(0)}) = (\hat{\theta}_1, \hat{\theta}_2)$, where $\hat{\theta}_1$ and $\hat{\theta}_2$ are the ML estimates of θ_1 and θ_2, respectively.

Step 2. Given $\theta_1^{(t-1)}$, generate θ_1^* from the positive truncated normal distribution, $N(\theta_1^{(t-1)}, \sigma_1^2)I_{\{\theta_1>0\}}$. Then $\theta_1^{(t)} = \theta_1^*$ with probability

$$\rho_1 = \min\left\{\frac{\pi_1^*(\theta_1^*|\boldsymbol{r})q_1(\theta_1^{(t-1)}|\theta_1^*)}{\pi_1^*(\theta_1^{(t-1)}|\boldsymbol{r})q_1(\theta_1^*|\theta_1^{(t-1)})}, 1\right\},$$

where $\pi_1^*(\cdot|\boldsymbol{r})$ is given in (4.3) and $q_1(x|b)$ is the density of $N(b, \sigma_1^2)I_{\{\theta_1>0\}}$, otherwise set $\theta_1^{(t)} = \theta_1^{(t-1)}$.

Step 3. Given $\theta_2^{(t-1)}$, generate θ_2^* from the positive truncated normal distribution, $N(\theta_2^{(t-1)}, \sigma_2^2)I_{\{\theta_2>0\}}$. Then $\theta_2^{(t)} = \theta_2^*$ with probability

$$\rho_2 = \min\left\{\frac{\pi_2^*(\theta_2^*|\boldsymbol{s})q_2(\theta_2^{(t-1)}|\theta_2^*)}{\pi_2^*(\theta_2^{(t-1)}|\boldsymbol{s})q_2(\theta_2^*|\theta_2^{(t-1)})}, 1\right\},$$

where $\pi_2^*(\cdot|\boldsymbol{s})$ is given in (4.4) and $q_2(x|b)$ is the density of $N(b, \sigma_2^2)I_{\{\theta_2>0\}}$, otherwise set $\theta_2^{(t)} = \theta_2^{(t-1)}$.

Step 4. Compute $R^{(t)}$ from (1.3) using the generated values of $\theta_1^{(t)}$ and $\theta_2^{(t)}$.

Step 5. Set $t = t + 1$ and repeat Steps 2, 3 and 4, T times where T is a large number. Thus, the set $\{R^{(1)}, R^{(2)}, \cdots, R^{(T)}\}$ is the generated sample.

In this paper, we take the parameters of the proposal distributions σ_1^2 and σ_2^2 to be the observed values of $\frac{1}{\widehat{I}_{11}}$ and $\frac{1}{\widehat{I}_{22}}$, respectively, where $\widehat{I}_{11}$ and $\widehat{I}_{22}$ are given in (3.1) and (3.2), respectively.

We may discard the first K generated values of the sample $\{R^{(1)}, R^{(2)}, \cdots, R^{(T)}\}$, where K is the burn-in period.

So we use the sample $\{R_1, \cdots, R_N\} = \{R^{(K+1)}, R^{(K+2)}, \cdots, R^{(T)}\}$, where $N = T - K$. Now, the approximate Bayes estimate of R under the SEL function is given by

$$\widetilde{R}_S = \frac{1}{N}\sum_{i=1}^{N} R_i,$$

Moreover, the approximate Bayes estimates of R under the LEL and GEL functions are given by

$$\widetilde{R}_L = -\frac{1}{c}\ln\Big(\frac{1}{N}\sum_{i=1}^{N} \mathrm{e}^{-cR_i}\Big),$$

and

$$\widetilde{R}_G = \Big(\frac{1}{N}\sum_{i=1}^{N} R_i^{-q}\Big)^{-\frac{1}{q}},$$

respectively.

4.3. Importance sampling technique

The importance sampling (IS) technique is another method of approximating Bayes point estimates and constructing credible intervals for the parameter of interest, (see [2], Section 5.9). The joint posterior density of θ_1 and θ_2 given $\boldsymbol{r}$ and $\boldsymbol{s}$, that is given in (4.2), can be rewritten as follows

$$\pi^*(\theta_1, \theta_2|\boldsymbol{r}, \boldsymbol{s}) = g_1(\theta_1|\boldsymbol{r}) \cdot g_2(\theta_2|\boldsymbol{s}) \cdot h(\theta_1, \theta_2; \boldsymbol{r}, \boldsymbol{s}),$$

where $g_1(\theta_1|\boldsymbol{r})$ is the density of the gamma distribution with parameters $2n + \alpha_1$ and $\dfrac{1}{r_n} + \beta_1$, $g_2(\theta_2|\boldsymbol{s})$ is the density of the gamma distribution with parameters $2m + \alpha_2$ and $\dfrac{1}{s_m} + \beta_2$ and

$$h(\theta_1, \theta_2; \boldsymbol{r}, \boldsymbol{s}) = \frac{\Gamma(2n + \alpha_1)\Gamma(2m + \alpha_2)}{K^*(\frac{1}{r_n} + \beta_1)^{2n+\alpha_1}(\frac{1}{s_m} + \beta_2)^{2m+\alpha_2}} \cdot h^*(\theta_1, \theta_2; \boldsymbol{r}, \boldsymbol{s}),$$

in which

$$h^*(\theta_1, \theta_2; \boldsymbol{r}, \boldsymbol{s}) = \left\{(1+\theta_1) \prod_{i=1}^{n-1} (\theta_1(1+r_i)+r_i)\right\}^{-1} \cdot \left\{(1+\theta_2) \prod_{j=1}^{m-1} (\theta_2(1+s_j)+s_j)\right\}^{-1}. \quad (4.11)$$

Now, we use the following algorithm.

Algorithm 3

Step 1 . Generate θ_{11} from $g_1(\theta_1|\boldsymbol{r})$.
Step 2 . Generate θ_{21} from $g_2(\theta_2|\boldsymbol{s})$.
Step 3 . Compute R_1 from (1.3) using the generated values of θ_{11} and θ_{21}.
Step 4 . Repeat Steps 1, 2 and 3, N times, where N is a large number, to obtain the samples $\{(\theta_{11}, \theta_{21}), \cdots, (\theta_{1N}, \theta_{2N})\}$ and $\{R_1, \cdots, R_N\}$.

Now, the approximate Bayes estimates of R under the SEL, LEL and GEL functions are given by

$$\widetilde{R}_S^* = \frac{\sum_{i=1}^N R_i h^*(\theta_{1i}, \theta_{2i}|\boldsymbol{r}, \boldsymbol{s})}{\sum_{i=1}^N h^*(\theta_{1i}, \theta_{2i}|\boldsymbol{r}, \boldsymbol{s})},$$

$$\widetilde{R}_L^* = -\frac{1}{c} \ln \left[\frac{\sum_{i=1}^N \mathrm{e}^{-cR_i} h^*(\theta_{1i}, \theta_{2i}|\boldsymbol{r}, \boldsymbol{s})}{\sum_{i=1}^N h^*(\theta_{1i}, \theta_{2i}|\boldsymbol{r}, \boldsymbol{s})}\right],$$

and

$$\widetilde{R}_G^* = \left[\frac{\sum_{i=1}^N R_i^{-q} h^*(\theta_{1i}, \theta_{2i}|\boldsymbol{r}, \boldsymbol{s})}{\sum_{i=1}^N h^*(\theta_{1i}, \theta_{2i}|\boldsymbol{r}, \boldsymbol{s})}\right]^{-\frac{1}{q}},$$

respectively, where $h^*(\theta_1, \theta_2; \boldsymbol{r}, \boldsymbol{s})$ is defined in (4.11).

4.4. Bayesian credible intervals for R

In this subsection, due to the results of [18], the Chen and Shao shortest width credible intervals (CSSW CrIs) for R are discussed, based on the generated samples using the M-H and IS methods. First, we discuss the CSSW CrI based on the set $\{R_1, \cdots, R_N\} = \{R^{(K+1)}, R^{(K+2)}, \cdots, R^{(T)}\}$ generated by an M-H technique. In this regard, we shall use the following algorithm.

Algorithm 4

Step 1 . Generate an MCMC sample $\{R_i; i = 1, 2, \cdots, N\}$ using Algorithm 2.
Step 2 . Sort $\{R_i; i = 1, 2, \cdots, N\}$ in order of magnitude. Let $\{R_{(1)}, R_{(2)}, \cdots, R_{(N)}\}$ be the sorted set of $\{R_i; i = 1, 2, \cdots, N\}$ namely $R_{(1)} \leq R_{(2)} \leq \cdots \leq R_{(N)}$.
Step 3 . Set $K_j = (R_{(j)}, R_{(j+[(1-\gamma)N])})$ for $j = 1, 2, \cdots, N - [(1-\gamma)N]$.
Step 4 . Set $L_j = R_{(j+[(1-\gamma)N])} - R_{(j)}$ for $j = 1, 2, \cdots, N - [(1-\gamma)N]$.
Step 5 . Select j^* such that $L_{j^*} = \min\{L_j : j = 1, 2, \cdots, N - [(1-\gamma)N]\}$.
Step 6 . Report K_{j^*} as the $100(1-\gamma)\%$ CSSW Crl for R.

Note that K_j's in Step 3 of Algorithm 4 are $100(1-\gamma)\%$ credible intervals for R.

Next, we discuss the CSSW CrI based on the sets $\{(\theta_{11}, \theta_{21}), \cdots, (\theta_{1N}, \theta_{2N})\}$ and $\{R_1, \cdots, R_N\}$ generated by an IS method. In this regard, we shall use the following algorithm.

Algorithm 5

Step 1 . Generate samples $\{(\theta_{1i}, \theta_{2i}); i = 1, 2, \cdots, N\}$ and $\{R_i; i = 1, 2, \cdots, N\}$ using Algorithm 3. Set

$$w_i = \frac{h^*(\theta_{1i}, \theta_{2i}|\boldsymbol{r}, \boldsymbol{s})}{\sum_{j=1}^{N} h^*(\theta_{1j}, \theta_{2j}|\boldsymbol{r}, \boldsymbol{s})}, \quad i = 1, \cdots, N,$$

where $h^*(\theta_1, \theta_2; \boldsymbol{r}, \boldsymbol{s})$ is defined in (4.11).
Step 2 . Sort $\{R_i; i = 1, 2, \cdots, N\}$ in order of magnitude. Let $\{R_{(1)}, R_{(2)}, \cdots, R_{(N)}\}$ be the sorted set of $\{R_i; i = 1, 2, \cdots, N\}$ namely $R_{(1)} \leq R_{(2)} \leq \cdots \leq R_{(N)}$. Let further $\{w_{(1)}, w_{(2)}, \cdots, w_{(N)}\}$ be the corresponding values of the set $\{w_i; i = 1, 2, \cdots, N\}$ such that $w_{(i)}$ corresponds to $R_{(i)}$ for $i = 1, 2, \cdots, N$. Note that $w_{(i)}$ does not imply an ordered value of w_i.
Step 3 . Set $K_j = (R^{(j)}, R^{(j+[(1-\gamma)N])})$ for $j = 1, 2, \cdots, N - [(1-\gamma)N]$, where $R^{(\gamma)} = R_{(i)}$ if $\sum_{j=1}^{i-1} w_{(j)} < \gamma \leq \sum_{j=1}^{i} w_{(j)}$.
Step 4 . Set $L_j = R^{(j+[(1-\gamma)N])} - R^{(j)}$ for $j = 1, 2, \cdots, N - [(1-\gamma)N]$.
Step 5 . Select j^* such that $L_{j^*} = \min\{L_j : j = 1, 2, \cdots, N - [(1-\gamma)N]\}$.
Step 6 . Report K_{j^*} as the $100(1-\gamma)\%$ CSSW Crl for R.

5. A simulation study

In this section, we evaluate the behavior of the various estimators of the stress-strength parameter with the help of a simulation study. The numbers of records are taken to be $n = 3, 4, 5$ and $m = 3, 4, 5$ such that $n \leq m$. The exact values of the parameter R are chosen to be $R = 0.12, 0.28, 0.50$ and 0.87 that correspond to $(\theta_1, \theta_2) = (0.5, 2), (1, 2), (1, 1)$ and $(2, 0.5)$, respectively. In the Bayesian part, the hyperparameters are considered to be $(a_1, b_1, a_2, b_2) = (0.01, 0.01, 0.01, 0.01)$ as an approximate noninformative joint prior. The number of replications is $N^* = 1000$. In each replication, we generate n records from $ILD(\theta_1)$ and m records from $ILD(\theta_2)$ and then we obtain the point estimates of R, such as the ML estimate and approximate Bayes estimates under the SEL, LEL (with $c = -0.2$ and 0.2) and GEL (with $q = -0.2$ and 0.2) functions using the M-H, IS and TK methods. Besides, we construct the 95% MACI, Boot-P CI and Norm-Boot CI, as well as the 95% CSSW CrIs using the generated samples that are obtained from the M-H and IS approaches. For bootstrap intervals, we use $B = 999$ bootstrap samples. Let $\hat{R}$ be an estimator of R and $\hat{R}_i$ be the corresponding estimate obtained in the ith iteration. Then, the estimated bias (bias for short), the estimated mean squared error (EMSE) and the estimated risks (ERs) of $\hat{R}$ under the LEL and GEL functions are given by

$$Bias(\hat{R}) = \frac{1}{N^*}\sum_{i=1}^{N^*}(\hat{R}_i - R),$$

$$EMSE(\hat{R}) = \frac{1}{N^*}\sum_{i=1}^{N^*}(\hat{R}_i - R)^2,$$

$$ER_L(\hat{R}) = \frac{1}{N^*}\sum_{i=1}^{N^*}\left[e^{c(\hat{R}_i - R)} - c(\hat{R}_i - R) - 1\right],$$

and

$$ER_G(\hat{R}) = \frac{1}{N^*}\sum_{i=1}^{N^*}\left[\left(\frac{\hat{R}_i}{R}\right)^q - q\ln\left(\frac{\hat{R}_i}{R}\right) - 1\right],$$

respectively.

We have computed the EMSEs of all kinds of the considered estimators and the results are given in Table 1. Note that the EMSE is equivalent to the estimated risk under the SEL function. We have also calculated the ERs of the approximate Bayes estimators under the LEL and GEL functions, according to their own corresponding loss functions, whereas for the ML estimators, we have obtained both ER_Ls and ER_Gs and the related results are reported in Table 2.

Table 3 is devoted to the computed biases for all kinds of the considered estimators.

Moreover, the calculated average widths (AWs) and estimated coverage probabilities (CPs for short) of the 95% intervals are tabulated in Table 4.

From Tables 1–4, we extract the following conclusions:

- The approximate Bayes estimators under the SEL function possess smaller EMSEs than the ML estimators for the cases of $R = 0.28$, and 0.50, but the reverse is true for the cases of $R = 0.12$ and 0.87.
- Among the approximate Bayes estimators, the EMSEs of those under the SEL function are either the smallest or close to the smallest values in most cases, as expected.
- Among the Bayesian approximation methods, for the cases of $R = 0.12, 0.28$ and 0.50, the best and worst performances in terms of EMSE are observed with the TK and IS methods, respectively, in most cases. In contrast, for the case of $R = 0.87$, the M-H and TK strategies produce the smallest and largest EMSEs, respectively, in most cases.
- The ER_Ls and ER_Gs are decreasing w.r.t. the number of records in most cases. However, the biases do not have a special behavior w.r.t. the number of records.
- The approximate Bayes estimators under the GEL function perform the best among all kinds of the considered estimators in terms of bias when $R = 0.12$ and 0.28, whereas the ML estimators demonstrate the best performance in this respect for the case of $R = 0.87$.
- The computed biases for all kinds of the considered estimators are negative when $R = 0.87$.
- The AWs have the biggest values for the case of $R = 0.5$. As the number of records increases, the AWs decrease in most cases.
- The Boot-P interval estimators outperform all other kinds of interval estimators in the sense of CP (one exception exists), whereas the smallest values of CP belong to the MACIs in most cases.

- The CSSW CrIs based on the IS method possess the smallest AWs in most cases except for the case of $R = 0.87$. For the case of $R = 0.87$, the MACIs are the shortest intervals on average (one exception exists).

Table 1. Computed EMSEs of the point estimators of R.

(θ_1,θ_2)	(n,m)	Exact value of R	ML	IS Method					M-H Method					TK Method				
					LEL		GEL			LEL		GEL			LEL		GEL	
				SEL	$c=-0.2$	$c=0.2$	$q=-0.2$	$q=0.2$	SEL	$c=-0.2$	$c=0.2$	$q=-0.2$	$q=0.2$	SEL	$c=-0.2$	$c=0.2$	$q=-0.2$	$q=0.2$
$(0.5,2)$	$(3,3)$	0.12	0.0170	0.0190	0.0186	0.0195	0.0128	0.0145	0.0190	0.0185	0.0195	0.0127	0.0144	0.0175	0.0197	0.0205	0.0118	0.0134
	$(3,4)$		0.0143	0.0154	0.0150	0.0157	0.0106	0.0119	0.0152	0.0148	0.0155	0.0106	0.0118	0.0145	0.0161	0.0168	0.0101	0.0113
	$(3,5)$		0.0159	0.0168	0.0164	0.0172	0.0113	0.0129	0.0166	0.0163	0.0170	0.0113	0.0128	0.0161	0.0177	0.0184	0.0109	0.0124
	$(4,4)$		0.0109	0.0123	0.0120	0.0125	0.0086	0.0096	0.0124	0.0121	0.0127	0.0086	0.0096	0.0118	0.0128	0.0134	0.0082	0.0092
	$(4,5)$		0.0119	0.0134	0.0131	0.0136	0.0097	0.0107	0.0132	0.0129	0.0134	0.0094	0.0104	0.0126	0.0135	0.0140	0.0090	0.0100
	$(5,5)$		0.0095	0.0111	0.0109	0.0113	0.0082	0.0090	0.0108	0.0106	0.0110	0.0078	0.0086	0.0104	0.0110	0.0114	0.0075	0.0083
$(1,2)$	$(3,3)$	0.28	0.0313	0.0274	0.0270	0.0279	0.0256	0.0257	0.0275	0.0270	0.0280	0.0255	0.0257	0.0262	0.0262	0.0268	0.0244	0.0246
	$(3,4)$		0.0317	0.0277	0.0273	0.0282	0.0254	0.0257	0.0277	0.0272	0.0282	0.0254	0.0258	0.0265	0.0269	0.0276	0.0243	0.0247
	$(3,5)$		0.0284	0.0249	0.0245	0.0253	0.0231	0.0233	0.0242	0.0239	0.0246	0.0226	0.0228	0.0238	0.0242	0.0249	0.0221	0.0223
	$(4,4)$		0.0249	0.0229	0.0226	0.0233	0.0214	0.0216	0.0229	0.0225	0.0232	0.0214	0.0216	0.0218	0.0218	0.0224	0.0204	0.0206
	$(4,5)$		0.0252	0.0230	0.0226	0.0233	0.0214	0.0216	0.0230	0.0226	0.0233	0.0213	0.0216	0.0221	0.0222	0.0228	0.0204	0.0207
	$(5,5)$		0.0188	0.0185	0.0182	0.0188	0.0170	0.0173	0.0177	0.0174	0.0180	0.0162	0.0165	0.0170	0.0171	0.0176	0.0157	0.0159
$(1,1)$	$(3,3)$	0.50	0.0424	0.0338	0.0338	0.0338	0.0409	0.0378	0.0338	0.0338	0.0338	0.0408	0.0378	0.0336	0.0303	0.0304	0.0409	0.0378
	$(3,4)$		0.0380	0.0308	0.0308	0.0307	0.0376	0.0346	0.0307	0.0307	0.0306	0.0375	0.0345	0.0304	0.0280	0.0280	0.0373	0.0343
	$(3,5)$		0.0325	0.0267	0.0267	0.0267	0.0326	0.0300	0.0266	0.0267	0.0266	0.0324	0.0299	0.0261	0.0244	0.0244	0.0320	0.0294
	$(4,4)$		0.0323	0.0269	0.0269	0.0270	0.0316	0.0296	0.0266	0.0266	0.0266	0.0313	0.0293	0.0263	0.0246	0.0246	0.0312	0.0291
	$(4,5)$		0.0302	0.0258	0.0258	0.0258	0.0306	0.0285	0.0253	0.0253	0.0253	0.0301	0.0280	0.0250	0.0236	0.0236	0.0299	0.0278
	$(5,5)$		0.0279	0.0239	0.0239	0.0239	0.0276	0.0260	0.0239	0.0239	0.0239	0.0276	0.0260	0.0234	0.0224	0.0223	0.0271	0.0255
$(2,0.5)$	$(3,3)$	0.87	0.0159	0.0180	0.0185	0.0176	0.0245	0.0219	0.0177	0.0181	0.0173	0.0240	0.0215	0.0190	0.0195	0.0187	0.0259	0.0232
	$(3,4)$		0.0124	0.0148	0.0151	0.0144	0.0197	0.0178	0.0145	0.0148	0.0141	0.0192	0.0173	0.0153	0.0156	0.0149	0.0205	0.0184
	$(3,5)$		0.0117	0.0143	0.0146	0.0139	0.0188	0.0170	0.0144	0.0147	0.0140	0.0189	0.0171	0.0149	0.0152	0.0145	0.0197	0.0178
	$(4,4)$		0.0113	0.0130	0.0133	0.0127	0.0166	0.0152	0.0128	0.0131	0.0125	0.0163	0.0150	0.0133	0.0138	0.0132	0.0172	0.0157
	$(4,5)$		0.0104	0.0125	0.0128	0.0123	0.0156	0.0145	0.0121	0.0124	0.0119	0.0152	0.0141	0.0127	0.0130	0.0125	0.0161	0.0148
	$(5,5)$		0.0098	0.0114	0.0116	0.0112	0.0138	0.0129	0.0110	0.0113	0.0108	0.0135	0.0126	0.0113	0.0116	0.0112	0.0139	0.0130

Table 2. Computed ER_Ls and ER_Gs of the ML estimators and the approximate Bayes estimators of R.

(θ_1, θ_2)	(n, m)	Exact value of R	ER_L $(c = +0.2)$				ER_L $(c = -0.2)$				ER_G $(q = +0.2)$				ER_G $(q = -0.2)$			
			ML	IS	M-H	TK	ML	IS	M-H	TK	ML	IS	M-H	TK	ML	IS	M-H	TK
$(0.5, 2)$	$(3, 3)$	0.12	0.0003	0.0004	0.0004	0.0004	0.0003	0.0004	0.0004	0.0004	0.0145	0.0149	0.0149	0.0147	0.0160	0.0143	0.0144	0.0143
	$(3, 4)$		0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0119	0.0128	0.0127	0.0125	0.0127	0.0121	0.0119	0.0117
	$(3, 5)$		0.0003	0.0003	0.0003	0.0004	0.0003	0.0003	0.0003	0.0004	0.0116	0.0119	0.0119	0.0116	0.0121	0.0116	0.0115	0.0112
	$(4, 4)$		0.0002	0.0002	0.0002	0.0003	0.0002	0.0002	0.0002	0.0003	0.0103	0.0107	0.0107	0.0105	0.0110	0.0103	0.0102	0.0100
	$(4, 5)$		0.0002	0.0003	0.0003	0.0003	0.0002	0.0003	0.0003	0.0003	0.0095	0.0100	0.0098	0.0096	0.0099	0.0096	0.0094	0.0092
	$(5, 5)$		0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0089	0.0094	0.0092	0.0090	0.0094	0.0092	0.0089	0.0087
$(1, 2)$	$(3, 3)$	0.28	0.0006	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0005	0.0090	0.0098	0.0098	0.0098	0.0101	0.0094	0.0093	0.0094
	$(3, 4)$		0.0006	0.0006	0.0006	0.0005	0.0006	0.0006	0.0006	0.0005	0.0072	0.0079	0.0079	0.0078	0.0077	0.0074	0.0074	0.0073
	$(3, 5)$		0.0006	0.0005	0.0005	0.0005	0.0006	0.0005	0.0005	0.0005	0.0071	0.0081	0.0079	0.0079	0.0077	0.0077	0.0075	0.0075
	$(4, 4)$		0.0005	0.0005	0.0005	0.0004	0.0005	0.0005	0.0005	0.0004	0.0069	0.0074	0.0075	0.0074	0.0075	0.0072	0.0072	0.0071
	$(4, 5)$		0.0005	0.0005	0.0005	0.0004	0.0005	0.0005	0.0005	0.0005	0.0063	0.0067	0.0067	0.0067	0.0067	0.0064	0.0065	0.0064
	$(5, 5)$		0.0004	0.0004	0.0004	0.0003	0.0004	0.0004	0.0004	0.0003	0.0048	0.0053	0.0051	0.0050	0.0051	0.0051	0.0049	0.0049
$(1, 1)$	$(3, 3)$	0.50	0.0008	0.0007	0.0007	0.0006	0.0008	0.0007	0.0007	0.0006	0.0055	0.0066	0.0065	0.0067	0.0063	0.0063	0.0063	0.0065
	$(3, 4)$		0.0008	0.0006	0.0006	0.0006	0.0008	0.0006	0.0006	0.0006	0.0043	0.0054	0.0054	0.0055	0.0048	0.0051	0.0051	0.0052
	$(3, 5)$		0.0007	0.0005	0.0005	0.0005	0.0006	0.0005	0.0005	0.0005	0.0034	0.0044	0.0044	0.0044	0.0036	0.0041	0.0040	0.0040
	$(4, 4)$		0.0006	0.0005	0.0005	0.0005	0.0006	0.0005	0.0005	0.0005	0.0037	0.0043	0.0043	0.0044	0.0040	0.0041	0.0041	0.0042
	$(4, 5)$		0.0006	0.0005	0.0005	0.0005	0.0006	0.0005	0.0005	0.0005	0.0031	0.0039	0.0038	0.0038	0.0034	0.0037	0.0036	0.0036
	$(5, 5)$		0.0006	0.0005	0.0005	0.0004	0.0006	0.0005	0.0005	0.0004	0.0030	0.0035	0.0035	0.0035	0.0032	0.0033	0.0034	0.0033
$(2, 0.5)$	$(3, 3)$	0.87	0.0003	0.0004	0.0004	0.0004	0.0003	0.0004	0.0004	0.0004	0.0006	0.0010	0.0010	0.0011	0.0007	0.0010	0.0009	0.0010
	$(3, 4)$		0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0004	0.0008	0.0007	0.0008	0.0005	0.0007	0.0007	0.0007
	$(3, 5)$		0.0002	0.0003	0.0003	0.0003	0.0002	0.0003	0.0003	0.0003	0.0004	0.0007	0.0007	0.0008	0.0004	0.0007	0.0007	0.0007
	$(4, 4)$		0.0002	0.0003	0.0003	0.0003	0.0002	0.0003	0.0003	0.0003	0.0004	0.0006	0.0006	0.0007	0.0004	0.0006	0.0006	0.0006
	$(4, 5)$		0.0002	0.0003	0.0002	0.0003	0.0002	0.0002	0.0002	0.0003	0.0004	0.0006	0.0006	0.0006	0.0004	0.0005	0.0005	0.0006
	$(5, 5)$		0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0003	0.0005	0.0005	0.0005	0.0004	0.0005	0.0005	0.0005

Table 3. The estimated biases of the point estimators of R.

(θ_1,θ_2)	(n,m)	Exact value of R	ML	IS Method					M-H Method					TK Method				
					LEL		GEL			LEL		GEL			LEL		GEL	
				SEL	$c=-0.2$	$c=0.2$	$q=-0.2$	$q=0.2$	SEL	$c=-0.2$	$c=0.2$	$q=-0.2$	$q=0.2$	SEL	$c=-0.2$	$c=0.2$	$q=-0.2$	$q=0.2$
$(0.5,2)$	$(3,3)$	0.12	0.0331	0.0598	0.0582	0.0615	0.0131	0.0288	0.0600	0.0584	0.0617	0.0130	0.0288	0.0545	0.0661	0.0696	0.0101	0.0250
	$(3,4)$		0.0280	0.0489	0.0476	0.0503	0.0072	0.0214	0.0486	0.0472	0.0500	0.0069	0.0210	0.0463	0.0543	0.0572	0.0057	0.0195
	$(3,5)$		0.0432	0.0598	0.0584	0.0611	0.0191	0.0332	0.0596	0.0582	0.0610	0.0190	0.0329	0.0585	0.0652	0.0681	0.0185	0.0322
	$(4,4)$		0.0219	0.0434	0.0422	0.0446	0.0067	0.0189	0.0441	0.0429	0.0453	0.0069	0.0193	0.0417	0.0481	0.0508	0.0056	0.0176
	$(4,5)$		0.0299	0.0478	0.0467	0.0490	0.0139	0.0254	0.0485	0.0473	0.0496	0.0135	0.0253	0.0468	0.0518	0.0542	0.0125	0.0241
	$(5,5)$		0.0219	0.0407	0.0397	0.0416	0.0108	0.0208	0.0404	0.0394	0.0414	0.0091	0.0195	0.0394	0.0435	0.0457	0.0086	0.0189
$(1,2)$	$(3,3)$	0.28	0.0234	0.0425	0.0398	0.0451	−0.0127	0.0067	0.0428	0.0401	0.0455	−0.0124	0.0070	0.0352	0.0485	0.0531	−0.0184	0.0004
	$(3,4)$		0.0377	0.0500	0.0476	0.0525	−0.0003	0.0176	0.0501	0.0477	0.0526	−0.0004	0.0175	0.0450	0.0553	0.0596	−0.0050	0.0127
	$(3,5)$		0.0311	0.0393	0.0372	0.0415	−0.0075	0.0092	0.0391	0.0369	0.0414	−0.0088	0.0083	0.0364	0.0451	0.0493	−0.0109	0.0059
	$(4,4)$		0.0196	0.0362	0.0341	0.0384	−0.0076	0.0077	0.0360	0.0338	0.0382	−0.0082	0.0071	0.0318	0.0393	0.0434	−0.0120	0.0032
	$(4,5)$		0.0285	0.0386	0.0367	0.0406	−0.0016	0.0125	0.0409	0.0389	0.0429	−0.0004	0.0141	0.0372	0.0431	0.0471	−0.0038	0.0106
	$(5,5)$		0.0219	0.0365	0.0347	0.0383	0.0005	0.0130	0.0372	0.0353	0.0391	−0.0012	0.0121	0.0337	0.0385	0.0423	−0.0041	0.0090
$(1,1)$	$(3,3)$	0.50	0.0027	0.0026	−0.0006	0.0058	−0.0491	−0.0303	0.0020	−0.0012	0.0052	−0.0493	−0.0306	−0.0064	−0.0004	0.0047	−0.0587	−0.0397
	$(3,4)$		0.0010	−0.0053	−0.0082	−0.0024	−0.0529	−0.0355	−0.0048	−0.0077	−0.0018	−0.0526	−0.0352	−0.0102	−0.0042	0.0007	−0.0590	−0.0412
	$(3,5)$		0.0069	−0.0030	−0.0058	−0.0002	−0.0481	−0.0316	−0.0023	−0.0052	0.0005	−0.0481	−0.0314	−0.0070	−0.0014	0.0035	−0.0536	−0.0365
	$(4,4)$		0.0023	0.0034	0.0007	0.0061	−0.0374	−0.0229	0.0027	0.0000	0.0054	−0.0387	−0.0238	−0.0035	−0.0006	0.0041	−0.0454	−0.0305
	$(4,5)$		−0.0003	−0.0045	−0.0070	−0.0020	−0.0429	−0.0292	−0.0035	−0.0060	−0.0010	−0.0427	−0.0286	−0.0077	−0.0047	−0.0001	−0.0473	−0.0330
	$(5,5)$		0.0013	0.0008	−0.0015	0.0031	−0.0329	−0.0210	0.0007	−0.0017	0.0030	−0.0340	−0.0217	−0.0026	−0.0009	0.0032	−0.0377	−0.0253
$(2,0.5)$	$(3,3)$	0.87	−0.0304	−0.0578	−0.0595	−0.0562	−0.0760	−0.0692	−0.0573	−0.0590	−0.0556	−0.0755	−0.0687	−0.0622	−0.0673	−0.0639	−0.0814	−0.0743
	$(3,4)$		−0.0199	−0.0486	−0.0501	−0.0472	−0.0641	−0.0584	−0.0481	−0.0495	−0.0466	−0.0634	−0.0577	−0.0520	−0.0559	−0.0528	−0.0684	−0.0623
	$(3,5)$		−0.0231	−0.0521	−0.0535	−0.0507	−0.0667	−0.0613	−0.0525	−0.0540	−0.0511	−0.0674	−0.0619	−0.0559	−0.0595	−0.0563	−0.0716	−0.0658
	$(4,4)$		−0.0224	−0.0446	−0.0458	−0.0434	−0.0565	−0.0522	−0.0449	−0.0462	−0.0437	−0.0573	−0.0528	−0.0477	−0.0513	−0.0487	−0.0605	−0.0559
	$(4,5)$		−0.0231	−0.0461	−0.0472	−0.0449	−0.0571	−0.0532	−0.0458	−0.0470	−0.0446	−0.0572	−0.0530	−0.0486	−0.0516	−0.0490	−0.0606	−0.0562
	$(5,5)$		−0.0203	−0.0393	−0.0403	−0.0384	−0.0483	−0.0451	−0.0392	−0.0402	−0.0382	−0.0486	−0.0453	−0.0412	−0.0437	−0.0417	−0.0508	−0.0474

Table 4. The AWs and CPs of the 95% interval estimators of R.

(θ_1,θ_2)	(n,m)	Exact value R	MACI		Boot-P CI		Norm-Boot CI		CSSW CrI M-H Method		CSSW CrI IS Method	
			AW	CP	AW	CP	AW	CP	AW	CP	AW	CP
$(0.5,2)$	$(3,3)$	0.12	0.3680	0.8500	0.4550	0.9470	0.3899	0.9230	0.4053	0.9210	0.4024	0.9120
	$(3,4)$		0.3497	0.8750	0.4392	0.9450	0.3762	0.9360	0.3718	0.9200	0.3663	0.9130
	$(3,5)$		0.3715	0.8870	0.4531	0.9330	0.3966	0.9380	0.3768	0.9180	0.3662	0.8960
	$(4,4)$		0.3305	0.8750	0.3950	0.9370	0.3478	0.9210	0.3545	0.9180	0.3459	0.9060
	$(4,5)$		0.3356	0.8910	0.3953	0.9460	0.3536	0.9330	0.3476	0.9230	0.3306	0.9060
	$(5,5)$		0.3148	0.8870	0.3604	0.9420	0.3286	0.9260	0.3270	0.9210	0.3069	0.8860
$(1,2)$	$(3,3)$	0.28	0.5863	0.8470	0.6045	0.9410	0.5757	0.8750	0.5631	0.9000	0.5522	0.8910
	$(3,4)$		0.5830	0.8660	0.5960	0.9470	0.5735	0.8950	0.5450	0.9090	0.5287	0.8900
	$(3,5)$		0.5677	0.8670	0.5805	0.9390	0.5614	0.8890	0.5249	0.9060	0.4986	0.8800
	$(4,4)$		0.5514	0.8590	0.5495	0.9430	0.5387	0.8760	0.5169	0.8990	0.4955	0.8740
	$(4,5)$		0.5426	0.8760	0.5393	0.9440	0.5323	0.8810	0.5032	0.9000	0.4722	0.8780
	$(5,5)$		0.5314	0.9080	0.5172	0.9560	0.5176	0.9110	0.4938	0.9330	0.4530	0.8870
$(1,1)$	$(3,3)$	0.50	0.6917	0.8260	0.6667	0.9420	0.6586	0.8450	0.6300	0.8890	0.6214	0.8820
	$(3,4)$		0.6792	0.8400	0.6445	0.9490	0.6451	0.8480	0.6122	0.8880	0.5981	0.8820
	$(3,5)$		0.6787	0.8790	0.6337	0.9480	0.6409	0.8860	0.6068	0.9090	0.5825	0.8990
	$(4,4)$		0.6653	0.8800	0.6206	0.9460	0.6296	0.8810	0.5949	0.9130	0.5715	0.8970
	$(4,5)$		0.6465	0.8760	0.6012	0.9550	0.6116	0.8780	0.5791	0.9130	0.5500	0.8850
	$(5,5)$		0.6253	0.8750	0.5803	0.9450	0.5926	0.8760	0.5591	0.8970	0.5258	0.8840
$(2,0.5)$	$(3,3)$	0.87	0.3644	0.8530	0.4588	0.9490	0.3869	0.9130	0.4037	0.9190	0.3994	0.9050
	$(3,4)$		0.3363	0.8420	0.4077	0.9520	0.3508	0.8970	0.3781	0.9190	0.3740	0.9000
	$(3,5)$		0.3358	0.8550	0.3926	0.9490	0.3453	0.9040	0.3789	0.9340	0.3683	0.9160
	$(4,4)$		0.3307	0.8700	0.4001	0.9440	0.3480	0.9150	0.3565	0.9220	0.3446	0.9080
	$(4,5)$		0.3249	0.8770	0.3759	0.9420	0.3362	0.9150	0.3470	0.9280	0.3342	0.8870
	$(5,5)$		0.3104	0.8860	0.3607	0.9480	0.3237	0.9180	0.3234	0.9240	0.3051	0.8920

6. Application

In this section, as an example, we consider two independent record data to illustrate the inferential procedures developed in the paper. Crowder [22] considered data sets reflecting the lifetimes of steel specimens (in units of 1000 cycles) subjected to different stress amplitudes, see also [33, 38]. The specimens were tested at 14 distinct stress levels: $32.0, 32.5, 33.0, \ldots, 38.0, 38.5$. Taniş et al. [55] analyzed the data for 33.0 and 32.0 stress levels. In this analysis, we focus on the data for stress amplitudes of 32.0 (Data Set I) and 32.5 (Data Set II), which are divided by 1000. These data can also be found in [36] (page 574). Data Set I and Data Set II are reported in Tables 5 and 6, respectively.

Table 5. Data Set I: (The level stress: 32.0).

1.144	0.231	0.523	0.474	4.510	3.107	0.815	6.297
1.580	0.605	1.786	0.206	1.943	0.935	0.283	1.336
0.727	0.370	1.056	0.413	0.619	2.214	1.826	0.597

Table 6. Data Set II: (The level of stress: 32.5).

4.257	0.879	0.799	1.388	0.271	0.308	2.073	0.227
0.347	0.669	1.154	0.393	0.250	0.196	0.548	0.475
1.705	2.211	0.975	2.925				

According to [55], Data Set I and Data Set II are independent. The inverse Lindley distribution is fitted to the above data sets separately using the Kolmogorov-Smirnov (K-S) test. Note that the old version of the K-S test is not suitable here as the parameter estimates are obtained from the same data sets, see [51, 54]. So we use a bootstrapped version of the K-S test, described in detail in [16]. For Data Set I, the K-S test statistic and its corresponding bootstrapped p-value with $\hat{\theta}_1 = 0.9892$ are given by $D = 0.0917$ and $p = 0.9286$, respectively and for Data Set II, the K-S test statistic and its corresponding bootstrapped p-value with $\hat{\theta}_2 = 0.8089$ become $D = 0.0862$ and $p = 0.9864$, respectively. Therefore, the inverse Lindley model fits both data sets quite well.

We have extracted the first three lower records from Data Set I as follows:

$$1.144,\ 0.231,\ 0.206.$$

Besides, we extracted the first six lower records from Data Set II as follows:

$$4.257,\ 0.879,\ 0.799,\ 0.271,\ 0.227,\ 0.196.$$

Here, we calculated the point and 95% interval estimates under the same settings of the simulation study (stated in Section 5) and the results are reported in Table 7. Since the point estimates are less than 0.5, we may conclude that the probability that the lifetime under stress 32.0 becomes more than the lifetime under stress 32.5 is not high.

7. Concluding remarks

The parameter $R = P(Y < X)$ is one of the important measures to compare two populations. Besides, this parameter can be applied to stress strength models and that is why it is called the stress-strength parameter and can be used in reliability experiments and life testing. In this paper, we assume that the populations under study possess the inverse Lindley distribution with parameters θ_1 and θ_2. As the inverse Lindley distribution possesses a UTB-shaped hazard function, this model can play a key role in many real-life experiments. We worked on the point and interval estimation of R through classical and Bayesian procedures. A simulation study was performed to assess the estimators

Table 7. The numerical results of the example.

	point estimates		95% interval estimates	
ML		0.3210	MACI	(0.0000 , 0.6600)
TK	SEL	0.3269	Boot-P CI	(0.1059 , 0.7236)
	LEL, $c = -0.2$	0.3370		
	LEL, $c = +0.2$	0.3423		
	GEL, $q = -0.2$	0.2738	Norm-Boot CI	(0.0000 , 0.6528)
	GEL, $q = +0.2$	0.2928		
M-H	SEL	0.3424	M-H CSSW CrI	(0.0637 , 0.6846)
	LEL, $c = -0.2$	0.3396		
	LEL, $c = +0.2$	0.3453		
	GEL, $q = -0.2$	0.2832		
	GEL, $q = +0.2$	0.3048		
IS	SEL	0.3397	IS CSSW CrI	(0.0875 , 0.5816)
	LEL, $c = -0.2$	0.3374		
	LEL, $c = +0.2$	0.3421		
	GEL, $q = -0.2$	0.2941		
	GEL, $q = +0.2$	0.3101		

developed in the paper. From the simulation study, we deduce that among the Bayesian approximation methods, the TK procedure outperforms the IS and M-H methods in most cases in terms of the EMSE and estimated risk, when R is small. In the context of interval estimation, despite the fact that the Boot-P CIs do not possess the smallest AWs among the other considered intervals in the simulation, we recommend using the Boot-P CIs as their CPs are large enough to be near the nominal value 0.95 which is the considered confidence level of interval estimators.

We also presented a real data example, that involved two real data sets of lifetimes of steel specimens subjected to two distinct stress amplitudes. Here, the parameter R can help us compare the lifetimes under different stress levels. Suppose that a researcher is interested in knowing the probability that a lifetime under a specified level exceeds a lifetime under another stress level, so we see that the measure R plays a key role in this regard. Furthermore, suppose the researcher has only access to the observed lower records. Here, from Table 7, we may conclude that R is approximately, for example, 0.33 as most point estimates are close to this value and the interval estimates also contain this value. For further analysis, we can use hypothesis testing to determine whether R equals this number or not. So, assuming that the true value of R coincides with 0.33, the researcher may expect that a specimen under stress level 32.0 fails sooner than another specimen under stress level 32.5.

One can also work on other types of estimators of R for the inverse Lindley distribution, such as the preliminary test estimator, Bayesian shrinkage estimator, E-Bayesian estimator, empirical Bayes estimator, Bayesian hierarchical estimator and so on. Moreover, the problem of estimation of R based on other types of information samples, such as progressively Type I and Type II censored data, hybrid censored data and other different types of censored data, record ranked set samples, k-record values, upper records and inter-record times and so on. Besides, we can focus on the estimation of R for the newly introduced extensions of the inverse Lindley distribution, like the weighted inverse Lindley distribution [49]. Working on the above-suggested topics may become under progress by the authors. All the computations were done using the statistical software R [47] and the packages coda [45, 46], LindleyR [40] and truncnorm [41] therein.

Acknowledgements

We would like to express our sincere gratitude to the reviewers for their thorough and insightful feedback on this manuscript. We appreciate the time and effort they invested in providing constructive critiques and enhancing the clarity and rigor of this research.

Author contributions. All the co-authors have contributed rather equally in all aspects of the preparation of this submission.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Data availability. The data sets used in the article are provided in the manuscript.

References

[1] G.A. Abd-Elmougod and H.H. Abu-Zinadah, *Estimation of $P(Y < X)$ for two-parameter bathtub shape distribution using records: Bayesian and non-Bayesian approaches*, J. Comput. Theor. Nanosci. **14** (2), 1127-1135, 2017.

[2] J. Albert, *Bayesian Computation with R*, 2nd Edition, Dordrecht: Springer, 2009.

[3] R. Alotaibi, M. Nassar and A. Elshahhat, *Statistical analysis of inverse Lindley data using adaptive Type-II progressively hybrid censoring with applications*, Axioms **12** (5), 427, 2023.

[4] B.C. Arnold, N. Balakrishnan and H.N. Nagaraja, *Records*, New York: John Wiley & Sons, 1998.

[5] A. Asgharzadeh, M. Alizadeh and M.Z. Raqab, *Inverse Lindley distribution: different methods for estimating their PDF and CDF*, J. Stat. Comput. Simul. **94** (3), 604-623, 2024.

[6] A. Asgharzadeh, R. Valiollahi and M.Z. Raqab, *Estimation of $Pr(Y < X)$ for the two-parameter generalized exponential records*, Comm. Statist. Simulation Comput. **46** (1), 379-394, 2017.

[7] A. Baklizi, *Estimation of $Pr(X < Y)$ using record values in the one and two parameter exponential distributions*, Comm. Statist. Theory Methods **37** (5), 692-698, 2008.

[8] A. Baklizi, *Likelihood and Bayesian estimation of $Pr(X < Y)$ using lower record values from the generalized exponential distribution*, Comput. Statist. Data Anal. **52** (7), 3468-3473, 2008.

[9] A. Baklizi, *Bayesian inference for $Pr(Y < X)$ in the exponential distribution based on records*, Appl. Math. Model. **38** (5-6), 1698-1709, 2014.

[10] A. Baklizi, *Interval estimation of the stress-strength reliability in the two-parameter exponential distribution based on records*, J. Stat. Comput. Simul. **84** (12), 2670-2679, 2014.

[11] S. Basu, S.K. Singh and U. Singh, *Parameter estimation of inverse Lindley distribution for Type-I censored data*, Comput. Statist. **32** (1), 367-385, 2017.

[12] S. Basu, S.K. Singh and U. Singh, *Estimation of inverse Lindley distribution using product of spacings function for hybrid censored data*, Methodol. Comput. Appl. Probab. **21** (4), 1377-1394, 2019.

[13] S. Bennett, *Log-logistic regression models for survival data*, J. Roy. Statist. Soc. Ser. C **32** (2), 165-171, 1983.

[14] R. Calabria and G. Pulcini, *An engineering approach to Bayes estimation for the Weibull distribution*, Microelectron. Reliab. **34** (5), 789-802, 1994.

[15] R. Calabria and G. Pulcini, *Point estimation under asymmetric loss functions for left-truncated exponential samples*, Comm. Statist. Theory Methods **25** (3), 585-600, 1996.

[16] R. Castro, Lectures 2 and 3 - Goodness-of-fit (GoF) Tests, 2013, https://rmcastro.win.tue.nl/AppStat2013/files/lectures23.pdf.

[17] A. Chaturvedi, S.K. Singh and U. Singh, *Maximum product spacings estimator for fuzzy data using inverse Lindley distribution*, Austrian J. Stat. **52** (2), 86-103, 2023.

[18] M.H. Chen and Q.M. Shao, *Monte Carlo estimation of Bayesian credible and HPD intervals*, J. Comput. Graph. Statist. **8** (1), 69-92, 1999.

[19] J.Y. Chiang, N. Jiang, T.R. Tsai and Y.L. Lio, *Inference of $\delta = P(X < Y)$ for Burr XII distributions with record samples*, Comm. Statist. Simulation Comput. **47** (3), 822-838, 2018.

[20] J.D. Church and B. Harris, *The estimation of reliability from stress-strength relationships*, Technometrics **12** (1), 49-54, 1970.

[21] F. Çiftci, B. Saraçoğlu, N. Akdam and Y. Akdoğan, *Estimation of stress-strength reliability for generalized Gompertz distribution under progressive type-II censoring*, Hacet. J. Math. Stat. **52** (5), 1379-1395, 2023.

[22] M.J. Crowder, *Tests for a family of survival models based on extremes*, in: Recent Advances in Reliability Theory: Methodology, Practice, and Inference, Eds. N. Limnios and M. Nikulin, Birkhäuser, Boston, 307-321, 2000.

[23] A.C. Davison and D.V. Hinkley, *Bootstrap Methods and Their Application*, Cambridge: Cambridge University Press, 1997.

[24] S. Dey, M. Nassar, D. Kumar, A. Alzaatreh and M.H. Tahir, *A new lifetime distribution with decreasing and upside-down bathtub-shaped hazard rate function*, Statistica **79** (4), 399-426, 2019.

[25] S. Dey and M. Saha, *Bootstrap confidence intervals of the difference between two generalized process capability indices for inverse Lindley distribution*, Life Cycle Reliab. Saf. Eng. **7** (2), 89-96, 2018.

[26] B. Efron, *The Jackknife, the Bootstrap and Other Resampling Plans*, CBMS-NSF Regional Conference Series in Applied Mathematics **38**, Philadelphia: Society for Industrial and Applied Mathematics (SIAM), 1982.

[27] B. Etemad Golestani, E. Ormoz and S.M.T.K. MirMostafaee, *Statistical inference for the inverse Lindley distribution based on lower record values*, REVSTAT, 2024, https://revstat.ine.pt/index.php/REVSTAT/article/view/559.

[28] M.K. Hassan, M.I. Alohali and F.A. Alojail, *A new application of $R = P[Y < X]$ for the inverse Lindley distribution using ranked set sampling*, J. Stat. Manag. Syst. **24** (8), 1713-1731, 2021.

[29] W.K. Hastings, *Monte Carlo sampling methods using Markov chains and their applications*, Biometrika **57** (1), 97-109, 1970.

[30] S. Joo and J. Mi, *Some properties of hazard rate functions of systems with two components*, J. Statist. Plann. Inference **140** (2), 444-453, 2010.

[31] A. Joukar, M. Ramezani and S.M.T.K. MirMostafaee, *Estimation of $P(X > Y)$ for the power Lindley distribution based on progressively type II right censored samples,* J. Stat. Comput. Simul. **90** (2), 355-389, 2020.

[32] R.M. Juvairiyya and P. Anilkumar, *Estimation of stress-strength reliability for the Pareto distribution based on upper record values*, Statistica **78** (4), 397-409, 2018.

[33] A.C. Kimber, *Exploratory data analysis for possibly censored data from skewed distributions*, J. Roy. Statist. Soc. Ser. C **39** (1), 21-30, 1990.

[34] F. Kızılaslan and M. Nadar, *Statistical inference of $P(X < Y)$ for the Burr Type XII distribution based on records*, Hacet. J. Math. Stat. **46** (4), 713-742, 2017.

[35] A.O. Langlands, S.J. Pocock, G.R. Kerr and S.M. Gore, *Long-term survival of patients with breast cancer: a study of the curability of the disease*, Br. Med. J. **2** (6200), 1247-1251, 1979.

[36] J.F. Lawless, *Statistical Models and Methods for Lifetime Data*, 2nd Edition, Hoboken: John Wiley & Sons, 2003.

[37] I.S. Mabrouk, *Statistical inference for the parameter of the inverse Lindley distribution based on imprecise data with simulation study*, Int. J. Contemp. Math. Sci. **14** (4), 151-161, 2019.

[38] W.W. Maennig, *Bemarken zur Beurteilung des dauerschwring Festigkeitsverhaltens von Stahl und einige Untersuchungen zer Bestimmig des Dauerfestigkeitbereichs*, Materialpruf **12**, 124-131, 1967.

[39] M.A.W. Mahmoud, R.M. El-Sagheer, A.A. Soliman and A.H. Abd Ellah, *Bayesian estimation of $P[Y < X]$ based on record values from the Lomax distribution and MCMC technique*, J. Mod. Appl. Stat. Methods **15** (1), 488-510, 2016.

[40] J. Mazucheli, L.B. Fernandes and R.P. de Oliveira, LindleyR: The Lindley distribution and its modifications, R package version 1.0.0, https://CRAN.R-project.org/package=LindleyR, 2016.

[41] O. Mersmann, H. Trautmann, D. Steuer and B. Bornkamp, truncnorm: Truncated normal distribution. R package version 1.0-8, https://CRAN.R-project.org/package=truncnorm, 2018.

[42] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller and E. Teller, *Equations of state calculations by fast computing machine*, J. Chem. Phys. **21** (6), 1087-1092, 1953.

[43] M. Nadar and F. Kızılaslan, *Classical and Bayesian estimation of $P(X < Y)$ using upper record values from Kumaraswamy's distribution*, Statist. Papers **55** (3), 751-783, 2014.

[44] Z. Pakdaman and R. Alizadeh Noughabi, *On the study of the stress-strength reliability in Weibull-F models*, Hacet. J. Math. Stat. **53** (1), 269-288, 2024.

[45] M. Plummer, N. Best, K. Cowles and K. Vines, *CODA: Convergence diagnosis and output analysis for MCMC*, R News **6** (1), 7-11, 2006.

[46] M. Plummer, N. Best, K. Cowles, K. Vines, D. Sarkar, D. Bates, R. Almond and A. Magnusson, coda: Output analysis and diagnostics for MCMC. R package version 0.19-2, https://CRAN.R-project.org/package=coda, 2018.

[47] R Core Team, A Language and Envirenment for Statistical Computing. *R Foundation for Statistical Computing*, Vienna, Austria, 2020.

[48] M. Ramezani, A. Joukar and S.M.T.K. MirMostafaee, *Estimation of the stress-strength parameter for a decreasing failure rate model based on ranked set samples.* J. Test. Eval. **52** (6), https://doi.org/10.1520/JTE20240072, 2024.

[49] P.L. Ramos, F. Louzada, T.K. Shimizu and A.O. Luiz, *The inverse weighted Lindley distribution: Properties, estimation and an application on a failure time data*, Comm. Statist. Theory Methods **48** (10), 2372-2389, 2019.

[50] C.P. Robert and G. Casella, *Monte Carlo Statistical Methods*, 2nd Edition, New York: Springer, 2004.

[51] S. Sen, A.Z. Afify, H. Al-Mofleh and M. Ahsanullah, *The quasi xgamma-geometric distribution with application in medicine*, Filomat **33** (16), 5291-5330, 2019.

[52] J. Shao, *Mathematical Statistics*, 2nd Edition, New York: Springer, 2003.

[53] V.K. Sharma, S.K. Singh, U. Singh and V. Agiwal, *The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data*, J. Ind. Prod. Eng. **32** (3), 162-173, 2015.

[54] W. Stute, W.G. Manteiga and M.P. Quindimil, *Bootstrap based goodness-of-fit-tests*, Metrika **40** (1), 243-56, 1993.

[55] C. Taniş, B. Saraçoğlu, A. Asgharzadeh and M. Abdi, *Estimation of* $Pr(X < Y)$ *for exponential power records*, Hacet. J. Math. Stat. **52** (2), 499-511, 2023.

[56] B. Tarvirdizade and M. Ahmadpour, *Estimation of the stress-strength reliability for the two-parameter bathtub-shaped lifetime distribution based on upper record values*, Stat. Methodol. **31**, 58-72, 2016.

[57] B. Tarvirdizade and H. Kazemzadeh Gharehchobogh, *Inference on* $Pr(X > Y)$ *based on record values from the Burr Type X distribution*, Hacet. J. Math. Stat. **45** (1), 267-278, 2016.

[58] L. Tierney and J.B. Kadane, *Accurate approximations for posterior moments and marginal densities*, J. Amer. Statist. Assoc. **81** (393), 82-86, 1986.

[59] A. Tripathi, U. Singh and S.K. Singh, *Estimation of* $P(X < Y)$ *for Gompertz distribution based on upper records*, Int. J. Model. Simul. **42** (3), 388-399, 2022.

[60] H.R. Varian, *A Bayesian approach to real estate assessment*, in: Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage, Eds. S.E. Fienberg and A. Zellner, North-Holland Publishing Company, Amsterdam, 195-208, 1975.

[61] L.A. Wasserman, *All of Statistics: A Concise Course in Statistical Inference*, New York: Springer, 2004.

[62] A.C.M. Wong, and Y.Y. Wu, *A note on interval estimation of* $P(X < Y)$ *using lower record data from the generalized exponential distribution*, Comput. Statist. Data Anal. **53** (10), 3650-3658, 2009.

[63] A. Zellner, *Bayesian estimation and prediction using asymmetric loss functions*, J. Amer. Statist. Assoc. **81** (394), 446-451, 1986.

Appendix

A. SEL

$$\frac{\partial g^{1*}(\theta_1,\theta_2)}{\partial\theta_1} = \frac{1}{N}\Big[\frac{2n+\alpha_1+1}{\theta_1} - \frac{2}{1+\theta_1} - \frac{1}{r_n} - \beta_1 - \sum_{i=1}^{n-1}\frac{1+r_i}{\theta_1(1+r_i)+r_i} + \frac{B}{C} - \frac{3}{\theta_1+\theta_2}\Big],$$

$$\frac{\partial g^{1*}(\theta_1,\theta_2)}{\partial\theta_2} = \frac{1}{N}\Big[\frac{2m+\alpha_2-1}{\theta_2} - \frac{2}{1+\theta_2} - \frac{1}{s_m} - \beta_2 - \sum_{j=1}^{m-1}\frac{1+s_j}{\theta_2(1+s_j)+s_j} + \frac{D}{C} - \frac{3}{\theta_1+\theta_2}\Big],$$

$$g_{11}^{1*} = \frac{\partial^2 g^{1*}(\theta_1,\theta_2)}{\partial\theta_1^2} = \frac{1}{N}\Big[-\frac{2n+\alpha_1+1}{\theta_1^2} + \frac{2}{(1+\theta_1)^2} + \sum_{i=1}^{n-1}\frac{(1+r_i)^2}{[\theta_1(1+r_i)+r_i]^2} + \frac{2(1+\theta_2)}{C} - \frac{B^2}{C^2} + \frac{3}{(\theta_1+\theta_2)^2}\Big],$$

$$g_{22}^{1*} = \frac{\partial^2 g^{1*}(\theta_1,\theta_2)}{\partial\theta_2^2} = \frac{1}{N}\Big[-\frac{2m+\alpha_2-1}{\theta_2^2} + \frac{2}{(1+\theta_2)^2} + \sum_{j=1}^{m-1}\frac{(1+s_j)^2}{[\theta_2(1+s_j)+s_j]^2} + \frac{2(2\theta_1+3\theta_2+3)}{C} - \frac{D^2}{C^2} + \frac{3}{(\theta_1+\theta_2)^2}\Big],$$

$$g_{12}^{1*} = g_{21}^{1*} = \frac{\partial^2 g^{1*}(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} = \frac{1}{N}\Big[\frac{2(\theta_1+2\theta_2+2)}{C} - \frac{BD}{C^2} + \frac{3}{(\theta_1+\theta_2)^2}\Big],$$

where B, C and D are given in (3.5), (1.5), (3.6), respectively.

B. LEL

$$\frac{\partial g^{2*}(\theta_1,\theta_2)}{\partial\theta_1} = \frac{1}{N}\Big[\frac{2n+\alpha_1-1}{\theta_1} - \frac{1}{1+\theta_1} - \frac{1}{r_n} - \beta_1 - \sum_{i=1}^{n-1}\frac{1+r_i}{\theta_1(1+r_i)+r_i}\Big]$$
$$-\frac{c}{N}\Big[\frac{\theta_1(2C+\theta_1 B)}{A_{113}} - \frac{\theta_1^2 C(A_{013}+3A_{112})}{A_{113}^2}\Big],$$

$$\frac{\partial g^{2*}(\theta_1,\theta_2)}{\partial\theta_2} = \frac{1}{N}\Big[\frac{2m+\alpha_2-1}{\theta_2} - \frac{1}{1+\theta_2} - \frac{1}{s_m} - \beta_2 - \sum_{j=1}^{m-1}\frac{1+s_j}{\theta_2(1+s_j)+s_j}\Big]$$
$$-\frac{c}{N}\Big[\frac{\theta_1^2 D}{A_{113}} - \frac{\theta_1^2 C(A_{103}+3A_{112})}{A_{113}^2}\Big],$$

$$g_{11}^{2*} = \frac{\partial^2 g^{2*}(\theta_1,\theta_2)}{\partial\theta_1^2} = \frac{1}{N}\Big[-\frac{2n+\alpha_1-1}{\theta_1^2} + \frac{1}{(1+\theta_1)^2} + \sum_{i=1}^{n-1}\frac{(1+r_i)^2}{[\theta_1(1+r_i)+r_i]^2}\Big]$$
$$-\frac{c}{N}\Big[\frac{2(C+2\theta_1 B+\theta_1^2(1+\theta_2))}{A_{113}} - \frac{2\theta_1(2C+\theta_1 B)(A_{013}+3A_{112})}{A_{113}^2}$$
$$-\frac{6\theta_1^2 C(A_{012}+A_{111})}{A_{113}^2} + \frac{2\theta_1^2 C(A_{013}+3A_{112})^2}{A_{113}^3}\Big],$$

$$g_{22}^{2*} = \frac{\partial^2 g^{2*}(\theta_1,\theta_2)}{\partial\theta_2^2} = \frac{1}{N}\Big[-\frac{2m+\alpha_2-1}{\theta_2^2} + \frac{1}{(1+\theta_2)^2} + \sum_{j=1}^{m-1}\frac{(1+s_j)^2}{[\theta_2(1+s_j)+s_j]^2}\Big]$$
$$-\frac{c}{N}\Big[\frac{2\theta_1^2(2\theta_1+3\theta_2+3)}{A_{113}} - \frac{2\theta_1^2 D(A_{103}+3A_{112})}{A_{113}^2}$$
$$-\frac{6\theta_1^2 C(A_{102}+A_{111})}{A_{113}^2} + \frac{2\theta_1^2 C(A_{103}+3A_{112})^2}{A_{113}^3}\Big],$$

$$g_{12}^{2*} = g_{21}^{2*} = \frac{\partial^2 g^{2*}(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} = -\frac{c}{N}\Big[\frac{2\theta_1(D+\theta_1(\theta_1+2\theta_2+2))}{A_{113}} - \frac{\theta_1(2C+\theta_1 B)(A_{103}+3A_{112})}{A_{113}^2}$$
$$-\frac{\theta_1^2[D(A_{013}+3A_{112})+C(A_{003}+3A_{012}+3A_{102}+6A_{111})]}{A_{113}^2}$$
$$+\frac{2\theta_1^2 C(A_{013}+3A_{112})(A_{103}+3A_{112})}{A_{113}^3}\Big],$$

where A_{ijk} is given in (1.4).

C. GEL

$$\frac{\partial g^{3*}(\theta_1,\theta_2)}{\partial\theta_1}=\frac{1}{N}\Big[\frac{2n+\alpha_1-1-2q}{\theta_1}-\frac{1-q}{1+\theta_1}-\frac{1}{r_n}-\beta_1-\sum_{i=1}^{n-1}\frac{1+r_i}{\theta_1(1+r_i)+r_i}-\frac{qB}{C}+\frac{3q}{\theta_1+\theta_2}\Big],$$

$$\frac{\partial g^{3*}(\theta_1,\theta_2)}{\partial\theta_2}=\frac{1}{N}\Big[\frac{2m+\alpha_2-1}{\theta_2}-\frac{1-q}{1+\theta_2}-\frac{1}{s_m}-\beta_2-\sum_{j=1}^{m-1}\frac{1+s_j}{\theta_2(1+s_j)+s_j}-\frac{qD}{C}+\frac{3q}{\theta_1+\theta_2}\Big],$$

$$g_{11}^{3*}=\frac{\partial^2 g^{3*}(\theta_1,\theta_2)}{\partial\theta_1^2}=\frac{1}{N}\Big[-\frac{2n+\alpha_1-1-2q}{\theta_1^2}+\frac{1-q}{(1+\theta_1)^2}+\sum_{i=1}^{n-1}\frac{(1+r_i)^2}{[\theta_1(1+r_i)+r_i]^2}$$
$$-\frac{2q(1+\theta_2)}{C}+\frac{qB^2}{C^2}-\frac{3q}{(\theta_1+\theta_2)^2}\Big],$$

$$g_{22}^{3*}=\frac{\partial^2 g^{3*}(\theta_1,\theta_2)}{\partial\theta_2^2}=\frac{1}{N}\Big[-\frac{2m+\alpha_2-1}{\theta_2^2}+\frac{1-q}{(1+\theta_2)^2}+\sum_{j=1}^{m-1}\frac{(1+s_j)^2}{[\theta_2(1+s_j)+s_j]^2}$$
$$-\frac{2q(2\theta_1+3\theta_2+3)}{C}+\frac{qD^2}{C^2}-\frac{3q}{(\theta_1+\theta_2)^2}\Big],$$

$$g_{12}^{3*}=g_{21}^{3*}=\frac{\partial^2 g^{3*}(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2}=-\frac{q}{N}\Big[\frac{2(\theta_1+2\theta_2+2)}{C}-\frac{BD}{C^2}+\frac{3}{(\theta_1+\theta_2)^2}\Big].$$